Computational Commutative Algebra
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Lecture – 06
Initial ideals

Welcome to the 6th lecture on Computational Commutative Algebra. So, in this lecture we will look at what is called initial terms and Initial ideals and after having put all these things together we will prove the version of Hilbert basis theorem that we are after.

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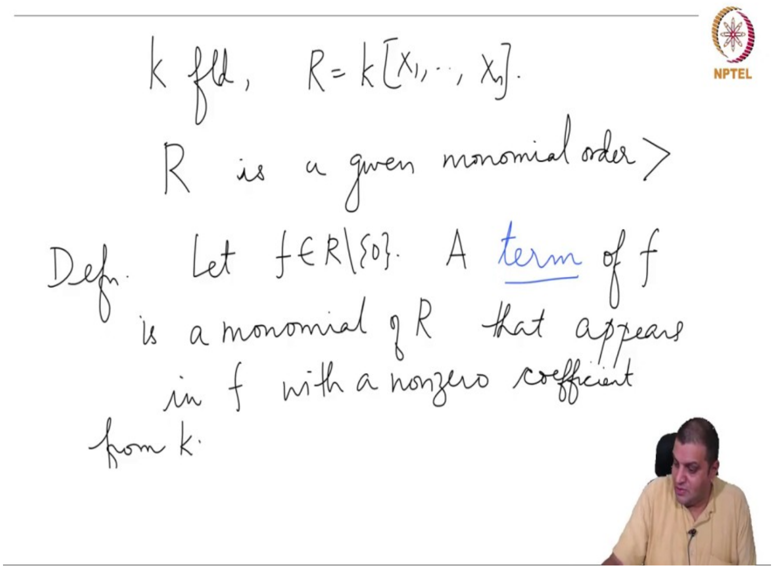

So, recall that we are working over a polynomial ring over a field with finitely many variables. And we now R is given a monomial order could be lex could be graded-lex or something completely different. So, it has a monomial order.

Now, definition let f be a nonzero polynomial. A term of f is a monomial of R that appears in f with a nonzero coefficient from k; what exactly we on emphasize with that.

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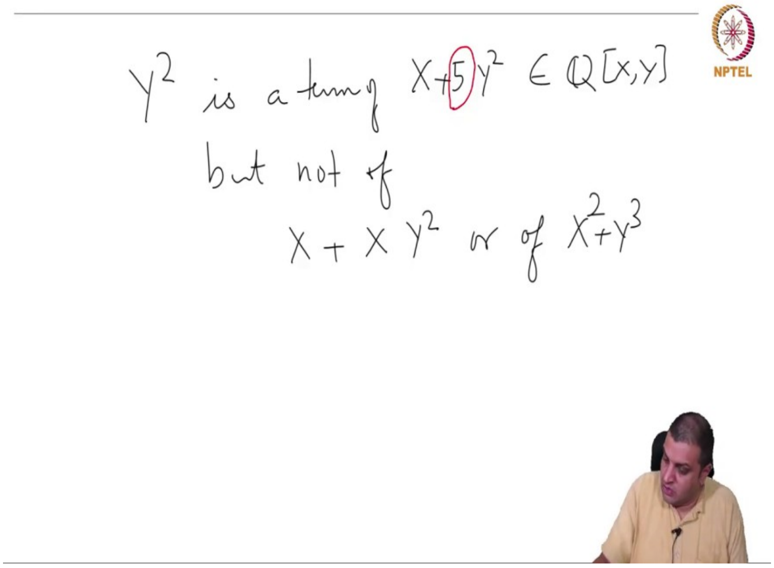

For example, Y^2 is a term of $X+5Y^2$; let us say this is a polynomial in $Q[X,Y]$. I mean if you in characteristic 5 this coefficient is 0 so one has to its just two ok; that is a nonzero coefficient in characteristics different from 5. So, here we chose a field of characteristic 0

So, this is a term anything, but not of $X+XY^2$ or of X^2+Y^3. So, what multiplies Y^2 should be an element of the field and not another variable or another polynomial or of X^2+Y^3. So, I hope you understand what is meant by a term. So, its a monomial that appears with a nonzero coefficient.

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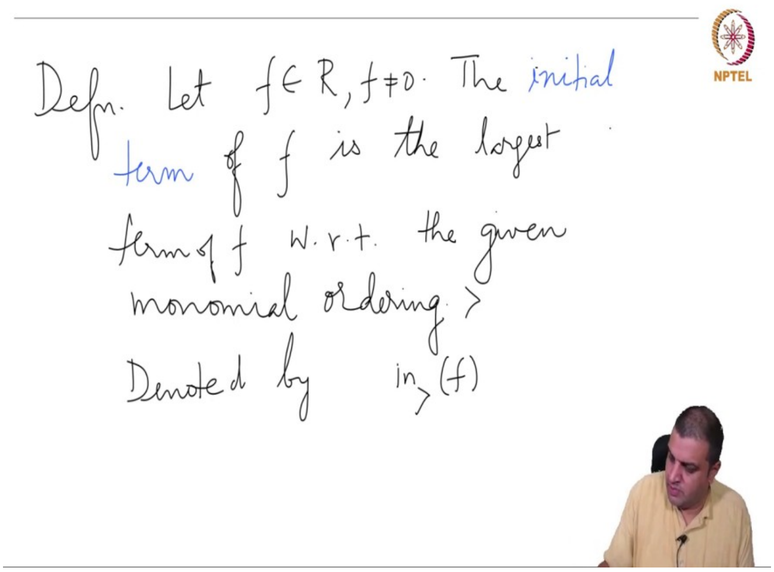

So, now we can define a initial term; let f be a nonzero polynomial. The initial term of f is the largest term of f with respect to the given monomial ordering. It is denoted by ¿f its also sometimes called. So, maybe just write here or may be ok, sometimes called leading term.

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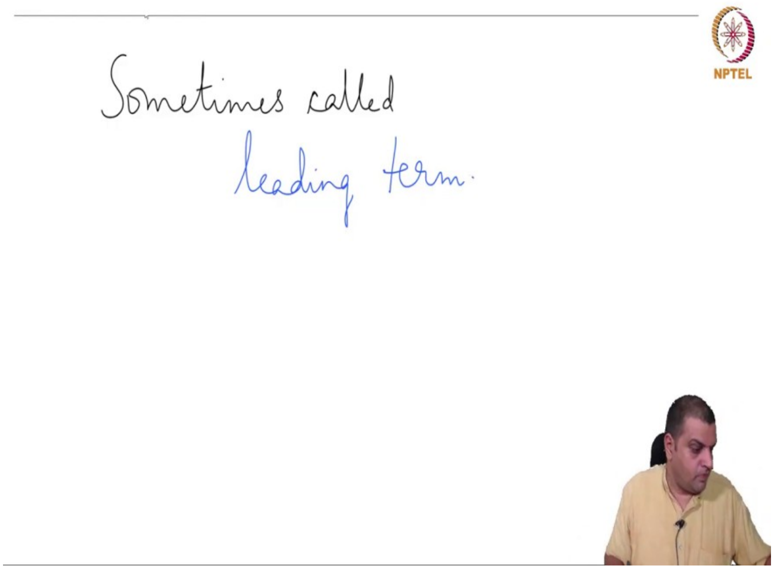

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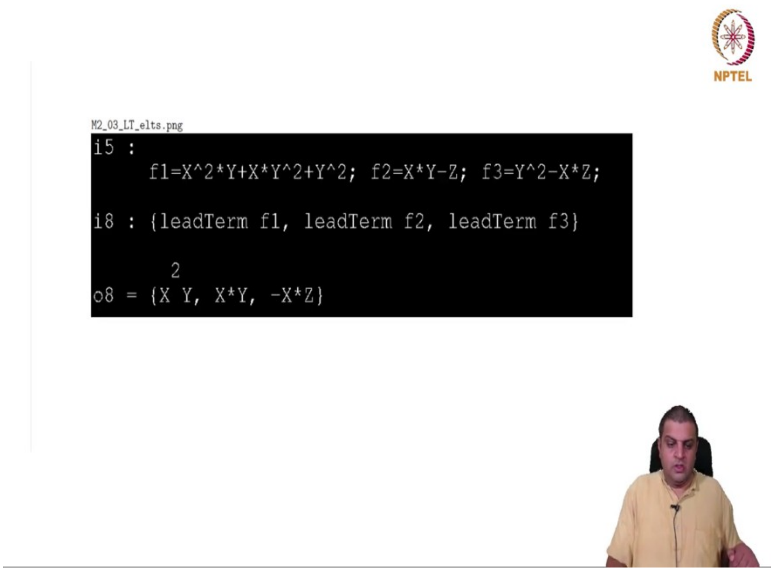

Now let us look at; so here are three polynomials this is again the same polynomial ring in three variables X,Y,Z and order is graded-lex. So, here f_1 there are three terms X^2Y, XY^2 and Y^2.

So, if you compare these three these two are degree three polynomials this is a quadratic polynomial. So the leading terms going to be one of these things. Here is X^2Y and this is XY^2 and in lex so there is the same degrees so we just use lex to compare we would just get X^2. Here in f_2 there is an XY which is degree 2 is a Z. So, this is the one that is in that is bigger in glex.

So, XY and here both of them have the same degree Y^2 and XZ. So, we just have to use lex between them. This one involves X this does not involve X. So, if you write out the lex comparison for these two one would see immediately that this is the leading term.

So, just so the Macaulay command for finding the leading term is lead term with a capital T. So, one thing to keep in mind is that when you ask Macaulay for a leading term it will also put the coefficient. Typically when we discuss the problem we will assume without loss of generality that the coefficients are one

So, one has to just keep that in mind when that sometimes Macaulay is I mean what is convenient for us to discuss is probably not is what is convenient to be programmed. So, that one has to worry about. So, this is the this is an example.

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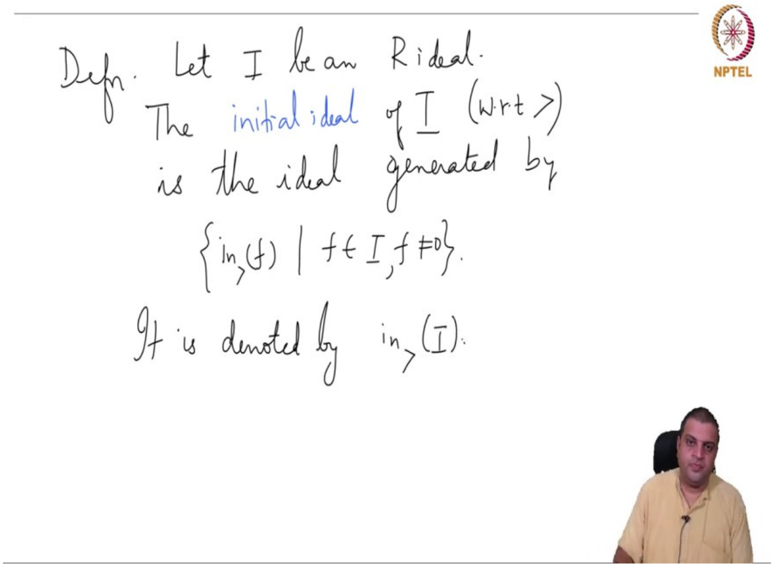

So, now let us extend this notion to ideals. Definition; let I be an R ideal, the initial ideal of I again all of this is with respect to the given order monomial order with respect to is the ideal generated by the initial terms of the nonzero polynomials in f in I

So, this set itself is just a set of monomials, its not an ideal. So, we look at the ideal in R generated by this set and it is denoted by ¿¿ I and a few remarks

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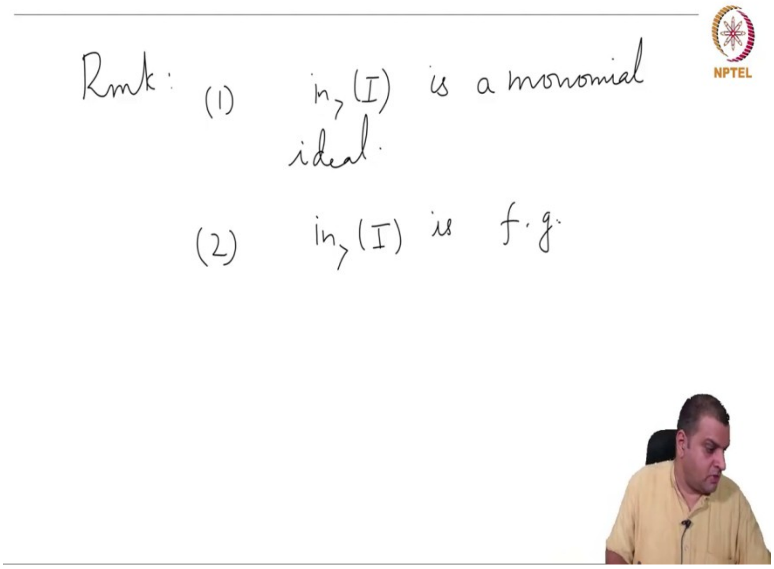

One, initial ideal of an ideal is a monomial ideal because it is generated by the initial terms. And two, because it is a monomial ideal it is also finitely generated this is what we proved in the previous lecture. So, this is a financially generated ideal. And now let us look at let us compute this in Macaulay and we will see some small surprise.

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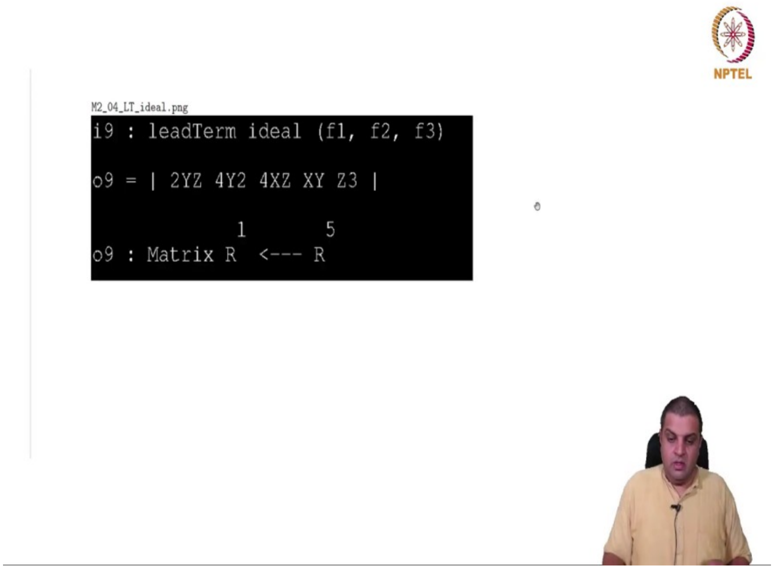

So, here we ask so the command for the computing initial ideal is also lead term. So, if you give lead term it will check its argument and decide what is to be computed. So, we ask lead term of ideal generated by f_1, f_2 and f_3 recall f_1, f_2 and f_3 are these and the lead terms are X^2Y, XY, XZ.

So, if you look at the ideal generated by them and ask for its initial ideal; then suddenly we see that there are more than three things here because there was X^2Y, XY, XZ, but now we see there is an XY but there is no X^2Y and then there is an XZ, but then there are three other things which are new which we did not know.

So, let us try to understand what happened. There are two issues to be understood one; X^2Y is not in this list and then in addition there are three other things. Let us understand these things one at a time X^2Y is not there because XY is there and X^2Y is a multiple of XY. So, any ideal which requires X^2Y is already taken care of by at this XY. So, that explains why X^2Y is not there in this list.

So, now let us try to understand some of these thing I mean we will not explain all three of them, but its a same reasoning. So, let us try to do this computation.

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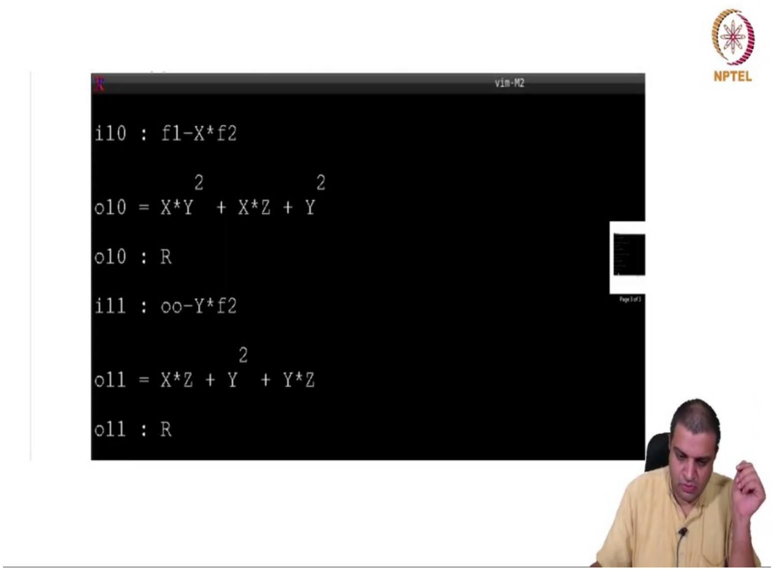

So, if we took $f_1 - Xf_2$. So, this is because how did we make this decision the leading term of f_1 was X^2Y and leading term of f_2 was XY; so I multiply by X. So, that the leading and then take the difference; so that the leading terms cancel each other that is what we did So, we get some polynomial whose leading term is XY^2 which is divisible by the leading term of f_2 and if you so which you need to multiply by Y.

So, we take this; so the word "oo" refers to just the previous output $oo-Yf_2$ will now cancel this term also and we have some we get some new expression.

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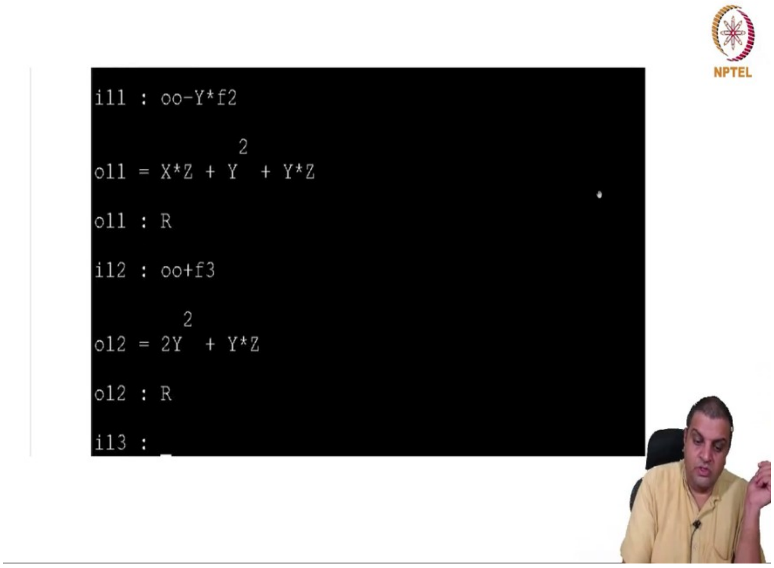

Now this one leading term is XZ and we know that the in the last polynomial f_3 there was a $-XZ$; so we add these two together and we get $2Y^2+YZ$. Now the leading term is Y^2, but you know because Macaulay2 has to keep track of you know when you write programs we cannot keep lose these we cannot lose these coefficients.

So, here in the algorithm that Macaulay was running it would give $4Y^2$, but the point is the same there is a polynomial in this ideal whose leading term is Y^2 which is not visible by just looking at the generators itself. So, that is how this term $4Y^2$ came and similarly one can work more and see where the XZ, Z^3 came in and the YZ came in.

So, this is the this is a what is meant by initial ideal and what issues one has to worry about computing going about computing them . So, with this we have now learned a way to convert an arbitrary ideal to an initial ideal to a monomial ideal and now we want to use this idea to prove Hilbert basis theorem. So, we call Hilbert basis theorem.

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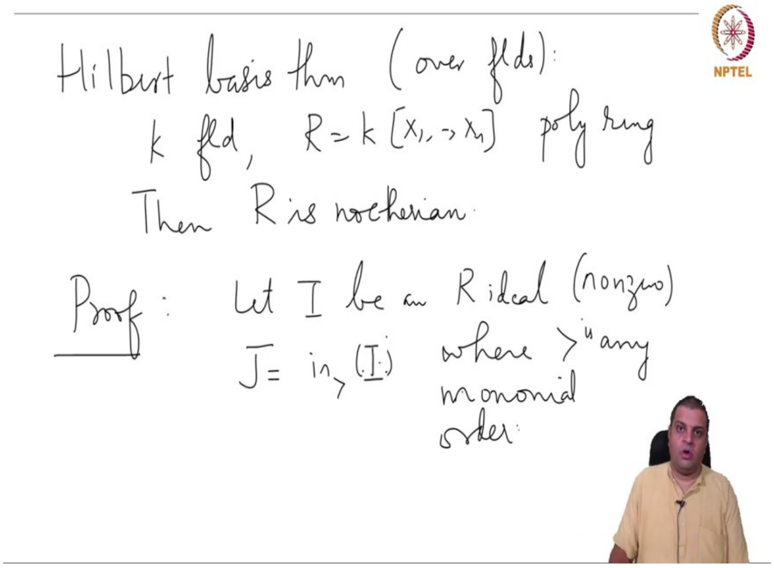

So, for fields the point that we want to prove is k field R is a polynomial ring in finitely many variables so that is then R is Noetherian. This is what we will now prove.

So, let I be an ideal we may as per you assume that I is a nonzero ideal because if it is 0 its always finitely generated. And we define J to be the initial ideal of I, ¿¿ I mean where this is any monomial order it does not matter for this proof what that is.

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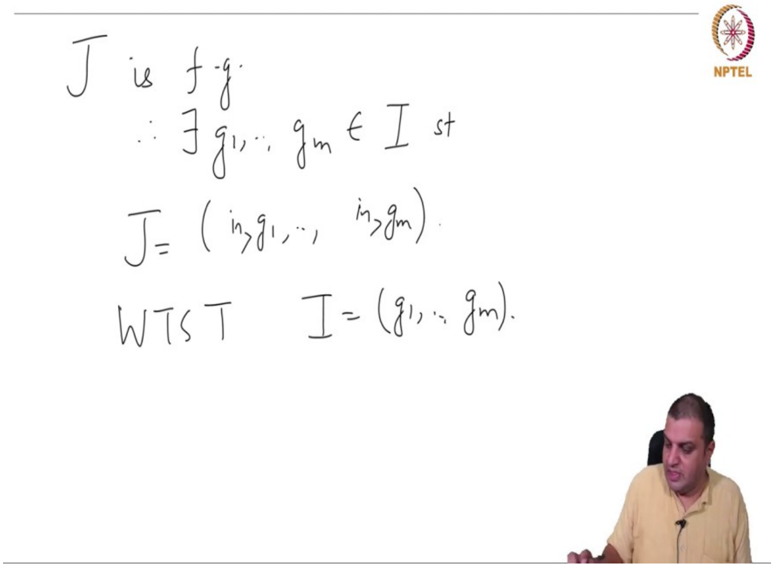

So, J is finitely generated that was we know that all we have seen that all monomial ideals are finitely generated as a consequence of Dicksonon's lemma.

So, J is generated by initial terms. So, therefore, there exist $g_1,\ldots,g_m$ inside I such that J is generated by the initial term of g_1 up to initial term of g_m. For a generating set for J is the initial terms and then there is a by Dickson's lemma there is a finite set and so they are initial terms of some elements from I, call those elements of I as $g_1,\ldots,g_m$.

And so we are claimed; so we wanted to show that J is generated by the initial terms of these then I is generated by those elements themselves. This need not be a minimal generating set which you will see some examples in the exercises, but it is generated by the same set whose initial terms generate the initial ideal.

So, we will prove it some more informally the reason is I would like you to understand what is going on in the argument rather than writing it out formally in some setup.

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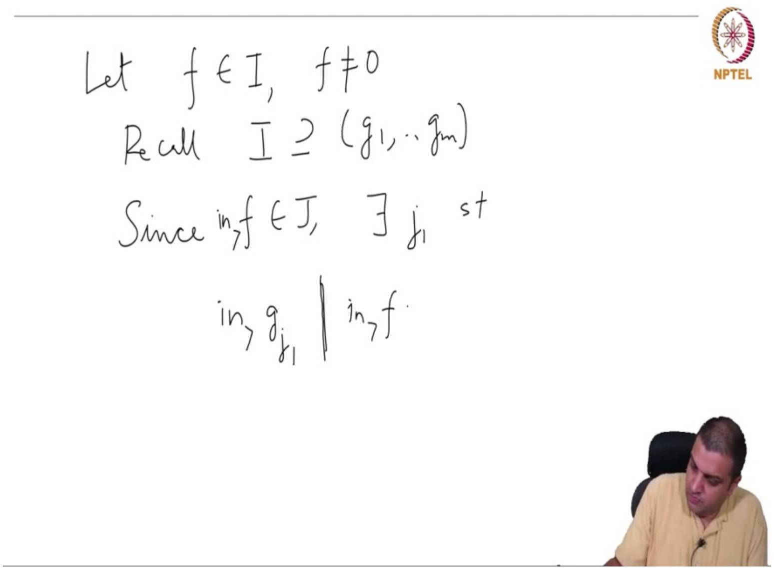

So, let $f \in I$ is nonzero. If it is zero then it is already in the ideal generated by the g's. So, we call that I contains the ideal generated by the g's. What we want to show is the other containment if f is 0 then its already here. So, we take a nonzero element we take an element here that is nonzero then prove that it is there.

Since f the initial term of f is inside J there exist a j_1 such that the initial term of g_{j_1} divides the initial term of f. So, this is the property about monomial ideals which is that if you have a monomial ideal and a monomial then a monomial of R is inside the ideal if and only if its divisible by one of the generators.

So, again you can do this as an exercise I will write this up and so one uses this. So, this is divisible by that.

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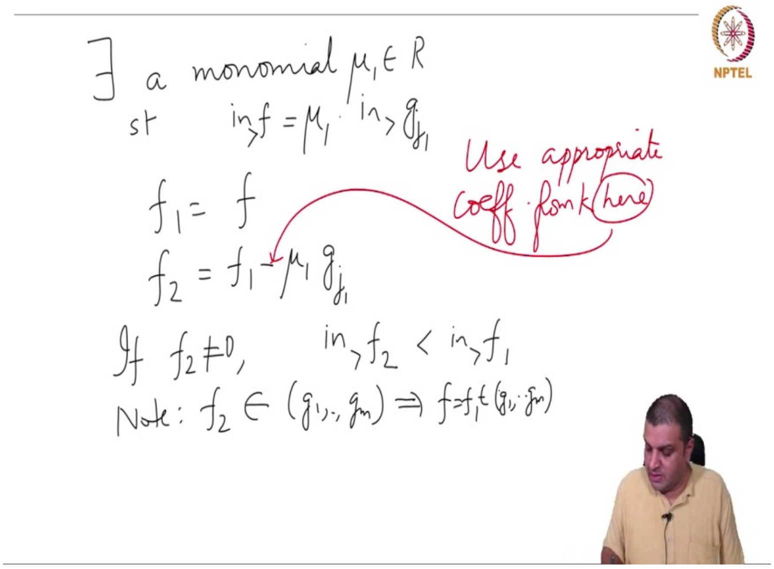

So, what does that mean? There exists a monomial $\mu_1 \in R$ such that the initial term of f is $¿¿ f = \mu_1 ¿¿ g_{j_1}$. So, the quotient I mean when you divide you get another monomial.

So, now we are going to proceed by induction let us define f_1 to be f itself. So, we are going to construct a sequence of polynomials. We define $f_2 = f_1 - \mu_1 ¿¿ g_{j_1}$. So, let me just restate what we did; we found j_1 such that initial term of g_{j_1}divides initial term of f_1 and μ_1 is the quotient that you have to multiply to get them equality. So, we just subtracted that from f_1. What is the property of this?

If f_2 is not zero then we can talk about the initial term of f_2 this has to be less than the initial term of f_1 because the initial term of I mean this was multiplied precisely to cancel the initial term here. So, let me we vague here need to take care of coefficients. So, when we say initial term we are saying we are taking it with the coefficient 1, but that may not work all the time. So, maybe I should just put this here; use appropriate coefficient from from k here.

It may not be just subtract it may not be minus 1, but if you appropriately multiply by a coefficient you can get this result you can cancel the initial term that is what one has to give so we get this. So, if f_2 is inside the ideal generated by the g's will now imply that f which is f_1 is also in the ideal generated by the g's.

So, its enough to prove that f_2 is inside the ideal generated by the g's. So, if f otherwise just continue. So, repeat the same procedure to f_2.

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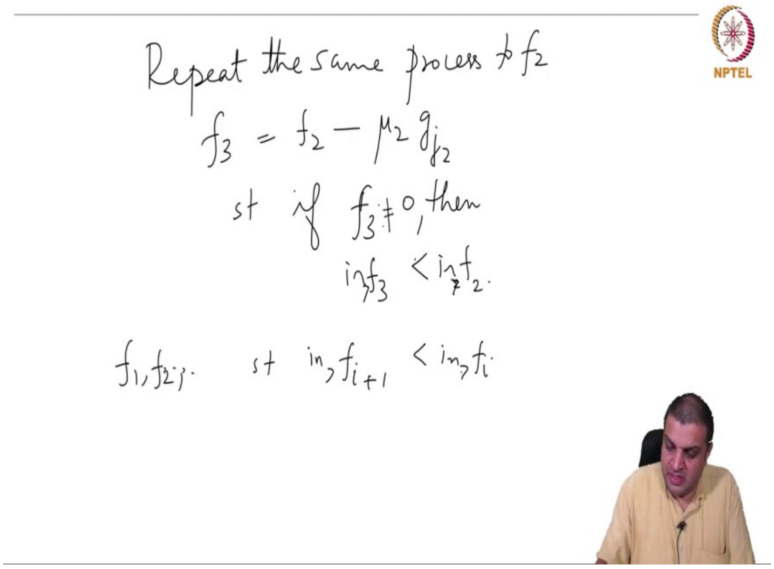

We would get an f_3 which is some $f_2 - \mu_2$¿¿g_{j_2} this values do not really matter or that matter is such that. So, again one has to find appropriate coefficient here not just minus 1 the same discussion that we have to if f_3 is non zero then in f_3 is this is initial with respect to there is only one ordered question is strictly less than in f_2.

So, this way we can get f_1 like this such that; So, assuming that this continues forever we would get f_{i+1} such that ¿¿$f_{i+1} <$¿¿f_i. Remember this question about initial ideal comes in only if the new polynomial that we just constructed is nonzero.

So, remember this is a descending sequence of monomials, ¿¿f_1 is the largest then is ¿¿f_2 then is ¿¿f_3 and so on. So, this construction can stop as soon as we hit an f_i which is inside g's.

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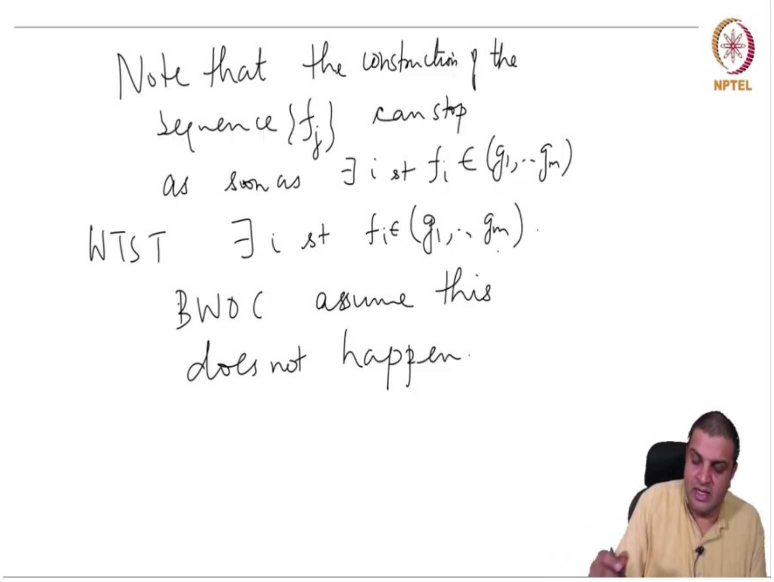

So, note that this the construction of the sequence f_j can stop as soon as there exists some i such that f_i is inside the ideal generated by the g's, because then as we explain how to go from f_2 to f_1. We can go backwards and prove that f is inside I the f is in generated by these things. So, let us continue like this.

So, we want to show that there exists some i such that f_i if you keep constructing like this we would end up with an i such that this is inside ideal generated by g_1 through g_m. So, this is what we want to prove. So, if this does not happen then f_i is nonzero. So, by way of contradiction assume this does not happen.

So, this is the assumption that; each time one constructs this thing one gets a polynomial which is not inside ideal generated by the g's. Let us understand that.

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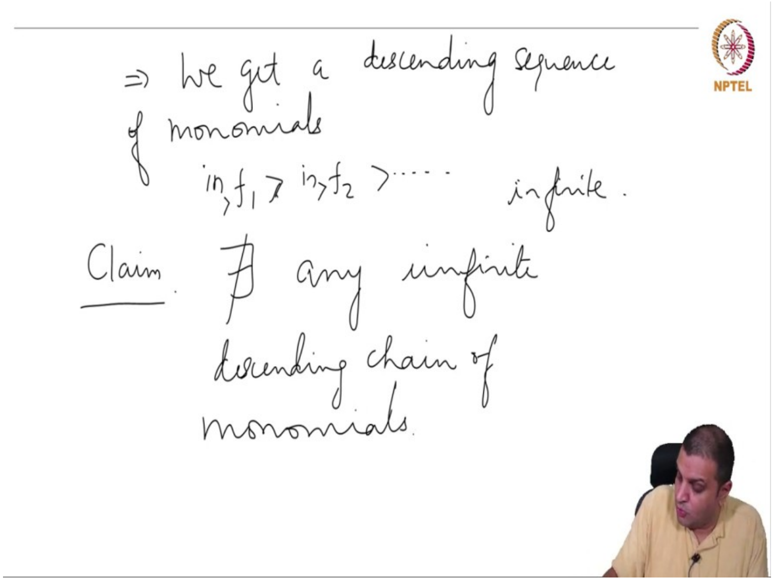

So, then first of all the f_i's have to be nonzero because any zero polynomial is already in the ideal generated by the g's. So, f_i is a non zero and we get a descending sequence of monomials with respect to the given order like this; descending in the sense that with respect to the given monomial order this is strictly descending.

So, then what we want to prove is the following. So, claim there does not exist any infinite descending change of monomials. Again descending change means with respect to the given monomial order

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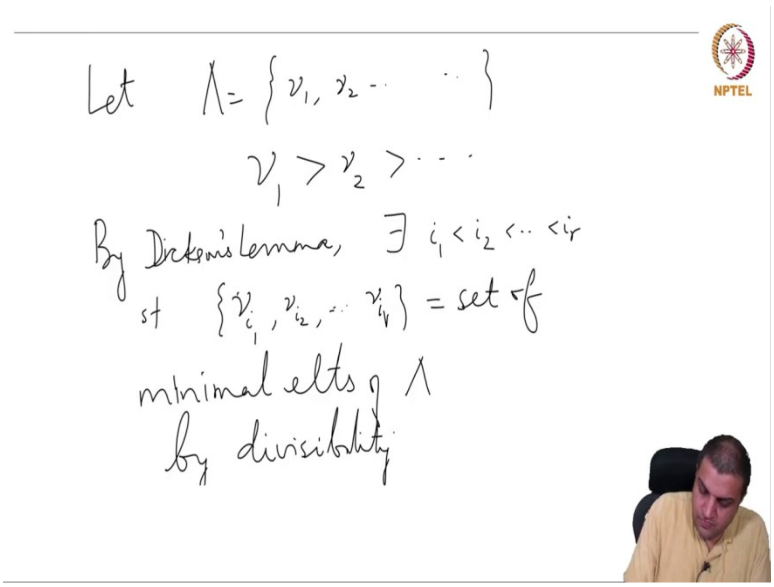

So, how do we prove this? So, let Λ be some monomials $\Lambda=\{v_1, v_2, \ldots\}$ such that $v_1 > v_2 > \ldots$.

Now, by Dickson's lemma this has the finite set of minimal elements. So, let us call them i_1 through i_r, that is there exists $i_1 < i_2 < \ldots < i_r$ such that $\{v_{i_1}, v_{i_2}, \ldots, v_{i_r}\}$ is the set of minimal elements of Λ by divisibility.

So, in this sentence we are using order and divisibility simultaneously minimal here means with by divisibility. So, what does that mean?

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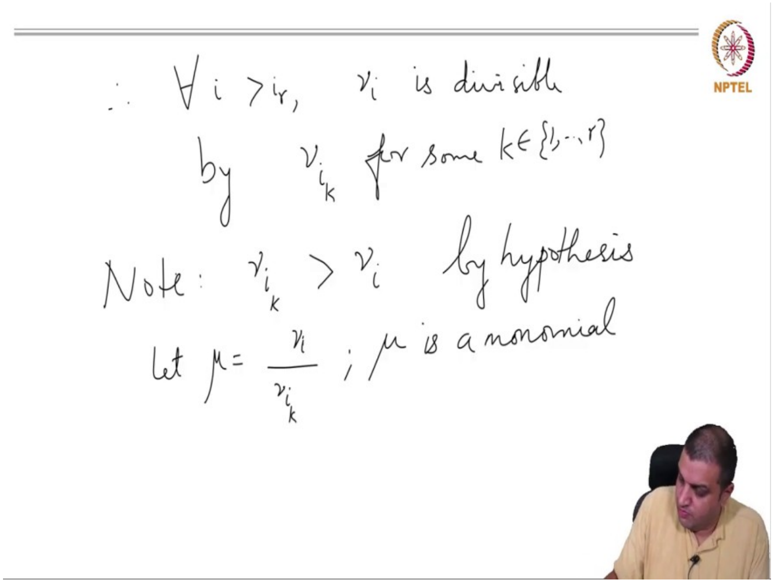

Therefore for all $i > i_r$, v_i is divisible by v_{i_k} for some $k \in \{1, 2, \ldots, r\}$. That is $v_{i_1}, v_{i_2}, \ldots, v_{i_r}$ one of these things is going to divide that.

Now, this is a monomial that divides that. So, note that $v_{i_k} < v_i$, this is by hypothesis and this divides that. So, let $\mu = \frac{v_i}{v_{i_k}}$. So, this is a monomial. So, this is with respect to the given order and μ is a monomial.

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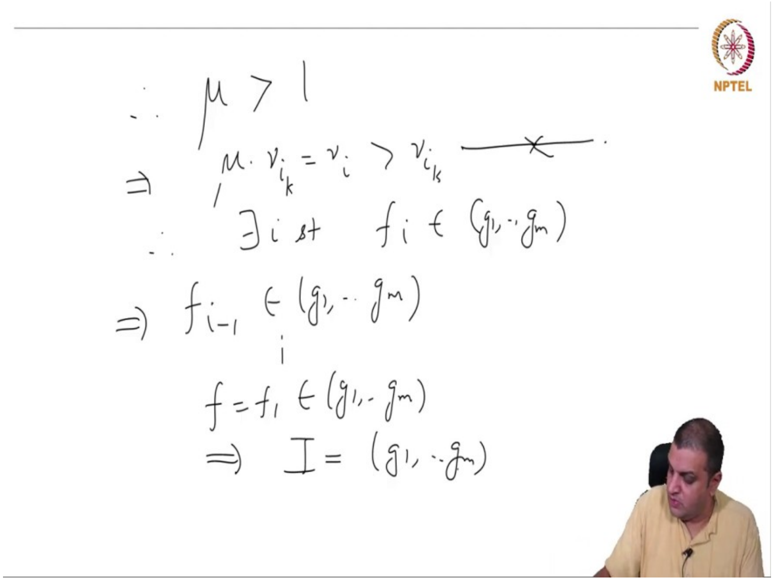

Therefore μ is bigger than 1 this is the definition of a monomial order. Now, let us look at put these two together μ is bigger than 1. Now if you multiply both sides by v_{i_k}, then $\mu . v_{i_k} = v_i$ which going to be bigger than v_{i_k}.

But hypothesis was the opposite that v_{i_k} is bigger than v_i, this is a contradiction. And what where did we get this contradiction? We got this contradiction by assuming for every i, f_i is not inside g's; so this was the assumption.

This assumption and when we proceeded with this assumption we got an infinite descending chain of monomials and then we put that is not possible. So, the contradiction is the assumption. Therefore, there exist i such that f_i is inside g_1 through g_m. And then we saw that this implies that f_{i-1} is inside g_1 through g_m and proceeding like this f which is f_1 is inside g_1 through g_m and this implies that I is the ideal generated by g's.

So, this is the proof of Hilbert basis theorem that we do using this idea of monomial ideal, monomial orders, initial ideal etc. So, this is the end of this lecture.

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