Computational Commutative Algebra
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Lecture – 59
Castelnuovo Mumford regularity - Part 1

Welcome, this is lecture 59; so in this lecture and in the next we will discuss Castelnuovo Mumford regularity which is another invariant about; numerical invariant of a module that we see from free resolution. There are other ways of seeing it; the original definitions were different this is a what we are using is an equivalent definition.

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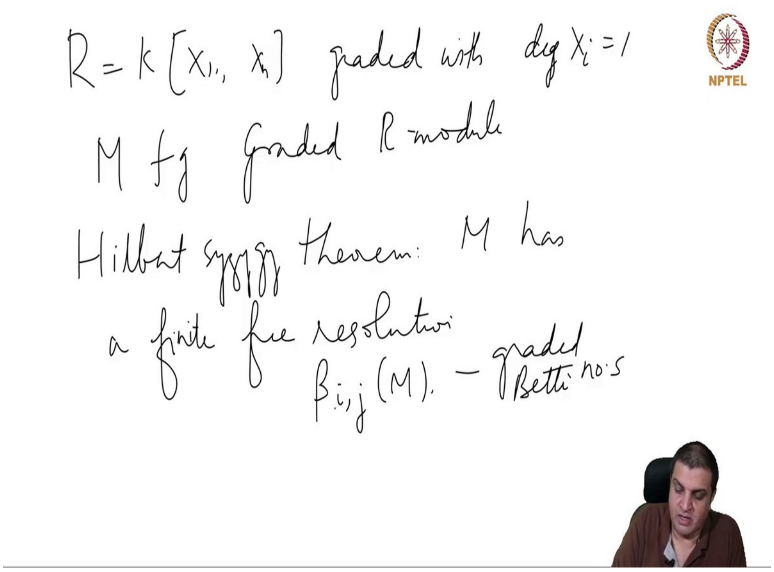

So, we will again treat work with polynomial ring $R=k[X_1,\ldots,X_n]$; graded with $deg(X_i)=1$ and M is a finitely generated graded R module. Then by Hilbert syzygy theorem ok; it is projective dimension is finite M has a finite free resolution.

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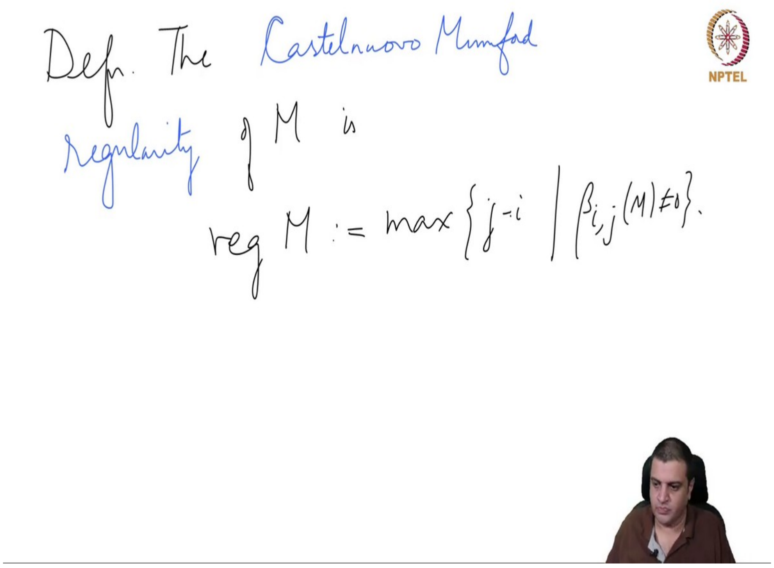

And we also know what is called graded Betti numbers. so these are the graded Betti numbers. The number of copies of R_j that appears in position i of the free resolution; the Castelnuovo Mumford regularity of M is; so which we will denote by $reg(M) := max\{j-i : \beta_{ij}(M) \neq 0\}$.

And because it has a finite free resolution, eventually for i > n, for every j, this is 0 and in any case for a given i; there is only finitely many non zero such; there are only finitely many j where this is non zero. So, this maximum is indeed a maximum.

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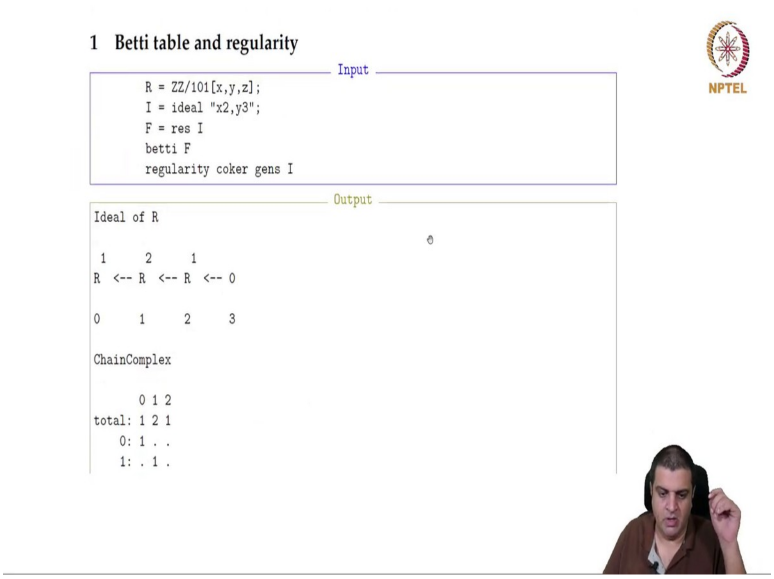

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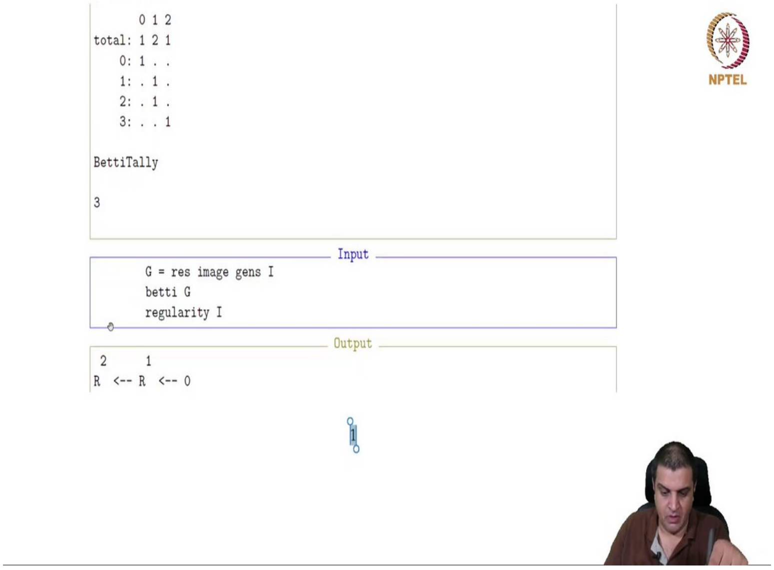

So, we can look at a Macaulay example. So, we construct we take a polynomial ring $k[x, y, z]$; then we ask for the ideal $I = (x^{2,} y^3)$; and then here is we ask for the resolution of I, then we ask for what is called the Betti table of I . So resolution of I is; this is a if you notice this is a one can show this is a regular sequence ; x^2 is a non zero divisor and in $\frac{R}{(x^2)}$, y^3 will be a non zero divisor.

So, this is a kausal complex; so what information do we need; do we see from here? So, the one here just means that it is a quotient of a rank one free module. So, when we ask for a resolution it resolves $\frac{R}{I}$ not I ok. Now, here this 1 here; this 1 here means that in one of the free resolution in degree 2; there is a basis element and here it says degree 3, there is a basis element.

So, that just corresponds to the x^2 and y^3 and then it says it is a relation in degree. So, their relation is in degree 5 which is y^3 times the basis element here minus x cube; x square times the basis element here. So, that relation will be in degree 5 and this is this is what it says and if you notice this; this is already taken the j - i difference .

Remember, in position column i row index j, it is already $\beta_{i,i+j}$. So, in other words regularity is just the index of this bottom row. So, one can look at the betti table of a module, then immediately tell what the regularity is; regularity here is 3, that is what we were asking here.

So, the difference is; so, here when we ask for regularity of I see; I and $\frac{R}{I}$ are different modules. For resolution, if you ask for I; it actually computes for $\frac{R}{I}$, but we asked for regularity of I, it computes the regularity of I, not the regularity of $\frac{R}{I}$ and we will see that they are little different ok. So, we have to actually ask for regularity of the co kernel of gens of I which is the same thing as saying $\frac{R}{I}$. So, and it is 3 which we see from this row this.

Now, suppose we want the regularity of the module which is actually I. So, we ask for a resolution of the image of the generators of I, if you ask for a resolution of I; it will just; it will just take the cokernel $\frac{R}{I}$, now we actually have to ask for that map.

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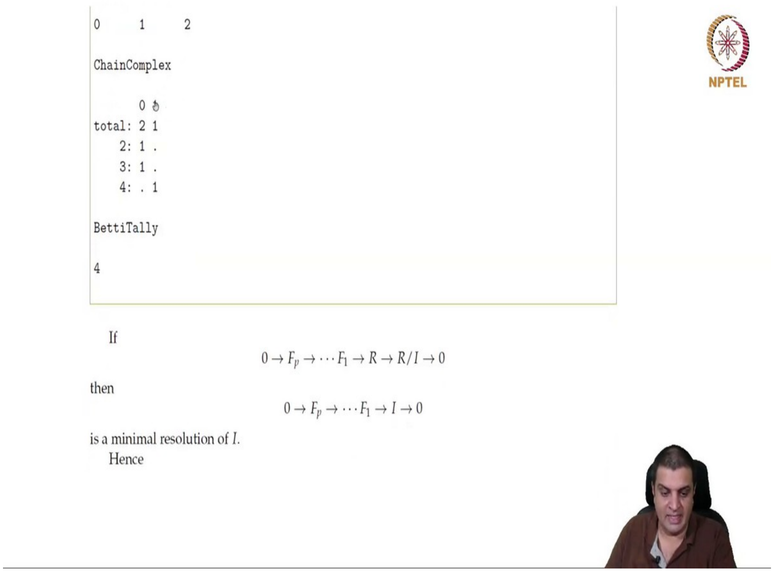

So, then we just get a map from $R^1 \to R^2$ and this is the Betti table. So, the difference between this Betti table and the other Betti table is; here there is an 1 here corresponding to R. There is one more difference; the row indexes have changed because the row index 1 here, in this column is degree 2, but in order to get that in degree in the row index will be 2. So, this one's regularity will be 4 .

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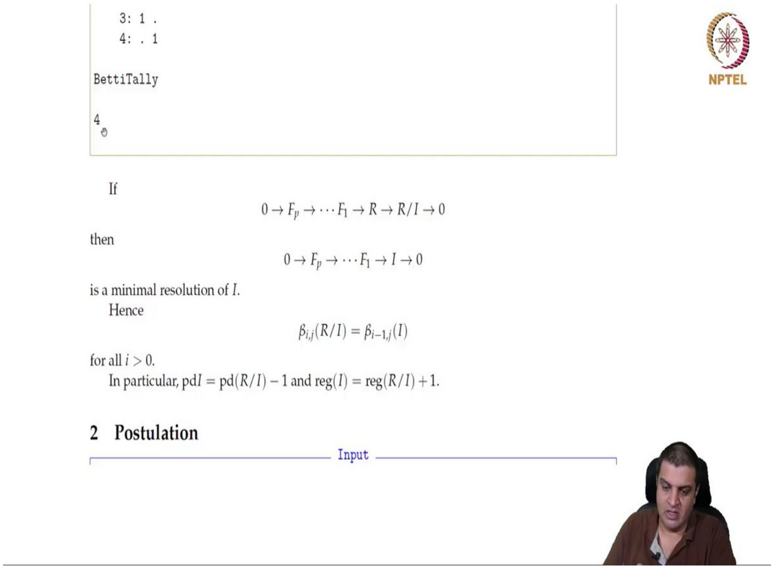

Regularity of the ideal is 4, regularity of the quotient is 3 . I mean it is indeed true; let us let us why. So, here suppose we have a free resolution of $\frac{R}{I}$; then just chop off this part, we get a free resolution of I. And conversely if you have minimal resolution of I, just put an R in here and then we get a free resolution of $\frac{R}{I}$.

So, if you look at the graded Betti numbers; $\beta_{i,j}\left(\frac{R}{I}\right)=\beta_{i-1,j}(I)$. So, hence the $pd(I)=pd\left(\frac{R}{I}\right)$ because the resolutions gone shorter. However, the regularity would be one more that is because you know we are subtracting j - i here; the same quantity here would be subtracted as j -(i- 1).

So, it would the regularity would be one more and that is all that we have observed here. So, we have to keep this in mind that unlike resolution; if you give resolution of an ideal, it computes the resolution of the quotient but you ask the regularity of the ideal it actually computes the regularity of the ideal.

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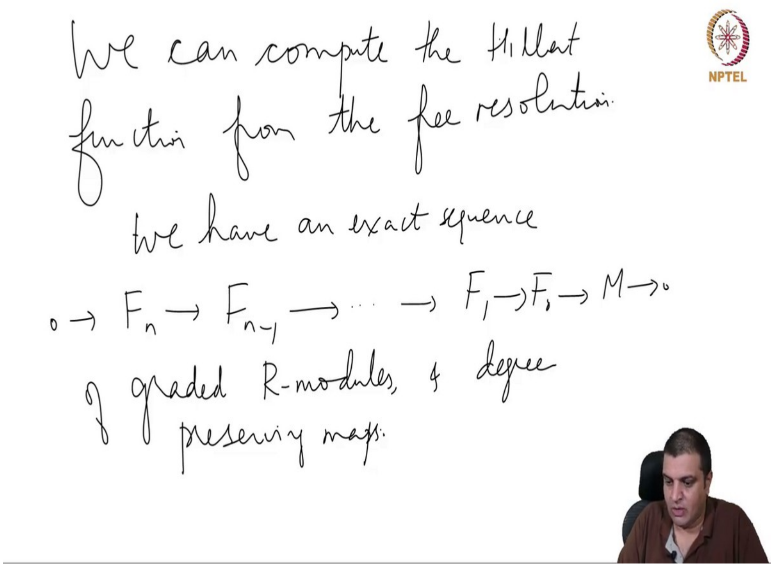

We can compute the Hilbert function from the free resolution. In fact, we can determine the Hilbert polynomial; Hilbert see Hilbert function is and then; so let us this is not very difficult to see. So, what is that we have? We have an exact; so we have an exact sequence,. I am not saying that these are non zero free modules, but we just know that it does not go beyond F_n; so we will just call it, we just stop there ok.

So, this could be 0; it does not matter; $0 \to F_n \to F_{n-1} \to ... \to F_1 \to F_0 \to M \to 0$. So, we have a free resolution; then we put this argument with this module with this with. So, exact sequence of graded R-modules and degree preserving maps. So, this is what we have from a free resolution .

So, we are not going to use explicitly now that these are; I mean we do not need to use it this is of minimal free resolution, but just to simplify the calculation, get a nicer expression or at least get some workable expression; we will assume that its minimal free resolution.

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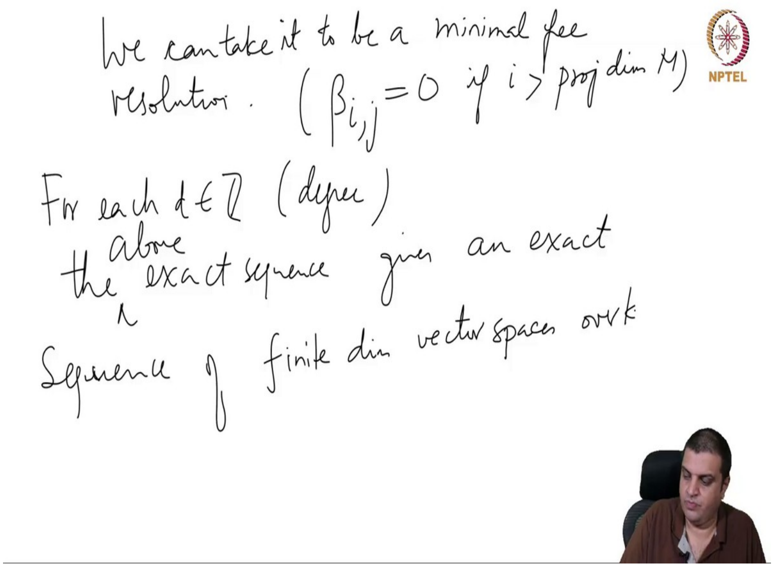

So, we can take this to a minimal free resolution . I mean they just set $\beta_{i,j}$; remember $\beta_{i,j}=0$if $i \geq pd(M)$. So, we can still treat it as. So then, from here it is because this is degree preserving maps; we can break this whole complex into its degree; so, in degree some d; in degree d, so the rank of this module. So, in each degree; this would be a complex, this would be an exact sequence of k-vector spaces; finite dimension k -ector spaces.

So, for each $d \in Z$ degree; the exact sequence the above exact sequence gives an exact sequence of finite dimensional vector spaces over k. So, then we can write down the rank of M in degree d.

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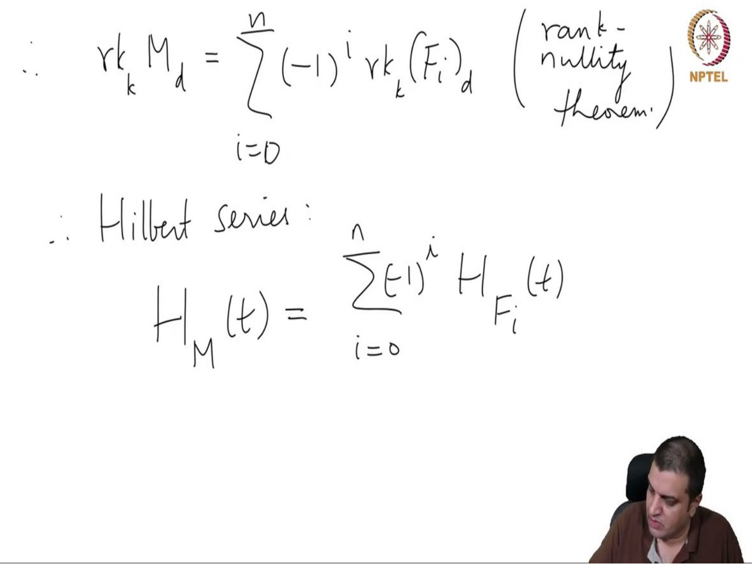

Therefore, $rk_k(M_d)=\sum_{i=1}^{n}(-1)^{-1} rk_i(F_i)_d$ is where this would be the alternating sum; . So, the ok; in other words the things at pose at 0, 1; 0, 2, 4 etcetera all get positive sign, other one is subtracted and this alternating sum will give the rank of M_d .

So, just write down an exact sequence of vector spaces and convince yourself that such an alternating sum this is or such things true. This is just I mean this comes down to rank nullity theorem; I mean it is just many applications over the rank nullity theorem. So, then; so we get this thing, so now we can just sum up to construct, multiply by some dummy variable t; powers of a dummy variable t and then sum it up to construct the Hilbert series.

So, what does that tell us? It gives us $H_M(t)=\sum_{i=1}^{n}(-1)^{-1} H_{F_i}(t)$

So, we are just multiplying this by t to the d and then summing up; we are changing some order of addition and sum, but this is we are not worried about some convergence.

So, we it is still, it is just a formal sum; . So, there is a the Hilbert series also has alternating sum property. So, we get like this, but this can be described. So, F_i has $\beta_{i,j}$ copies of R(-j).

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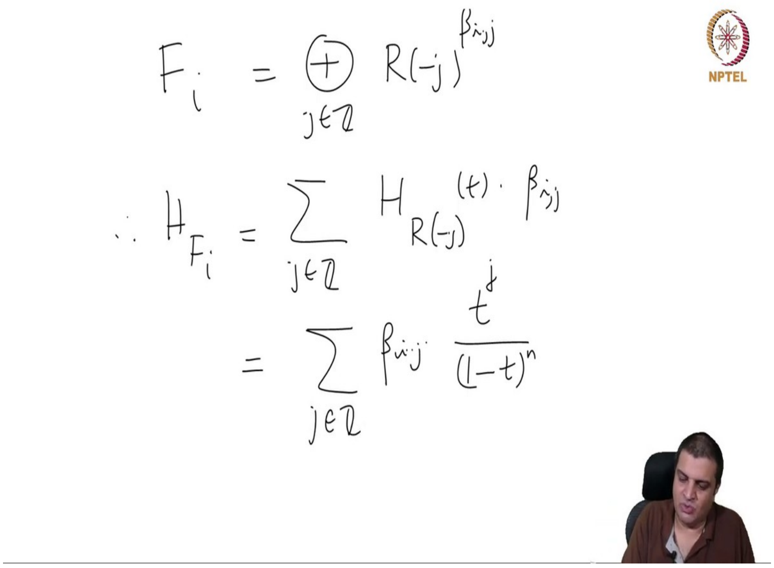

So, $F_i = . \oplus_{j \in Z} R(-j)^{\beta_{i,j}}$but only finitely many of these are non zero . ; so j in some finite set, but integers ; so this is what we have. Therefore, $H_M(t) = \sum_{i=1}^{n} (-1)^{-1} H_{F_i}(t) \beta_{i,j}$

; each copy will contribute one to this thing. But what is this number?

So, let me just rewrite it like this sum $H_{F_i} = \sum_{j \in Z}^{\square} \beta_{i,j} \frac{t^j}{(1-t)^n}$. . Remember, the polynomial ring in n variables, the Hilbert series looks like $\frac{1}{(1-t)^n}$. So, we write $j \in Z$, but only finitely many will show up.

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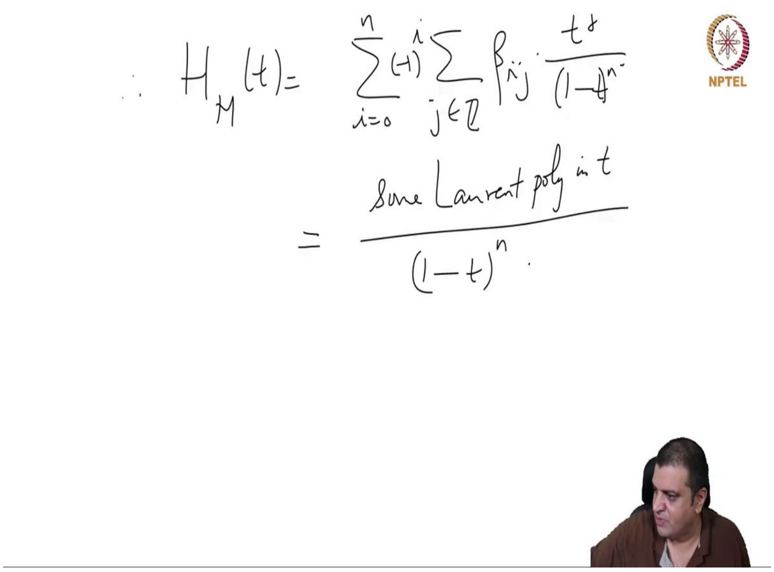

So, therefore, putting this together; we see that $H_M(t)=\sum_{i=1}^{n}(-1)^{-1}\sum_{j\in Z}\beta_{i,j}\frac{t^j}{(1-t)^n}$

. some Laurent series in t depend what j, if j is negative; it may not be a polynomial. So, this is going to be some Laurent polynomial in t divided by $(1-t)^n$; just put this in this common factor .

So, this is just a just a way to get, but you know it can be determined by the $\beta_{i,j}$ itself. So, in some sense the free resolution encodes more, it is a refinement of the Hilbert function or Hilbert series. Hilbert function Hilbert series encode the same information; it is just one is some written as a function, the other one is written as a generating function for that thing. So, this is this for this. So, where does regularity come into picture? So, it is not.

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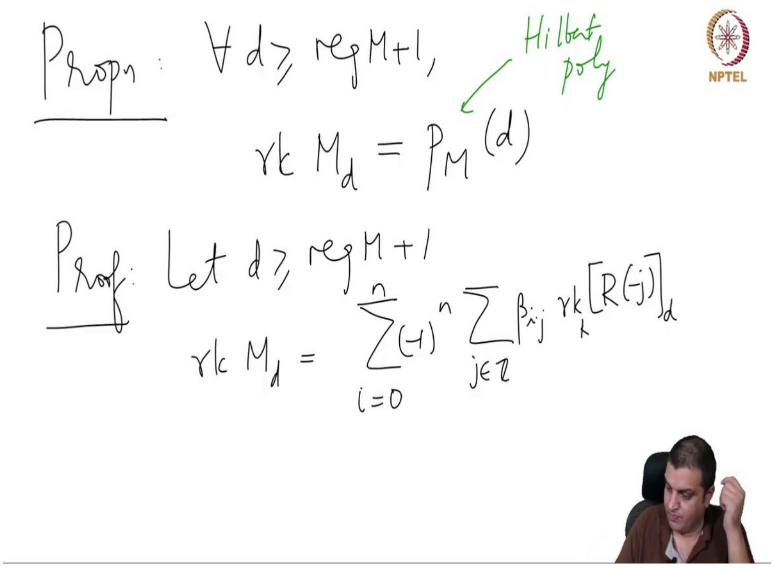

So, proposition for all $d \geq reg(M)+1$, the $rk(M_d)=P_M(d)$. Remember, this is the Hilbert polynomial; you do not need; I mean it is actually given by the polynomial. So, the difference between the Hilbert function, Hilbert polynomial will occur before it attains a regularity; after the regularity it is all before or yet, after attains regularity it will be given by the polynomial.

So, this is one place where. So, this is one place where if you knew regularity, then we know from there onwards or regularity plus one onwards what the Hilbert function is just by looking at the Hilbert polynomial. So, we will prove this. So, it goes back to looking at this expression.

So, let $d \geq reg(M)+1$; then $rk(M_d)=\sum_{i=1}^{n}(-1)^{-1}\sum_{j \in Z}\beta_{i,j} rk_k[R(-j)]_d$

this is; so, here we wrote the Hilbert series, but we can go back to this expression; this I mean we can go back to this expression and put this inside there to rewrite this.

So, they just $\beta_{i,j}$ times the rank of this in degree d that is all that we have written here. Now, what is this number? ; we need to determine that number.

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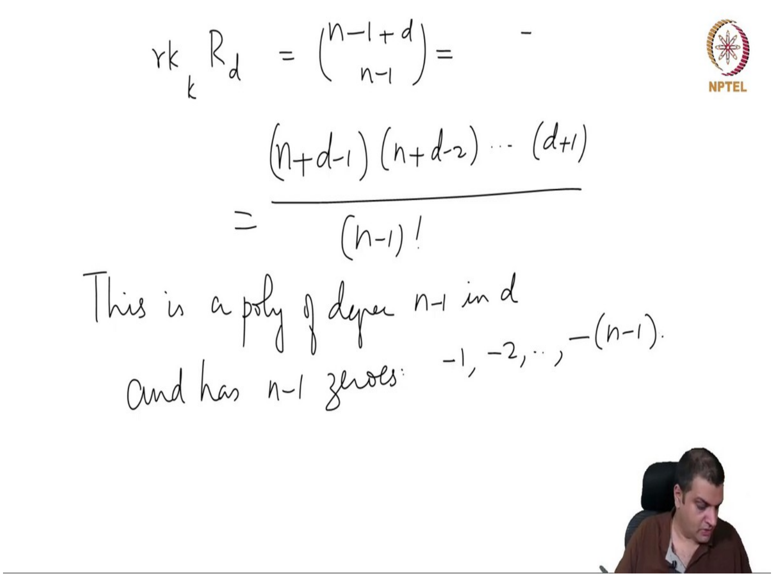

So, what is $rk(R_d)$? This is; so there are n variables. So, it is going to be $n-1+dchoosen-1$. So, this is one can prove by induction; so if you think about this function, so let us write this as a function in d. So, we are saying that this rank can be written as a polynomial in d. So, let us write this as a function in d; so it is going to be

$\frac{(n+d-1)(n+d-2)\ldots(d+1)}{(n-1)!}$. Remember, this is that number factorial divided by this factorial; times d factorial and we have just removed the d factorial from here. Now, if you think of it as a function in d. So, this is a polynomial in d; polynomial of degree n - 1 in d and has n - 1 zeroes; d = - 1, -2,…,-(n-1).

if you plug in any of those values of d, we would get 0. So, this is a polynomial which has these zeroes. So, now let us use this for our thing.

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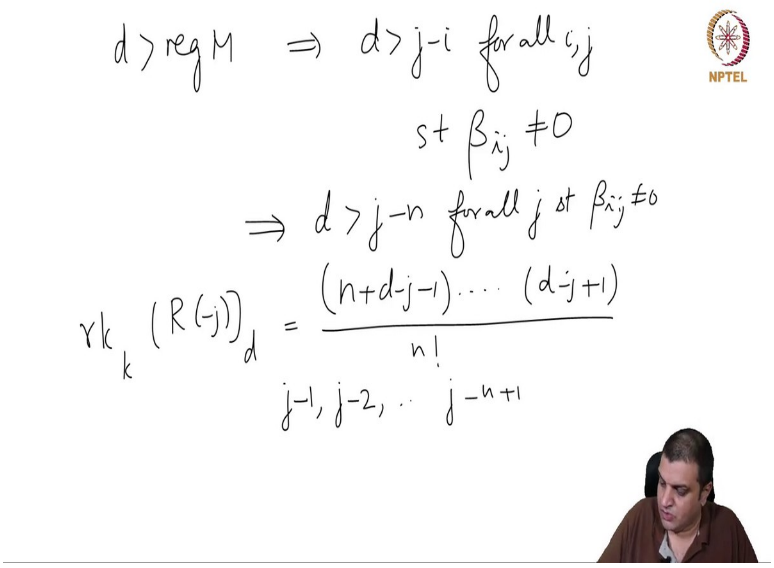

So, now our assumption is that $d > reg(M)$ which implies that $d > j - i$ for all i, j such that $\beta_{i,j} \neq 0$ which in particular implies that the maximum value of i is n; $d > j - n$ for all j such that $\beta_{i,j} \neq 0$. I mean j - i > j - n; so we get this condition. So, now, what does that say? so this is what we have.

So, we knew we know this; this is the $rk(R_d)$, what is the $rk_k[R(-j)]_d$; so this is

$$rk_k[R(-j)]_d = \frac{(n+d-j-1)(n+d-j-2)\ldots(d-j+1)}{n!}$$

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And this is ok; so this value is given by a polynomial for everything So, zeroes this polynomial are j - 1, j- 2,…, j-n + 1..

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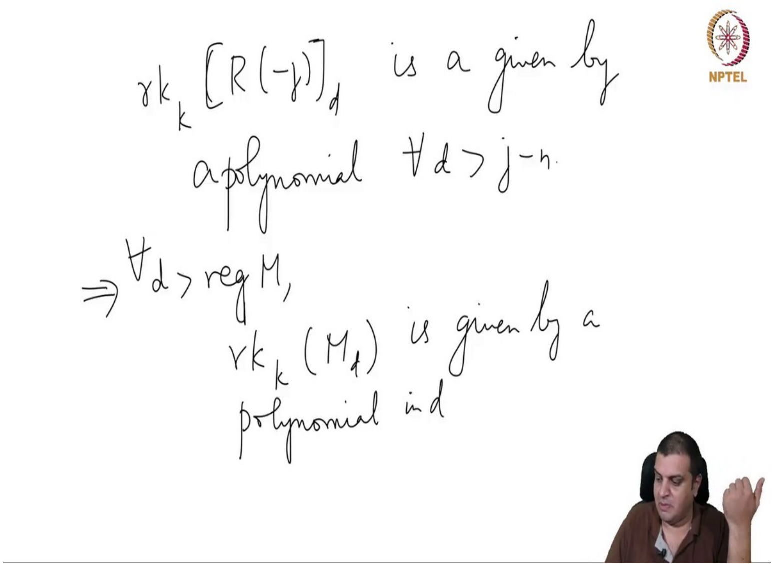

So this is given by a polynomial; so $rk_k[R(-j)]_d$ is given by a polynomial, for all d > j- n; that is because if it is less than j; I mean if it is like j - n + 1, the rank of this is 0. So, is the value of that polynomial .

So, this polynomial is such that from; let us go back to the polynomial for R. This polynomial is such that for degree 0 onwards; it has non-zero value and agrees with that rank. But, from – 1, -(n – 1) , it has value 0 and therefore, again it agrees with the values of this. I mean, this is 0, so is this; in this range.

So, in fact, for the polynomial ring; the Hilbert polynomial agrees with the Hilbert series from minus n + 1 onwards, not from 0.

It is just that the first part is; it is not very relevant, there are both the Hilbert function and the polynomial take value 0 ; so this is the point. Even though this module is generated in degree j and its Hilbert function is 0 when j<d , it is still given by this polynomial .

So therefore, this implies that; ; so remember, we got this condition because of d > n. Therefore, for all d >reg(M), $rk_k(M_d)$ is given by a polynomial in d . But, if there is a polynomial that gives it for every value greater than this; then it must be equal to the Hilbert polynomial.

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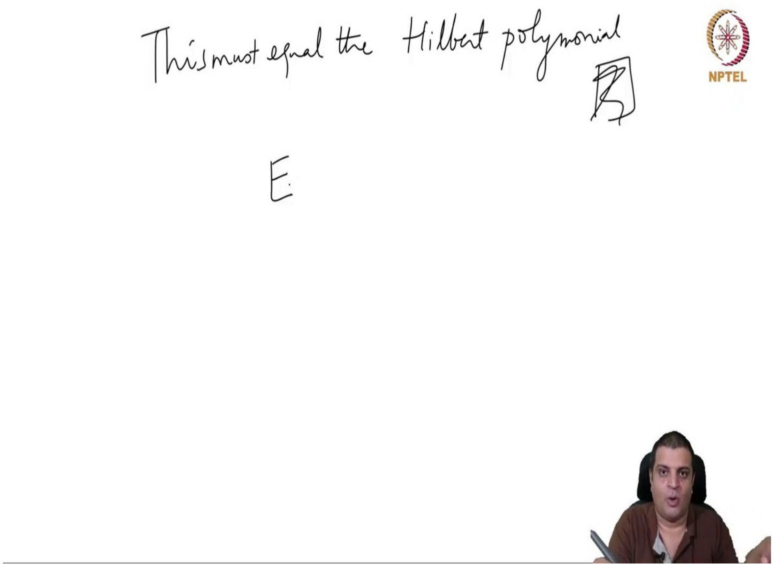

otherwise I mean, you cannot have two distinct polynomials like this which would agree for every integer sufficiently large, I mean that is not possible. So therefore, if it agrees like this; I mean if some polynomial shows this behavior, then it must be the Hilbert polynomial; so that is the end of this proof.

So, this is one application of regularity and this application will not goes back to Castelnuovo and in modern thing to Mumford; I mean who read it in modern, I think modern language I mean, but it is not written in terms of Hilbert, it is not written in terms of free resolutions that translation came later and so which is why this invariant is called Castelnuovo Mumford regularity.

So, this is the end of this lecture and in the next lecture which is the final lecture in this course. We will use this to look at a simple geometry problem which you know which is stated without reference to any of these things.

So, this is the end of this lecture.