MEASURE THEORY

PROF. INDRAVA ROY

Lecture 1: Finite Sets and Cardinality

Welcome to the first lecture on Measure Theory. When we say the word measure in its everyday meaning, we can think of measuring the length of an object or the area or the volume of an object. So this measures the size of the object or we can think about measuring the weight of an object or the temperature and so on. But at a fundamental level if you just think of it mathematically, we are just assigning numerical values to the various properties of the object. Now, this assignment of numerical values should follow certain reasonable intuitions, reasonable rules. For example, if we have two cups filled with water and you pour the water into a third empty vessel, then the amount of the water in the third vessel should be equal to the sum of the amounts of water in the two cups. So, these kinds of rules are reasonable to explain.

Outline of the lecture:

- Cardinality of finite sets
  - Proving that cardinality is well-defined
- Finite additivity property of cardinality

Today, we will see that if we get this numerical assignment for finite sets to be the cardinality of that set, then such a rule can be expected and it holds. This property is called the finite additivity of the cardinality. So first we will look at the definition of cardinality and we will try to show that this definition is indeed well posed and it is not meaningless. Secondly, we will show that according to this definition, we will have this finite additivity property in which when you take two disjoint finite sets and you take the union, then the cardinality of the union will be the sum of the cardinality of the individual sets.

Department of Mathematics, The Institute of Mathematical Science, Chennai.

Measure Theory - Lecture 1

6/17/2020

Definition (Finite sets and Cardinality):

A set A is called finite if

(i) A is empty

(ii) If $A \neq \phi$, and there exists a bijection between A and $\{1, 2, \ldots, n\}$ for some $n \in \mathbb{N}$.

In (i), the cardinality of $A = 0$

(ii), the cardinality of $A = n$.

Let us begin by recalling the definition of a finite set and its cardinality. A set A is called finite if either A is empty, (in this case we assume that empty sets are finite), or if A is not the empty set, there should exist a bijection between A and the set of positive integers from $1, 2, \ldots, n$ for some n in $\mathbf{N}$. So let me reread it. A set A is called finite either if it is empty or if it is not empty, then there should exist a bijection between the set A and $\{1, 2, \ldots, n\}$. So in the first case, the cardinality of A is said to be 0 and in the second case the cardinality of A is said to be n.

Now whenever we meet a definition, we should knock it from all sides to see whether it makes sense and whether there is no ambiguity in the definition. In this case, in the definition that I just provided, there is a bit of ambiguity. Because the natural number n, the cardinality, is not entirely determined by the set A itself. I can ask the following question.

Q: Given a non-empty set A, does there exist natural numbers $n, m \in \mathbb{N}$, $m \neq n$ such that $\exists$ bijections

$f_1 : A \longrightarrow \{1, 2, \ldots n\}$, and

$f_2 : A \longrightarrow \{1, 2, \ldots, m\}$

Thm: Suppose A is a non-empty set and there exists a bijection $f : A \longrightarrow \{1, 2, \ldots n\}$ for some $n \in \mathbb{N}$. Suppose $B \subset A$ is a proper subset of A. Then, provided B is not empty, there exists a bijection

$g : B \longrightarrow \{1, 2, \ldots m\}$ for some $m < n$.

and there is no bijection $h : B \longrightarrow \{1, 2, \ldots, n\}$

Given a non-empty set A, does there exist natural numbers n and m, $m \neq n$, such that there exist bijections $f_1 : A \longrightarrow \{1, 2, \ldots, n\}$ and $f_2 : A \longrightarrow \{1, 2, \ldots, m\}$?

So it is not clear from the definition that this situation is precluded and if this question has a positive answer, that is if there exist two such bijections with different ranges, then our notion of cardinality will not make sense. One should answer this question in the negative to be able to say that this definition is not meaningless, is not absurd. Okay, so for this we have the following theorem.

Theorem 0.1. *Suppose A is a non-empty set and there exists a bijection $f : A \longrightarrow \{1, 2, \ldots, n\}$ for some natural number n. Suppose also that B is a non empty proper subset of A, then there exists a bijection $g : B \longrightarrow \{1, 2, \ldots, m\}$ for some $m(< n)$.*

By this theorem, notice that there is no bijection $h : B \longrightarrow \{1, 2, \ldots, n\}$ for any proper subset B of A, here A has cardinality n. Hence we answer the above question in negative. This kind of results can be found in many books, so I will not prove it here. Let me just give the reference. See Munkres's book on Topology, chapter 1, section 6, where this is proved in detail.

Pf: [See Munkres 'Topology', Chapter 1, Section 6]

Corollary: (i) If A is a finite set, then there is no bijection of A with a proper subset of itself.

(ii) The cardinality of a finite set is uniquely determined by A.

Pf: (i) $\exists$ a bijection $f : A \to \{1, 2, \ldots, n\}$ for some $n \in \mathbb{N}$.
If $B \subset A$ is a proper subset and $g : A \to B$ is a bijection then $h : B \to \{1, 2, \ldots, n\}$ $\underbrace{h}_{\text{is a bijection}} := f \cdot g^{-1}$, a contradict to the thm.

Now this theorem has a few important consequences. Let us collect them in this corollary.

Corollary 0.2. *1. If A is a finite set, then there is no bijection of A with a proper subset of A.*
2. The cardinality of a finite set is uniquely determined by A.

This answers our question about the well posedness of the definition of cardinality of a finite set. Let us see of the short proofs of the above corollary. If A is a finite set, then there exists a bijection f from A to some subset of the form $\{1, 2, \dots, n\}$ for some n. Now if $B \subset A$, and $g : A \longrightarrow B$ is a bijection, then I am going to produce a bijection h between B and $\{1, 2, \dots, n\}$ and h is simply given by $f \circ g^{-1}$. So g^{-1} takes you from B to A, and f takes you from A to $\{1, 2, \dots n\}$. So our h, given by $f \circ g^{-1}$, is a bijection between B and $\{1, 2, \dots, n\}$. This is a contradiction to the theorem. This h we note that this is a bijection because it is composed of two bijective functions, therefore h itself is a bijection. So we have arrived at a contradiction.

(ii) If $f_1 : A \longrightarrow \{1,2,\dots,n\}$ is a bijection

and $f_2 : A \to \{1,2,\dots m\}$ is also a bijection

Suppose WLOG $m<n$.

$\Rightarrow$ $f_1 \circ f_2^{-1} : \{1,2,\dots m\} \longrightarrow \{1,2,\dots n\}$ is a bijection

this is a contradiction since $\{1,2,\dots m\} \subset \{1,2,\dots n\}$

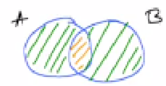
(follows from part (i) above).

For the second part, we have to show that if $f_1 : A \longrightarrow \{1, 2, \dots, n\}$ is a bijection and $f_2 : A \longrightarrow \{1, 2, \dots, m\}$ is also a bijection, then we have to arrive at a contradiction. Let us suppose that without loss of generality that m is less than n. Then $f_1 \circ f_2^{-1}$ is a map from $\{1, 2, \dots, m\}$ to $\{1, 2, \dots, n\}$. This is a bijection, but since m is strictly less than n, we have arrived at a contradiction. This is a contradiction since $\{1, 2, \dots m\}$ is a proper subset of $\{1, 2, \dots, n\}$ and this follows from the first part of our corollary. Therefore, we have shown that the notion of cardinality of a finite set is well defined.

Corollary: Finite unions of finite sets are finite sets.

Pf: $A, B \neq \emptyset$, If A and B are finite then $A \cup B$ is finite. (To show)

$$A \cup B = (A \Delta B) \cup (A \cap B)$$



Assume WLOG: A and B are disjoint

$\exists \quad f_A : A \longrightarrow \{1, 2, \ldots n_1\}$

$f_B : B \longrightarrow \{1, 2, \ldots n_2\}$

To produce $f_{A \cup B} : A \cup B \longrightarrow \{1, 2, \ldots, n_1 + n_2\}$

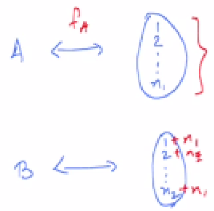
Now, there are further easy corollaries, one is as follows: *Finite unions of finite sets are also finite set.*

Let us see a proof. Let me just take two non-empty sets A, B, both are non empty. Then, I will prove that if A and B are finite, then $A \cup B$ is finite. Now we have seen that any proper subset of a finite set is also finite.We have $A \cup B = (A \Delta B) \cup (A \cap B)$, here $A \Delta B$ is the symmetric difference.

So now we have written $A \cup B$ as a union of two disjoint sets, first one is the symmetric difference of A and B and the second one is the intersection of A and B. So a quick Venn diagram will show that this is true. So this is A and B and the symmetric difference is the part without the intersection. The intersection is of course the overlap. So now, we can assume that without loss of generality that A and B are disjoint. So it is enough to show that if A and B are disjoint, and both are finite , then their union is also a finite set. Now we again go back to our definition of finite set. We are given there exists bijections $f_A : A \longrightarrow \{1, 2, \ldots, n_1\}$ and $f_B : B \longrightarrow \{1, 2, \ldots, n_2\}$, n_1 may be equal to n_2, but we do not care at this point. Now, we have to produce $f_{A \cup B} : A \cup B \longrightarrow \{1, 2, \ldots, n_1 + n_2\}$. Of course the expected cardinality is $n_1 + n_2$ and this is a type of conservation law that the cardinality of the union of two disjoint sets is the sum of the cardinalities of the individual sets.

$$f_{A\cup B} : A\cup B \longrightarrow \{1, 2, \dots, n_1+n_2\}$$

$$f_{A\cup B}(x) = \begin{cases} f_A(x) & \text{if } \underline{x\in A} \\ \underbrace{n_1 + f_B(x)} & \text{if } x \in B \end{cases}$$



Ex: Show that $f_{A\cup B}$ is a bijection.

We construct $f_{A\cup B} : A \cup B \longrightarrow \{1, 2, \dots, n_1 + n_2\}$ as follows.

$$f_{A\cup B}(x) = \begin{cases} f_A(x) & \text{if } x \in A, \\ n_1 + f_B(x) & \text{if } x \in B. \end{cases}$$

If $x \in A$, we land in $\{1, 2, \dots, n_1\}$ by f_A. Because our bijection f_A is between A and $\{1, 2, \dots, n_1\}$. For $x \in B$, if we just take here $f_B(x)$ rather than $n_1 + f_B(x)$, then we will end up in the set $\{1, 2, \dots, n_2\}$ and there will be overlaps in the range of this function $f_{A\cup B}$. We do not want this, so, I shift it by the value n_1. It is an easy exercise show that $f_{A\cup B}$ is a bijection.

So we have shown that if you take two disjoint sets, the cardinality of the union is the sum of the cardinalities of the respective sets.

I want to write this in the following notation. So let me denote the cardinality of A by the absolute value symbol $|A|$. Then in notational form, we have the following principle.

Notation: Denote the cardinality of a finite set A by $|A|$.

Principle of Conservation of Cardinality:
If A and B are disjoint sets, then

$$|A\cup B| = |A| + |B|$$

Generalization: If $A_1, A_2 \dots A_n$ are disjoint sets then

$$|A_1 \cup A_2 \cup \dots \cup A_n| = \sum_{i=1}^{n} |A_i|$$

$\rightsquigarrow$ finite additivity.

The principle of conservation of cardinality: if A and $B$are disjoint sets, then $|A \cup B| = |A| + |B|$.

So, I want you to remember this principle. This will be a guiding principle for us when we talk about infinite sets and subsets of $\mathbf{R}^n$. For the moment we can also generalize this from two sets to any arbitrary number of sets. So generalization, one can easily prove by induction, is that if $A_1, A_2, \ldots, A_n$ are disjoint sets, then the $|A_1 \cup A_2 \ldots A_n|$ is $\sum_{i=1}^{n} |A_i|$, the sum of the individual components. So, we will refer to this principle or rather this property as finite additivity.