`Computational Commutative Algebra

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Lecture – 60
Castelnuovo Mumford regularity - Part 2

Welcome, this is the lecture 60 and this is the final lecture in this course on Computational Communicative Algebra. So, we will, in this lecture we will look at some applications of Castelnuovo Mumford regularity in a problem in geometry problem that would be stated without any reference to any, it will just be about some functions and some points.

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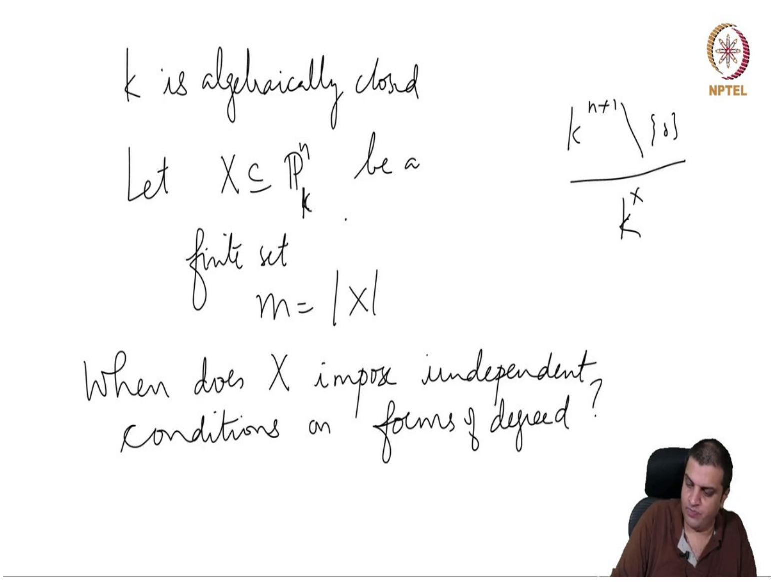

So, what is the question that we want to understand? So, throughout this lecture k is algebraically closed and we ask the following question; let $X \subset P^n$, P^n over k. So, we can think of this as lines in $\frac{k^{n+1} \setminus 0}{k \setminus 0}$, but take away the origin and then we let the units act by scalar multiplication.

So, the points in P_k^ncorrespond to such lines. So $X \subset P^n$ be a finite set; let us say m is the cardinality of X. So, we take m distinct points in $P^n{}_k$, and we ask when do these, when does X impose independent conditions on forms of degree d,?

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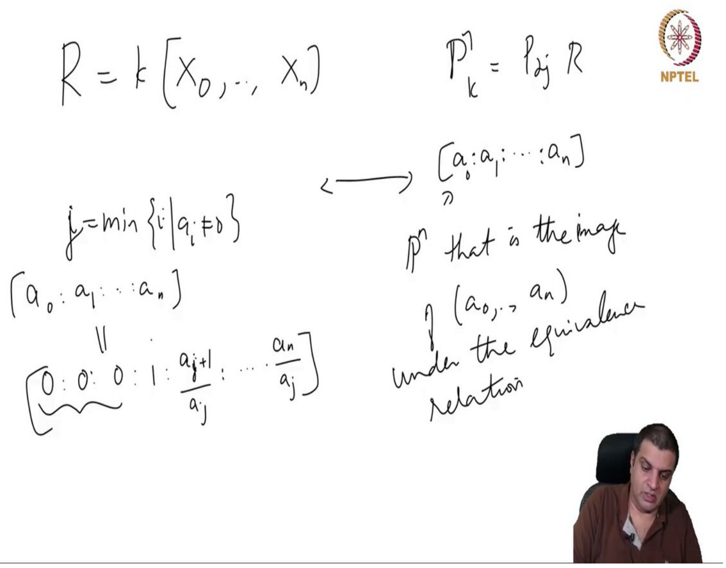

So, what does this mean? So, what does a question mean? So, if you think about the. So, we consider the polynomial ring in $k[x_{1,\ldots}, x_{n+1}]$; $P^n{}_k = Proj(R)$. Well, Proj here means homogeneous prime ideals that are maximal with respect to the property that they are properly inside the homogeneous maximal ideal.

So, points inside here will be actually equation will, in k^{n+1} it would just be given by a whole line. So, we can just write down the slope of the line$[a_0 : a_1 : \ldots . : a_n]$; we actually do not care about the actual values of $a_{0,}$ we only care about the ratio of these numbers, because if there is a $[b_0 : b_1 : \ldots . : b_n]$ that would be a same point if they have, if one can be obtained from the other by multiplication by a non zero element in k.

So, this is the point inside $P^n{}_k$ that is the image of $(a_{0,}, \ldots, a_n)$ under the equivalence relation equal. So, this is the equivalence class of this point. So, points in $P^n{}_k$ are not actual n +1 tuples, they are n + 1 tuples with an identification; the identification is that two n+1 tuples are considered to be same, if one is obtained by the other by a multiplication by a non-zero element of k .

So this one corresponds to the point, the maximal ideal in this thing which is; so, let us assume for simplicity that, I mean not all of them can be 0. So, let us assume for simplicity that, $j = \min\{i : a_i \neq 0\}$.

Because we are worried about ratios, whether a_i is non-zero does not depend on which point we take in the equivalence class,. So, a_i is non-zero and then we can write. So, then

$$[a_0 : a_1 : .... : a_n] = \left[0 : ... : 0 : 1 : \frac{a_{j+1}}{a_j} : ... : \frac{a_n}{a_j}\right]$$ 0 up to this is up to j - 1; .

So these two are the same points because, often we would write a point like this with a 1 somewhere, so just to clarify that they are the same points. But if you have a point like this; we can write.

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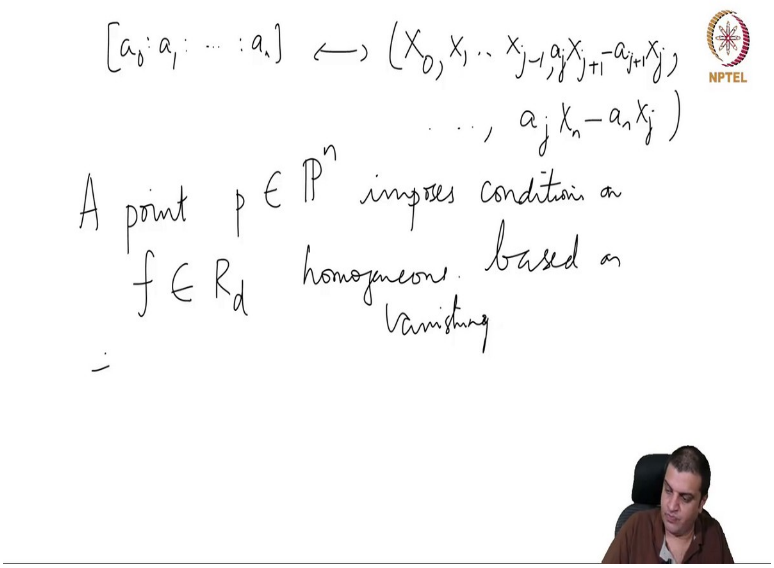

So, the point $[a_0 : a_1 : .... : a_n]$ corresponds to the prime ideal $(X_{0,}..., x_{j-1}, a_j x_{j+1} - a_{j+1} x_j., , a_j x_n - a_n x_j)$. .

So, this is a prime ideal of height n and therefore, a point inside Proj. So, this is the identification. So, now, what does the question say? A point $p \in P^n{}_k$ imposes conditions on f. So, we only have to worry about homogeneous polynomials; because if you try to evaluate f at p, then p is not a single point, p is a ratio alright.

So, which means if you, instead of taking p if you take some other point inside k^{n+1} representing this, f should take the same value and the only way that will happen is if f is homogeneous. So, f is homogeneous of some degree d. So, what does this mean? That is the coefficient; so imposes conditions based on vanishing more precisely,

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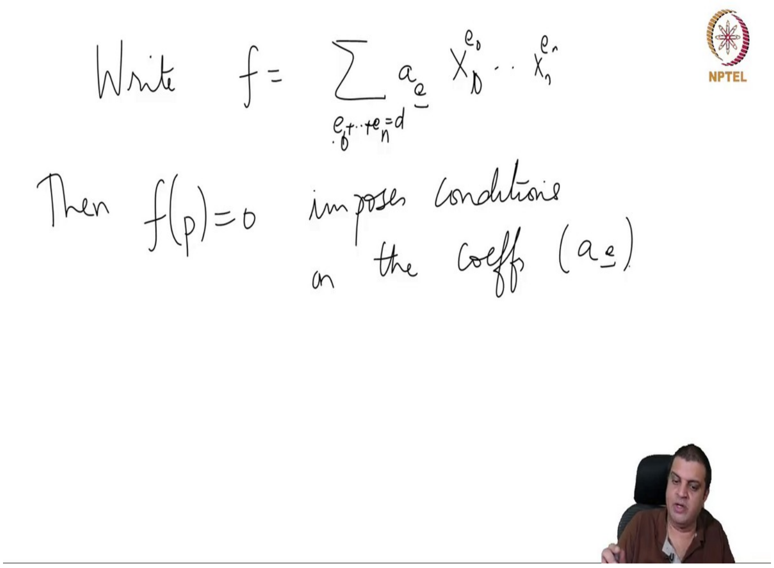

We can write $f=\sum_{e_0+\dots e_n=d} a_{\underline{e}} X^{e_0}\dots X^{e_n}$. So, f can be written like this. And now the condition $f(p)=0$ then imposes conditions on the in on the coefficients. So, let us look at this. So, each point will impose some condition and the question therefore is, when does X impose independent conditions on forms of degree d? So, let us look, let us look at what this means.

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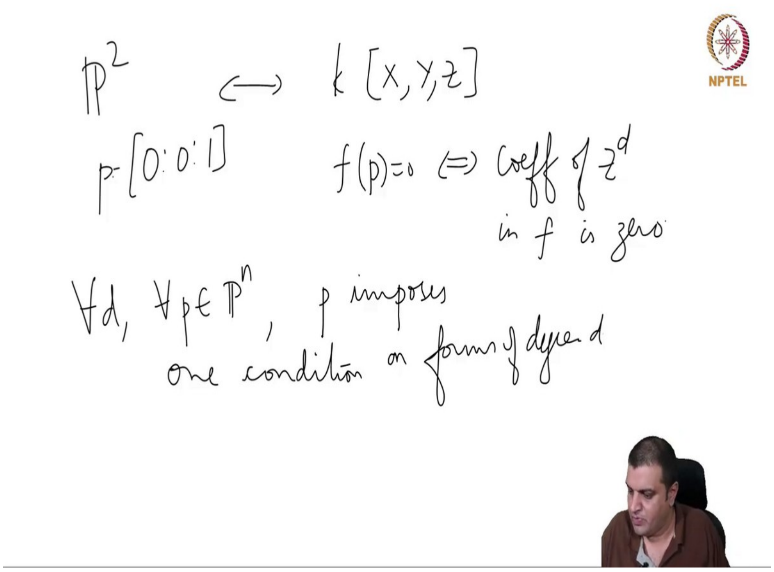

So, it is easy to do an example in P^2. So, the corresponding ring is $k[X,Y,Z]$. So, if you take the point p=[0: 0: 1]; so this is the point inside k 3 corresponding to 0, 0 and a non-zero third coordinate and all of them will correspond to the same point inside P^2,

$f(p)=0$ if and only if the coefficient of Z^d in f is zero. So, a single point imposes a single condition in every degree. And similarly we can do in any for other points also; it would be slightly little bit more complicated and we will see an example along the way.

So, every point imposes one condition, I mean conditions are coefficients independent, they are related; but given a degree d, on the forms of degree d a point will impose one condition, ok. So, for all d and for all $p \in P^n$; p imposes one condition on forms of degree d, form mean homogeneous polynomial. So now the question is, if you have m points in when do they impose independent conditions? So, if there are m points, when will they impose m conditions?

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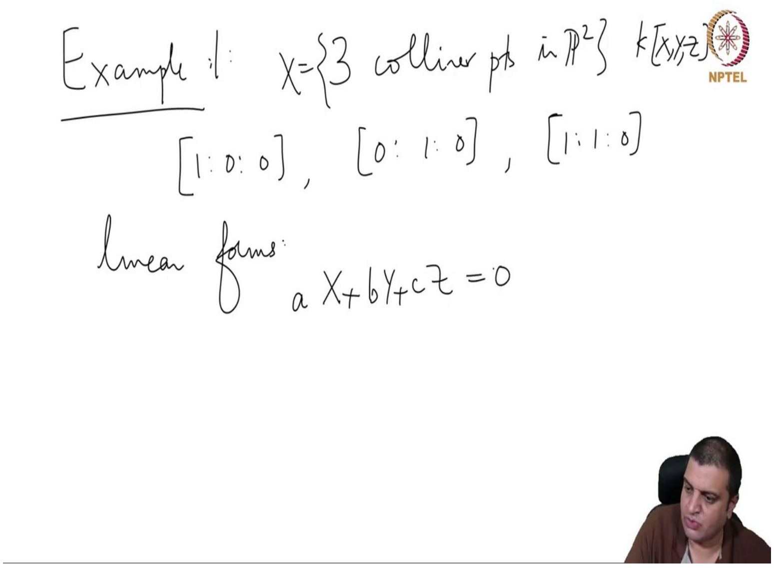

So, example 1: take 3 collinear points. So, X is 3 collinear points, X is a set of 3 collinear points in P^2. So, then we may as well assume that, I mean we can just do some change of coordinates. So, that it makes, it looks like this [1: 0: 0]; we can do some, this example is as general as the general case can get [0: 1: 0] and [1: 1: 0].

So now let us look at linear forms; So, we will write the way this is in P^2; so we will write the ring as k[X, Y, Z] not f = aX+bY+c Z. So, those are the variables; this one on this.

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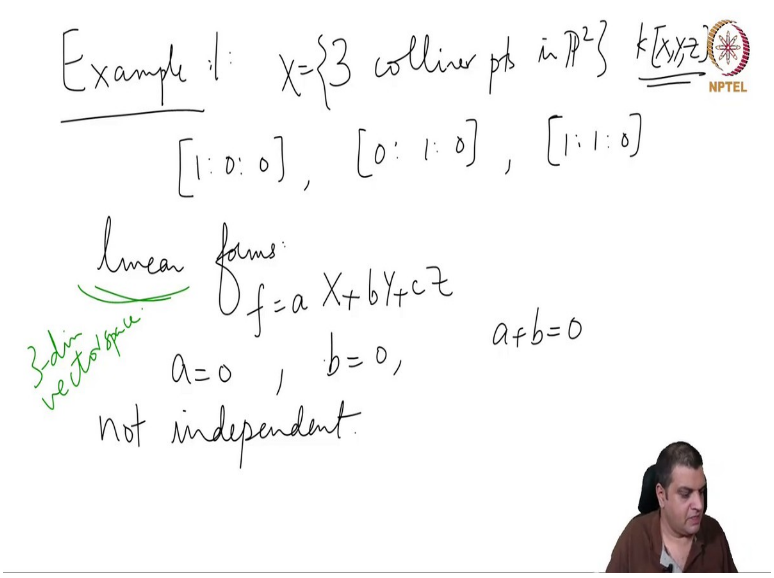

So, f has to vanish here means that a must be 0 that is from here it says b must be 0 and from here it says a + b must be 0 , but these are not independent conditions,. Because it only imposes two conditions and the for linear forms form a. So, this is a 3 dimensional vector space.

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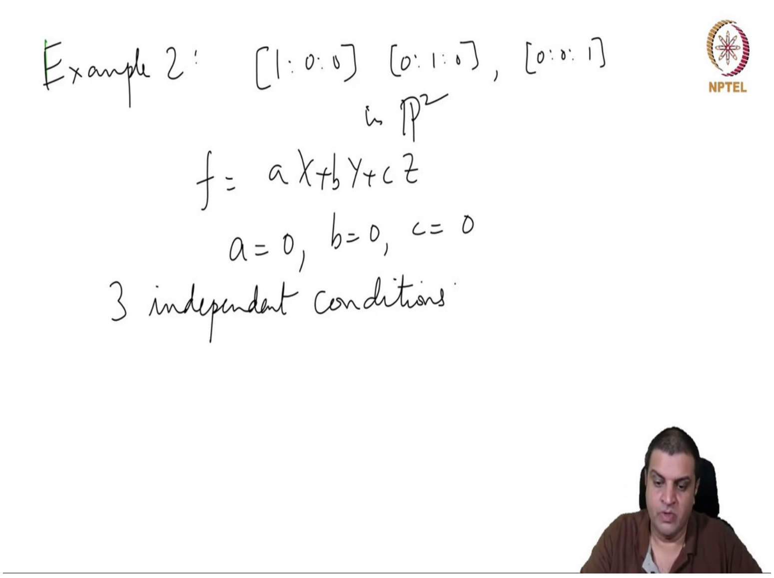

Now, we can look at a different example, example 2. The points are not collinear and we could take them to be like [1:0:0], [0: 1: 0], [0:0:1]. Again the example is still in P^2. And if you look at f = aX + b Y + c Z it imposes three conditions; it says a must be 0, b must be 0 and c must be 0; so, 3 independent conditions.

So here recall that in the first example of the points where 1, 0, 0, so that corresponds to vanishing of Y and Z; this is corresponds to vanishing of X and Z and that corresponds to the vanishing of Z and X being equal to Y.

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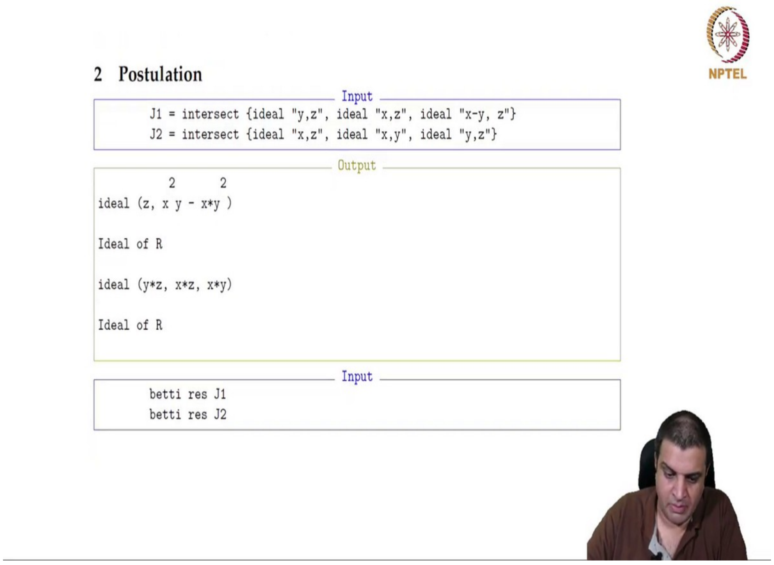

This is $J_1=(y,z)\cap(x,z)\cap(x-y,z)$; x minus y is to set that x is equal to y. So, this is the ideal the first point, this ideal is the second point, this ideal of the third point 1, 1, 0. So, we intersect that and then we get z and this one. So, the ideal is this.

In the next example, we the ideals are this is (Y, Z); this is defined by (X, Z) and this defined by (X,Y). not in that same order, but these three.

And we ask Macaulay to intersect and that would give the ideal of; in the first example the ideal of that set of points is this, second example the ideal set of points is this. So the ideals do look slightly different; collinear means that, there is a linear form in that ideal. So, it is a linear form and a quadratic and a cubic and here there are three quadrants. So, we ask for the resolution of $\frac{R}{J_1}$ and $\frac{R}{J_2}$.

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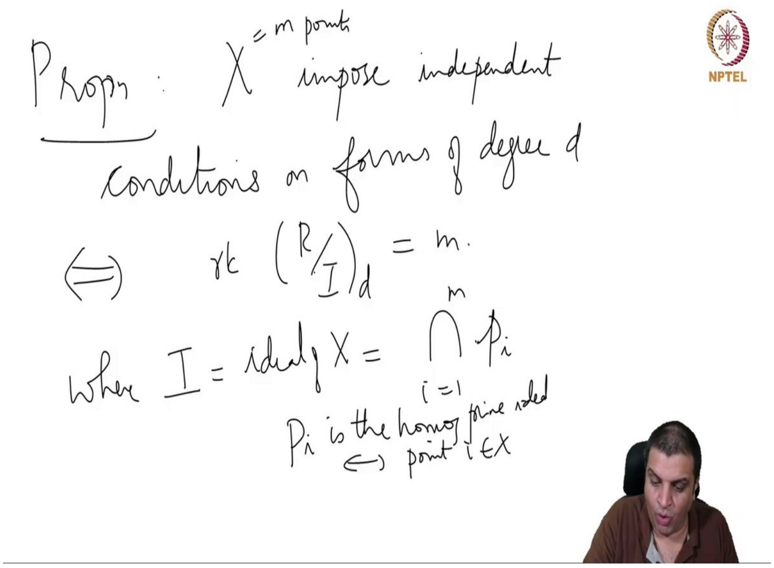

So, that we will look proposition, m points; so X which is m points impose independent conditions on forms of degree d if and only $rk\left(\frac{R}{I}\right)_d = m$, where $I = . \bigcap_{i=1}^{m} p_i$ is the ideal of X; in other words it is the intersection of the prime ideals corresponding to I goes from 1 through m, where P_i is the homogeneous prime ideal corresponding to point i in X. So, X contains m points; so there are m prime ideals intersect them. So, that is the ideal I and then rank of the Hilbert function is what we want here.

So, that is because, the question is; what sort of f will belong to I? That is what impose conditions, independent conditions mean. So, it is the checking of this Hilbert function that is required that we are I mean; this problem now has been translated to a problem about Hilbert function.

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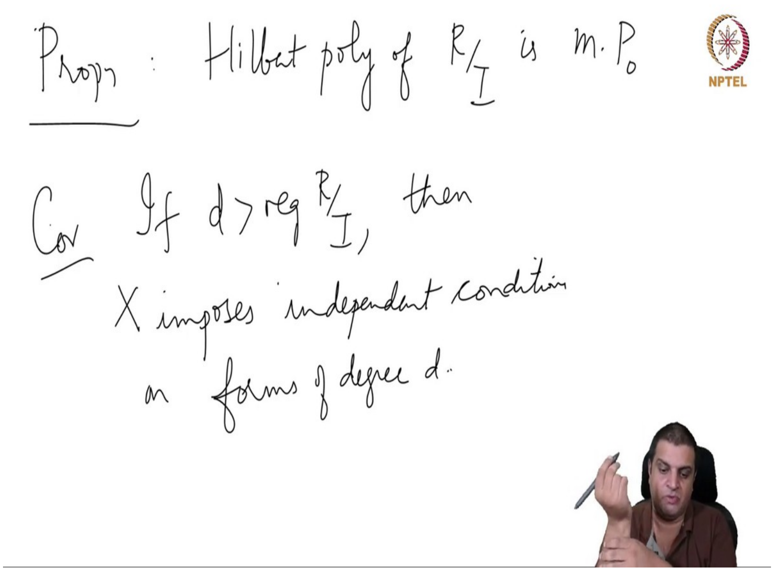

And just another observation, which we rehash in various things; Hilbert polynomial of $\frac{R}{I}$ is $m\,p_0$, where p_0 is the Hilbert polynomial of a single point that polynomial, the constant polynomial X choose 0. So, Hilbert polynomial is this.

So therefore, putting these two together from the previous lecture the conclusion is that corollary $d > reg\left(\frac{R}{I}\right)$; then X imposes independent condition on forms of degree d. in order to vanish, there it would impose conditions and the complement will also have the same.

This is an application to a problem which was a priori not stated in any of these thing cases; let us look at this the example. So, we computed these two ideals, the ideals of x; there are two there is an X_1 and an X_2, $X_1$is these three points and X_2 is the these three points. So in X_1 that is the ideal, in X_2 this is the ideal; so their generating such group different.

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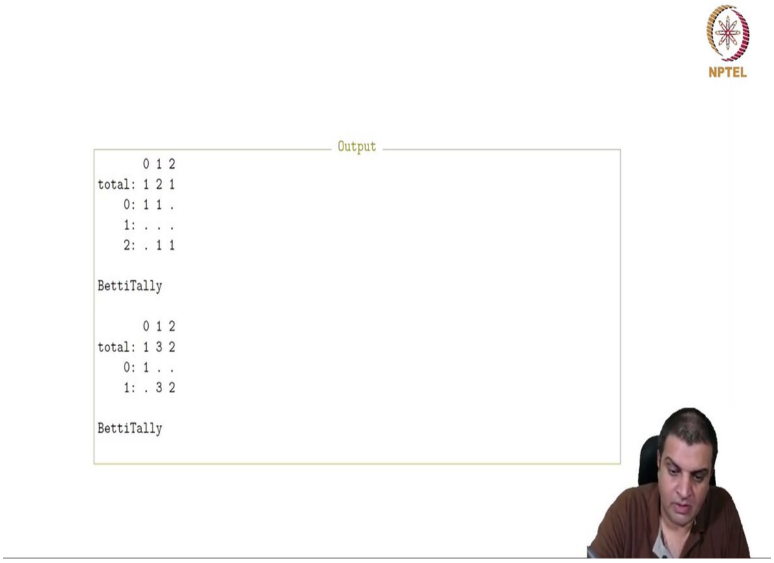

So, we ask for its resolution, this is for J_1; it has one; this is just the R. So, it has a linear form and a cubic and a second, I mean this is 1. In the other case, it has 3 quadratics and 1 linear syzygy among them.

So, the regularities are different; regularity of $\frac{R}{I}$, where I is the ideal of three collinear points is 2 where regularity of $\frac{R}{I}$, where I is the ideal of the three non collinear points is 1. So, the regularities are different.

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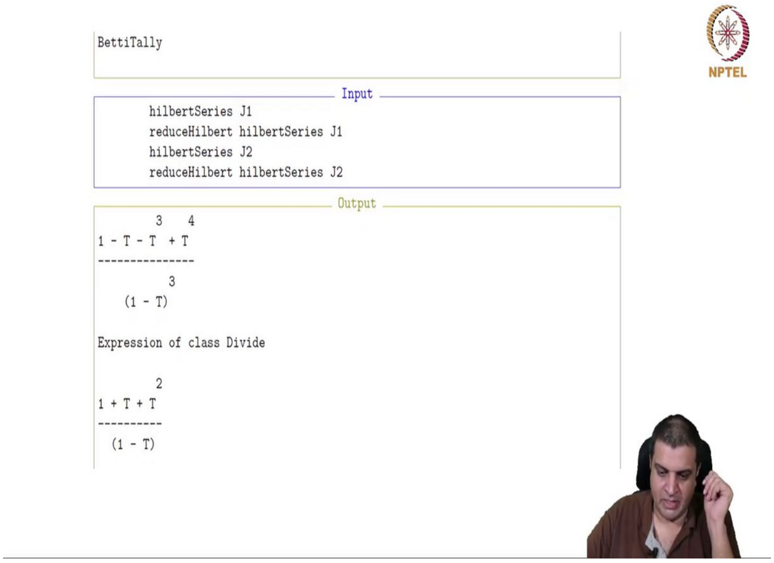

So from the proposition we; so we will come back to this in a minute.

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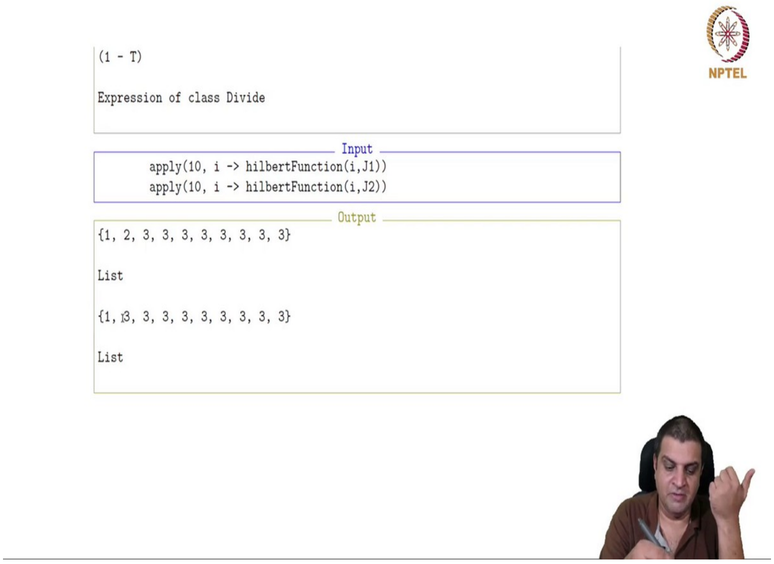

From the proposition, we know that the Hilbert function has to agree. So, the Hilbert polynomial is 3, the constant 3. So, Hilbert function has to become 3 after regularity. So, let us look at these two cases . So, this is degree 0. So, I am just applying for the integers i = 0 , …, 9 both included; Hilbert function of $\frac{R}{J_1}$ in degree i.

So if i is 0, it is just a field, both cases it is 1; if i is 1, then J_1 contains a linear form, so the quotient has only dimension 2. Here J_2 does not contain any linear form, so the quotient has dimension 3 and then for all higher things it is 3. So, in both cases we notice that.

So, if you notice that, even before we attain the regularity; the Hilbert function has attained the Hilbert polynomial. here this is degree 1; but the proposition we proved said if degree is greater than regularity, but regularity is also 1.

Similarly here, the regularity is 2; but already at degree 2 itself Hilbert function has agreed with Hilbert polynomial. So, this can happen, the proposition only said that if you go past the regularity, the function agrees with the polynomial. So, that we see from here.

So, why is it that, this has agreed a little earlier? In these two examples you can work it out and I will spell it out how to do it in an exercise. So, that estimate which said d greater than regularity is slightly loose in these examples and we will see why it is loose.

So this is the. So, this is the end of the course. So, I hope that you have learnt some aspects of computational communicative algebra and also be able to use Macaulay too. And hopefully you can use I mean; this will help you to use some other programs, which also have its own I mean like Singular and CoCoA, which also have and perhaps others also, which also have their own advantages and.

So, we as I mentioned at the beginning, most many of the theorems that we proved; we did not prove it in the most general form. Or for some theorems we chose, specific we purposely chose a roundabout route, so that we become familiar with computational, ideas behind computations Grobner bases or being able to use Macaulay to sort of explore.

So, thank you for listening to me in this course.