**Measure Theory**
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**Module No # 03**
**Lecture No # 11**
**Characterization of Jordan Measurable sets and Basic Properties of Jordan Measure – Part 2**

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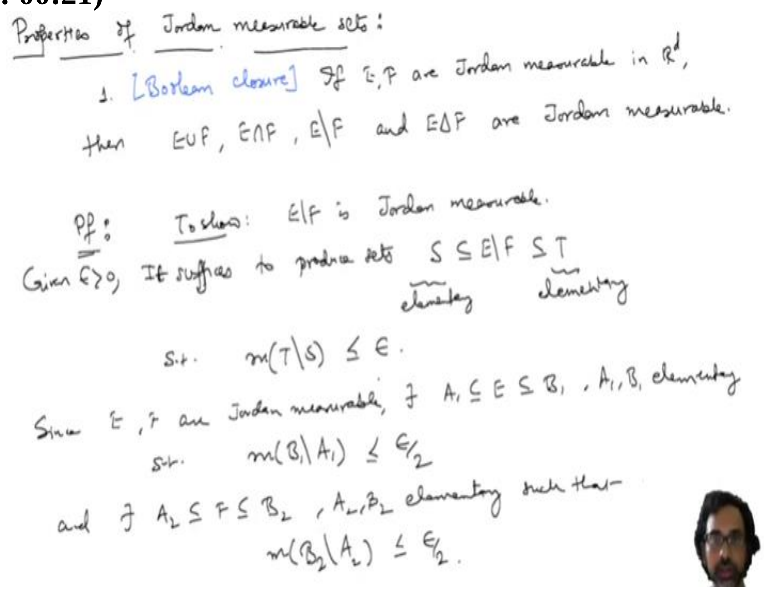


So the first of these properties is called Boolean closure and this is that if E and F are Jordan measureable subsets of Rd Jordan measureable in Rd then the union E union f the intersection and the said differences as well as the symmetric difference are all Jordan measureable. So let me prove one of that to illustrate the use of our equivalent conditions in the previous theorem. So I will only prove that E – F is Jordan measureable.

So to show this it suffices to produce sets S and T both elementary so S is elementary as well as T is elementary. So again given epsilon greater than 0 one can sandwich our set E – F between 2 elementary sets S and T says that the measure of T – S is less than equal to epsilon. So this was the second condition in our previous theorem which was in equivalent condition for Jordan measurability.

So I am going to use that E and F are themselves Jordan measurable so since EF are Jordan measureable. So we have there exist the elementary sets A1 and B1 elementary which sandwich

E such that the measure of B1 – A1 is less than or equal to epsilon / 2. And similarly for f they are exist A2 and B2 elementary such that the measure of B2 – A2 is less than or equal to epsilon over 2. Now I am going to use that this sets B1, B2, A1, A2 to produce our T and S.

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$$A_1 \setminus B_2 \subseteq E \setminus F \subseteq B_1 \setminus A_2$$

$$T = B_1 \setminus A_2, \qquad S = A_1 \setminus B_2$$

$$\begin{aligned} T \setminus S &= (\underbrace{B_1 \cap A_2^c}_{T}) \cap (\underbrace{A_1 \cap B_2^c}_{S})^c \\ &= (B_1 \cap A_2^c) \cap (A_1^c \cup B_2) \\ &= (B_1 \cap A_2^c \cap A_1^c) \cup (B_1 \cap A_2^c \cap B_2) \\ &= (B_1 \setminus (A_1 \cup A_2)) \cup ((B_1 \cap B_2) \setminus A_2) \\ &\subseteq (B_1 \setminus A_1) \cup (B_2 \setminus A_2) \end{aligned}$$

Now we have set inclusions our set E – F is included in B1 – A2 simply because is the subset of A1 and A2 is the subset of F. Similarly since A2 is the subset of E and f is the subset of B1 should be A1 – B2. So since A1 is the subset of E and f is the subset of B2. We have the first inclusions. So we will just rake T to be B1 – A2 and S to be A1 – B2 so let us see what is T – S, so this is B intersection A2 complement intersection with A1 intersection B2 complement and there is in another complement here.

So this set is S and you take T intersection S complement so we can write this as so first one is B1 intersection A2 complement intersection A1 complement union B2. So by using De Morgan’s law and now we can distribute this set inside the union so we will get B1 intersection A2 complement intersection A1 complement union B1 union B2. Now we can distribute this set B1 intersection A2 complement the A2 complement inside the union to get B1 intersection A2 complement intersection A1 complement union B1 intersection A2 complement intersection B2.

So see that the first one is B1 – A1 union A2 and this is the union with B1 intersection B2 – A2 now this is a subset of B1 – A1 and the second one is the subset of B2 - A2.

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2. Extension of properties of elementary measure:

Let $E, F \subseteq \mathbb{R}^d$ be Jordan measurable. Then

(i) [Non-negativity] $m(E) \geq 0$

(ii) [Finite-additivity] if $E$ & $F$ are disjoint then

$$m(E \cup F) = m(E) + m(F)$$

(iii) [Monotonicity] if $E \subseteq F$, then $m(E) \leq m(F)$

(iv) [Finite sub-additivity] $m(E \cup F) \leq m(E) + m(F)$

(v) [Translation-invariance] For any $x \in \mathbb{R}^d$, $E + x$ is Jordan measurable, and

$$m(E+x) = m(E)$$

Therefore this implies by monotonicity that the measure of T – S is less than or equal to the measure of B1 – A1 union B2 – A2 and by finite sub-additivity this is less than or equal to B1 – A1 + measure of B2- A2 which is epsilon. So this implies that the said difference E – F is Jordan measurable. Now we can state the definition for Jordan measure which is that given a Jordan measurable subset E of Rd then we know that the inner Jordan measure is equal to the outer Jordan measure and so we called the common value as the Jordan measure of E.

And it is denoted by m of E now the second part is just the extension of the properties of the elementary measure we have seen already. So for this we suppose that if E and F are Jordan measurable subsets of Rd then the first is non-negativity we have already seen this for elementary measure finite additivity monotonicity, finite sub-additivity and translation invariance.

So these are the properties we have already proved for elementary measure and we see that the Jordan measureable sets inherit this nice properties from our elementary measure. So these are quite easy to prove so let me just prove the second part which is finite additivity.

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(ii) Pf: Given $\epsilon > 0$, $\exists$ elementary sets $A_1 \subseteq E$, $A_2 \subseteq F$ s.t.

$$m_J(E) \leq m(A_1) + \epsilon/2$$

$$m_J(F) \leq m(A_2) + \epsilon/2$$

Since $A_1 \cup A_2 \subseteq E \cup F$,

$$m(E \cup F) = m_J(E \cup F) \geq \underbrace{m(A_1 \cup A_2)}_{\text{disjoint union}} = m(A_1) + m(A_2)$$

$$\geq \underbrace{m_J(E)}_{m(E)} + \underbrace{m_J(F)}_{m(F)} - \epsilon$$

$$\Rightarrow \quad m(E \cup F) \geq m(E) + m(F) - \epsilon$$

So for the proof of the second part; so give epsilon greater than 0 choose elementary sets. So given epsilon greater than 0 there exist elementary sets A1 inside E and A2 inside F such that the inner Jordan measure is less than or equal to m A1 + epsilon over 2 and the inner Jordan measure of F is less than or equal to m of A2 + epsilon / 2. So the inner Jordan measure of E union F. So now note that since A1 union A2 is inside the union A union F E union F so we get that inner Jordan measure of E union F is greater than or equal to the measure of A1 union A2.

But now we know that because E and F are disjoint these 2 are disjoint so this is the disjoint union and therefore this is equal to m A1 + m A2 by finite sub-additivity. And therefore this is greater than equal to by our 2 inequalities here this is greater than or equal to the inner Jordan measure of E + the inner Jordan measure of F – epsilon / 2 + epsilon / 2. So this is nothing but epsilon now note that E and F are Jordan measureable so this is nothing but the Jordan measure of E which is nothing but the Jordan measure of F.

And we have already seen that if 2 sets are Jordan measureable they are union is also Jordan measureable so this is nothing but Jordan measure of E union F. So we get that that the Jordan measure of E union F is greater than or equal to m E + m F – epsilon.

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Similarly, the same argument for the outer Jordan measure gives

$$m(E)+m(F)-\epsilon \leq m(E\cup F) \leq m(E)+m(F)+\epsilon$$

$$\Rightarrow \quad m(E\cup F) = m(E)+m(F)$$

So this we did for the inner Jordan measure similarly for the outer Jordan measure similarly the same argument for the outer Jordan measure which will give us that the measure of E union F is less than or equal to the measure of E + the measure of F + epsilon and from the inner Jordan measure condition we get m E + F – epsilon. So since epsilon is arbitrary we have an; equality so we that the finite additivity condition holds for E and F Jordan measureable and disjoint.

So we stop our lecture here and in the next class we will see what kind of specific examples of Jordan measureable subsets of Rd we can produce and we will see that there are plenty of them.