Measure Theory
Prof. Indrava Roy
Department of Mathematics
Institute of Mathematical Science

Module No # 02
Lecture No # 10
Characterization of Jordan Measurable sets and Basic Properties of Jordan Measure – Part 1

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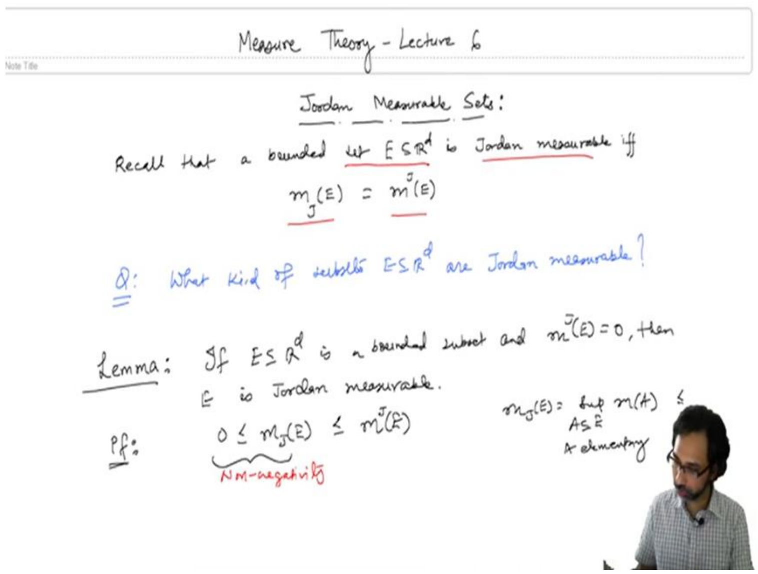

In the last lecture we have defined the notion of a Jordan measurable set and recall here the definition a bounded set E a subset of Rd is Jordan measurable if and only if the inner Jordan measure is equal to the outer Jordan measure of E. Now we ask ourselves the question the question is what kind of sets what kind of subsets of Rd are Jordan measurable. So remember that our main goal was to enlarge the class of object to which a nice notion of a measure can be defined.

And initially we started with elementary sets of Rd and now we want to enlarge the class of the sets on which a measure can be defined and this why we define the notion of a Jordan measurable set of subset of Rd. Now the usual the natural question is what kind of subsets are Jordan measurable? So let us begin with Lemma and easy Lemma says that if E is a bounded subset and the outer measure of E is 0 then E is Jordan measurable.

So the proof is just one so just one line proof this is because we already know that our inner Jordan measure is always non negative ok. This is the non-negativity property and then we have that the inner Jordan measure is bounded above by the outer Jordan measure of E. This is because remember the definition of a the inner Jordan measure which is that the inner Jordan measure is the supremum of the elementary sets subsets of E elementary and you take the elementary measure of this set A.

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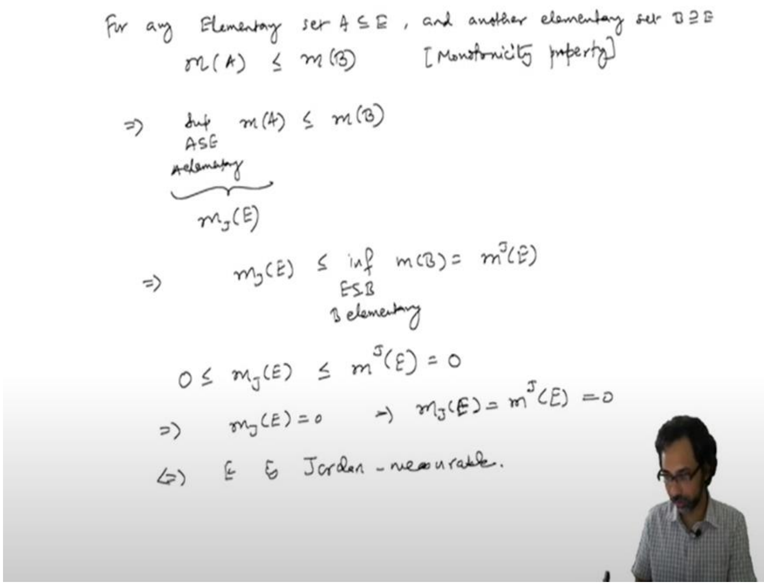

So now given that this is less than or equal to so first of all so for any subset elementary set A which is a subset of E and another elementary set B which is now a super set of A we have that the elementary measure of A is less than or equal to elementary measure of B. This is because of monotonicity property of the elementary measure. Now if you take the supremum on the left side first over all elementary subset of E this is less than or equal to the measure of B.

Now this is nothing but the inner Jordan measure of E. Now on the right hand side I take the infimum over all elementary subsets which are supersets of E and this is nothing but the outer Jordan measure of E. Therefore the inner Jordan measure is always bounded above by the outer Jordan measure. And so now that we know that by non-negativity we have 0 less than or equal to m super subscript j inner Jordan measure of E this is less than or equal to outer Jordan measure of E and by hypothesis this is 0.

So therefore the outer Jordan measure is also 0. So this implies that the inner Jordan measure is equal to the outer Jordan measure both are 0 therefore is Jordan measurable. Now let me give a theorem which gives a characterization for Jordan measurable sets.

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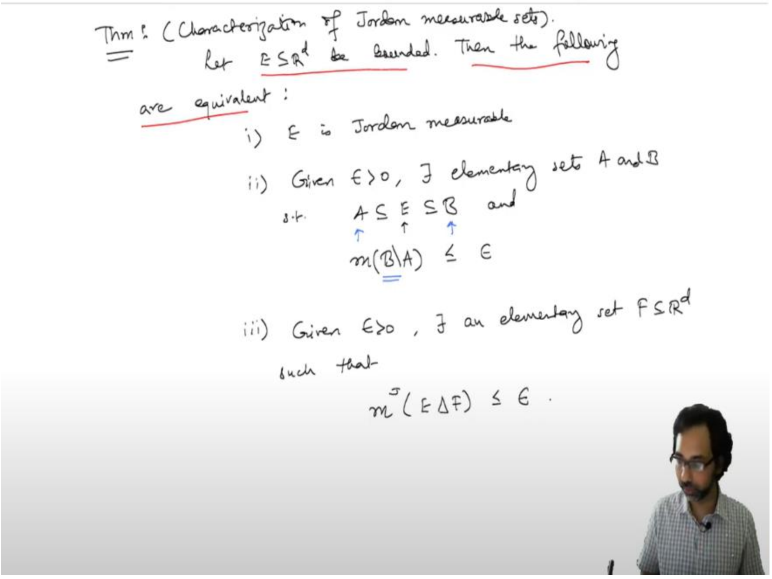

The statement of the theorem is as follows. So suppose that E is a bounded subset of Rd then the following are equivalent ok. So we will give 3 conditions for the Jordan measurability of E. So first is that E is Jordan measurable. So this is by the definition of Jordan measurability. Second is that given epsilon greater that 0 there exists elementary sets A and B such that A is a subset of E and E is the subset of B and the elementary measure of B - A is less than or equal to epsilon.

So this second condition says that the set E can be sandwiched between 2 elementary sets A and B such that the difference B - A has elementary measure as small as we like. So this says that the difference between the inner approximation and the outer approximation can be made as small as we like. Now the third condition says then given epsilon greater than 0 where exist an elementary set F such that the outer measure of E with the symmetric difference of F is less than or equal to epsilon.

So here note that this F may or may not lie completely within E or may not middle may not cover E completely as well. So we take the symmetric difference but it says that the outer Jordan measure of this symmetric difference is as small as when you like ok. So let us try to prove this.

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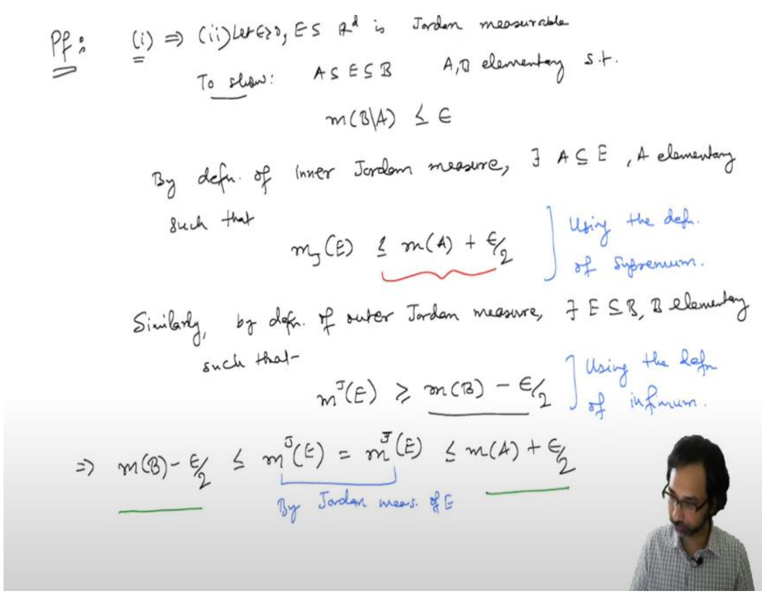

So proof so first is we will show 1 implies 2 now by Jordan measurability so we know that from one that as the set E is Jordan measurable and we will produce 2 elementary set 1 inside E and 1 covering E A B elementary such that, so let us fix first an epsilon greater than 0. And given this epsilon we have to find A and B elementary such that the measure of B - A is less than or equal to epsilon. So let us start with the definition of inner Jordan measure.

So by definition of inner Jordan measure we have that there exists an elementary subset of E such that the inner Jordan measure of E is less than or equal to m of A + epsilon over 2. Now this is simply by using the property of the supremum using the definition of supremum. So we can find an elementary subset of E such that the inner Jordan measure is less than or equal to the elementary measure of that subset plus some epsilon over 2.

Similarly by definition of outer Jordan measure there exist an elementary subset B covering E such that the outer Jordan measure of E is greater than or equal to mB - epsilon over 2. Now this is just using the definition of the infimum using the definition of infimum ok. So now we can write let us start with this one here so mB - epsilon over 2 is less than or equal to m superscript E j of E.

So our outer Jordan measure of E but now we know that since E is Jordan measurable these 2 the inner Jordan measure and the outer Jordan measure are equal so by Jordan measurability of E.

And next we have from the first inequality we have m A + epsilon over 2. So if we consider the first in this chain of equality and the last one then we have what we want.

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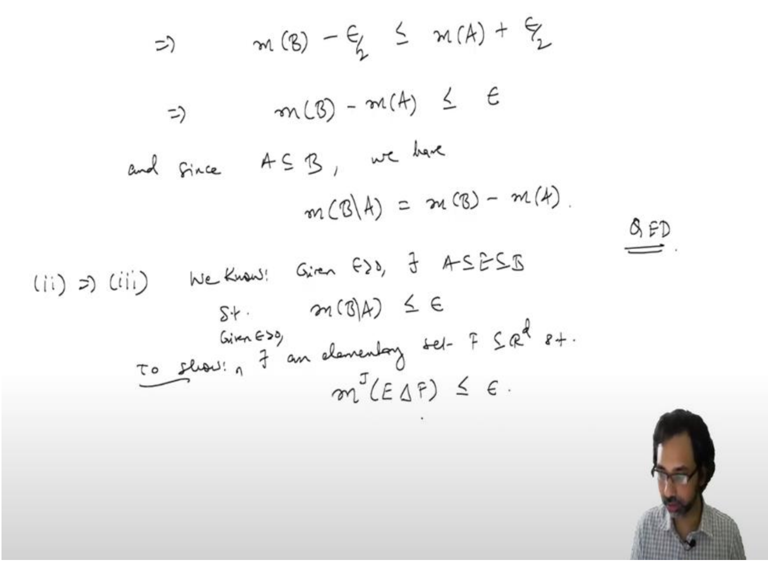

So this implies that m B - epsilon over 2 is less than or equal to m A + epsilon over 2. So all these things are finite so finite and non-negative so you can subtract on both sides is less than or equal to epsilon. And because A is a subset of B we have the measure of B - A is equal to the measure of B- the measure of A. So this is what we wanted so this proves that E is Jordan measurable implies that there exist elementary subsets A and B 1 within E and 1 without I mean covering E such that the difference has measure as small as we want ok.

So now we can go to the second part 2 implies 3 so here we know that given epsilon there exist A and B again such that the measure of the difference B- is less than or equal to epsilon and we have to show that there exists an elementary set E elementary set F sorry of Rd such that the outer Jordan measure of the symmetric difference is less than or equal to epsilon. So again so we fix here an epsilon and then we can say that there exists and elementary set f for which this is valid ok.

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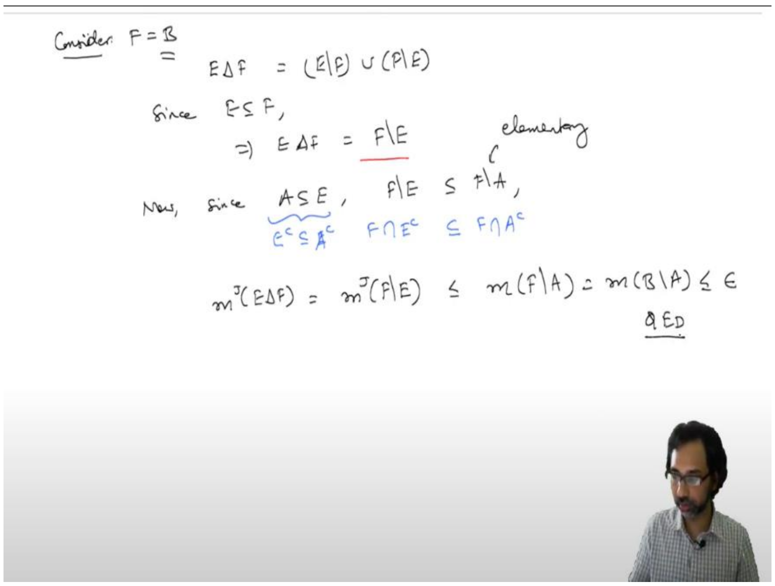

So now the obvious candidate to choose is F = B. So consider this set B itself then the symmetric difference is simply the union of E - F and F – E. But since E is a subset of F then this implies that the symmetric difference is simply the difference F - E. So this is because we have chosen F to be our set B ok. And now since A is a subset of E F - E is a subset of F - A ok. So we can just write this as F intersection, E compliment and this is F intersection A compliment and this inclusion implies that E compliment is a subset of A complement.

So therefore F - E is included in F - A. Now by the outer definition of the outer Jordan measure so if you take the definition of the outer Jordan measure of the symmetric difference E of E and F this is nothing but the outer Jordan measure of F - E because of our choice of F. And then this is less than or equal to measure of F - A because this set is now elementary and by the definition of the outer Jordan measure this is the infimum of all such sets.

Therefore this outer Jordan measure of F - E is less than or equal to the measure of F – A, but since F was equal to B this is B - A and we know that this is less than or equal to F E ok. So this shows that 2 implies 3.

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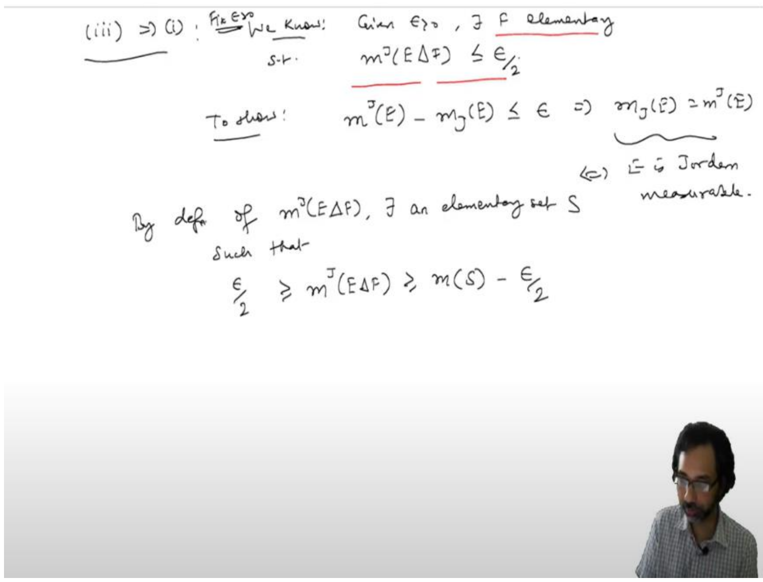

Now the most, tricky part is 3 implies 1 it is not too difficult but a bit tricky. So here we know that given epsilon greater than 0. We have an elementary set such that the outer Jordan measure is less than or equal to epsilon. So, we choose for given epsilon such set F. Now we need to show that the difference between the outer Jordan measure and the inner Jordan measure is less than or equal to epsilon.

So since epsilon was arbitrary this would imply that these 2 things are equal and this is the same as saying that E is Jordan measurable ok. So we start with what we know which is that there exist an elementary set F such that the outer Jordan measure of the symmetric difference of E and F is less than or equal to epsilon. So by definition of outer Jordan measure of this symmetric difference so again let us fix our epsilon.

So once the epsilon is fixed we choose our elementary set F and then we can say that by the definition of the outer Jordan measure there exists an elementary set, let us call it S such that the outer Jordan measure is greater than or equal to the measure of S- epsilon over 2. So in fact it is it will be convenient to have already that the outer Jordan measure is of E symmetric difference f is less than or equal to epsilon over 2. So then we have this is less than or equal to epsilon over 2.

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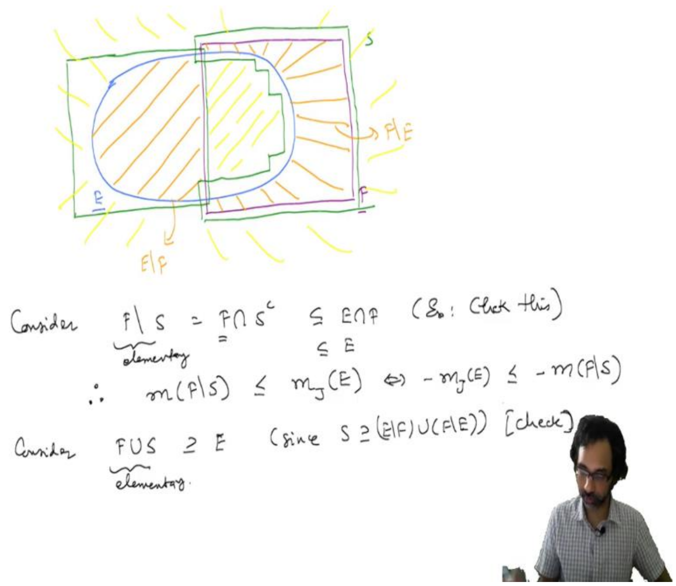

So let us look at the Venn diagram picture that we have here. So we consider our set E in blue and our elementary set F in violet color and now this part is E – F. So this is E - F and this part is F - E. Now our set S will cover the union of these 2 so it can be something like this so 1 part covers E - F and the other part will cover F - E so it could be something like this. So the set S will be a union of these 2 regions enclosed in the green boundaries.

Now if you consider F - S so which is simply F intersection S complement. So S compliment will have 2 parts 1 part is within E intersection F and the other part is without. So this is outside E union F. So I will consider there only the part which lies only within E intersection F which is why I have taken an intersection with F. So, one can easily show that this is inside E intersection F. So as an exercise you can check this using just an laws for intersection union and complements you can simply show that it lies within E intersection F.

Therefore the elementary measure of F - S is less than or equal to the inner Jordan measure of E because the set F - S is an elementary set which lies inside E intersection F. So it also lies inside E and so by the definition of the inner Jordan measure we have that we have this inequality. So I am going to write this as -MjE the inner Jordan measure is less than or equal to minus measure of F - S.

Now consider the union F of the union of F and S. Now one can show that this is in included sorry this includes the set E itself because S covers E - F and union F - E. So one can check again that this set F union S covers E and now this is again an elementary set.

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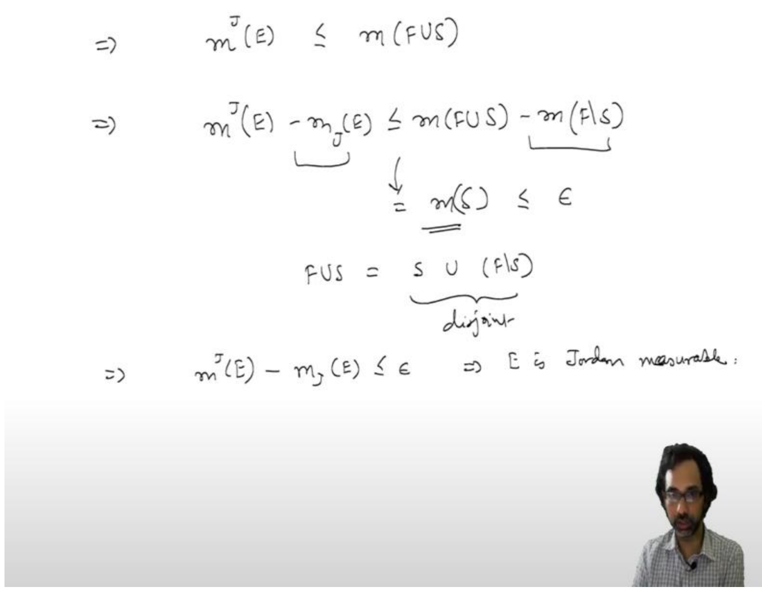

So therefore we can use the definition of the outer Jordan measure of E and this will be bounded above by the measure of F union S. So this implies that the measure the outer Jordan measure minus the inner Jordan measure of E is less than or equal to measure of F union S- measure of F - S. So this thing was less than or equal to this thing and we have added the 2 inequalities ok.

So now notice that this right hand side is equal to the measure of S itself because F union S can be written as S union F - S and this is the disjoint union and one can use finite additivity to get this this equality from m of F union S- measure of F - S = measure of S. But S was chosen to have measure less than equal to epsilon. So we have shown that outer measure of E- the inner measure of outer Jordan measure of E- inner Jordan measure of E is less than or equal to epsilon which shows that E is Jordan measurable.

So this these extra 2 equivalent condition for the Jordan measurability of E will come coming quite handy to prove if some given set given bounded subset of Rd is Jordan measurable or not. Now we will use these equivalent conditions to show some nice properties of the Jordan measurable sets which it inherits from elementary sets