Computational Commutative Algebra
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Lecture – 58
More on Koszul complexes

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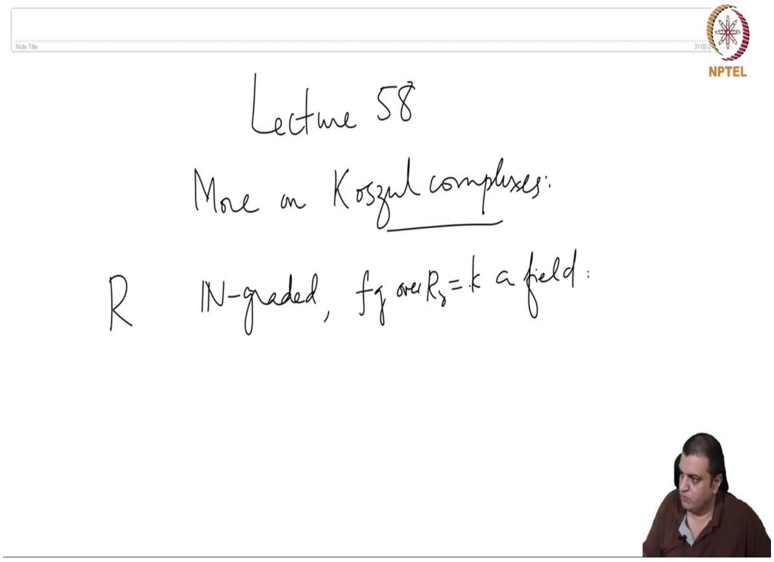

This is a continuation on Koszul Complexes. So, now, for this we can assume that R is non-negatively graded, finitely generated over R_0; which is a field, so it equals k a field, finitely generated over algebra over a field.

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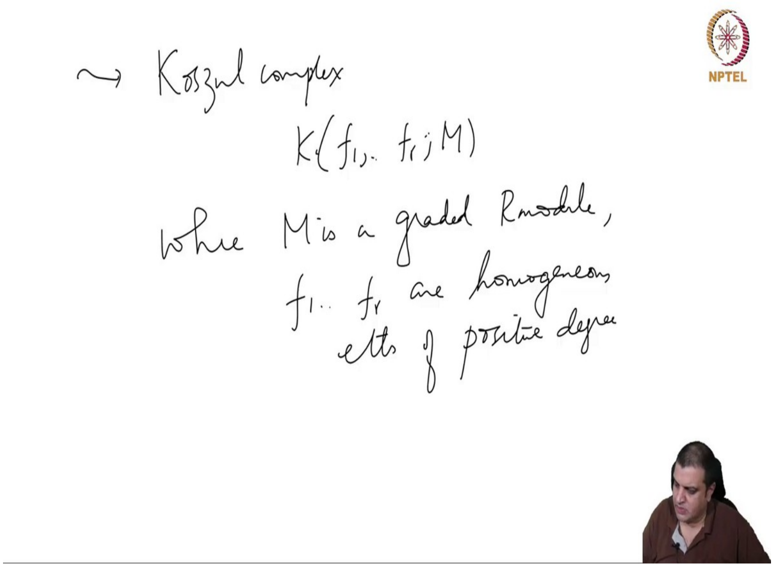

So, we can extend the same construction from last time to get Koszul complex $k(f_{1,} \ldots, f_r; M)$, where M is a graded module. And $f_{1,} \ldots, f_r$ are homogeneous elements positive degree, not units.

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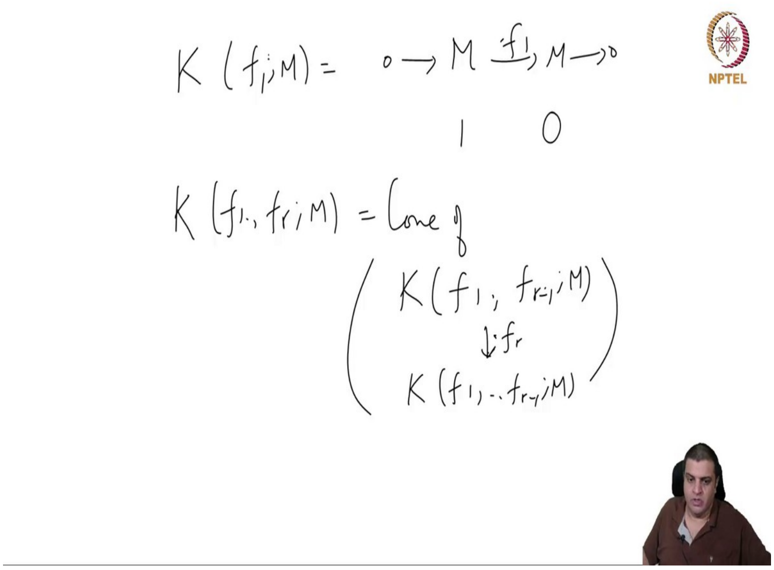

The thing is we start with $K(f_1; M)$ as $0 \to M \to M \to 0$ multiplication by f_1. This is in position 0 and this is position 1. And then we just define inductively $k(f_{1,} \ldots, f_r; M)$ to be the mapping cone of the map in which we take r- 1 of them and multiply.

So, if you just go back and do the same thing the complexes in this will be just direct sums of M with itself, and therefore, we can talk about multiplication by f_r and then this. So, the cone of this map we can define it like this. So, it is entirely analogous construction. Just repeating whatever we did.

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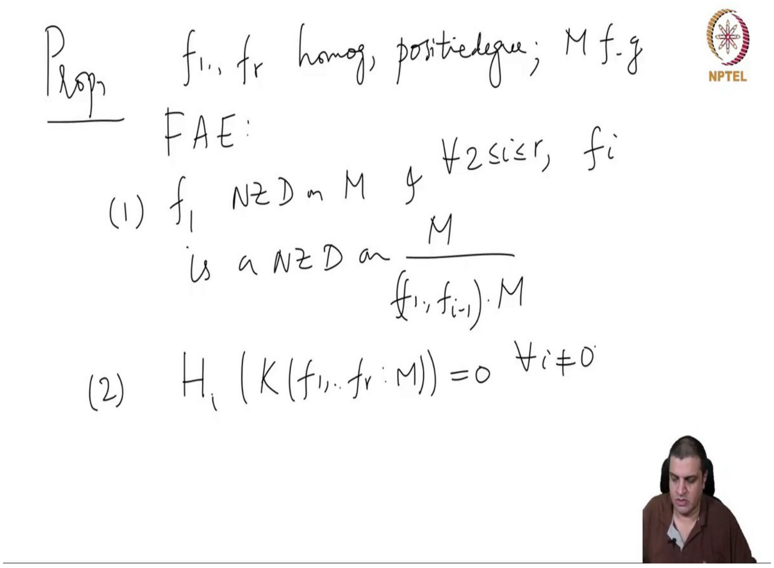

So, now, we want to prove want to describe an important use of this this Koszul complex. So, just to remind ourselves $f_1,\dots,f_r$ are homogeneous non-units, so positive degree. M finitely generated, then the following are equivalent.

1) f_1 is a nonzero divisor on M and for all $2\le i\le r$, f_i is a nonzero divisor on $\frac{M}{(f_1,\dots,f_{i-1})}M$,.

So, this is one condition. 2, the homology of the Koszul complex $H_i(k(f_1,\dots,f_r;M))=0$ for all $i\neq 0$. This is the thing.

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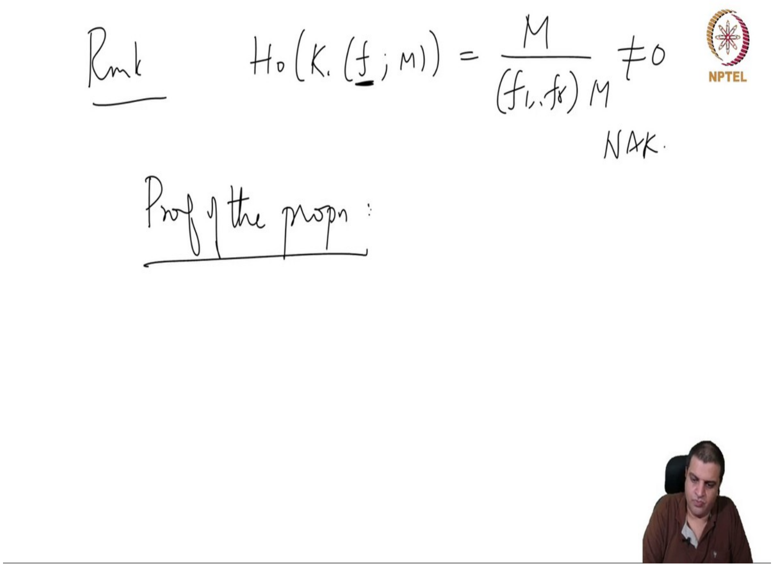

Just a remark here. $H_0\left(k\left(f_{1,}\ldots,f_r;M\right)\right)=\frac{M}{\left(f_{1,}\ldots,f_r\right)M}\neq 0$. because by Nakayama’s lemma. I mean an ideal of elements of positive degree cannot kill a module,. I mean $M\neq IM$, if every element of I has positive degree. So, this is what we have.

So, let us prove this I mean I will we will only give a brief sketch going back to the previous lecture the last lecture in which we said that the mapping cone constructs the homology for induced by the map. So, what does that mean for us? So, proof of the proposition.

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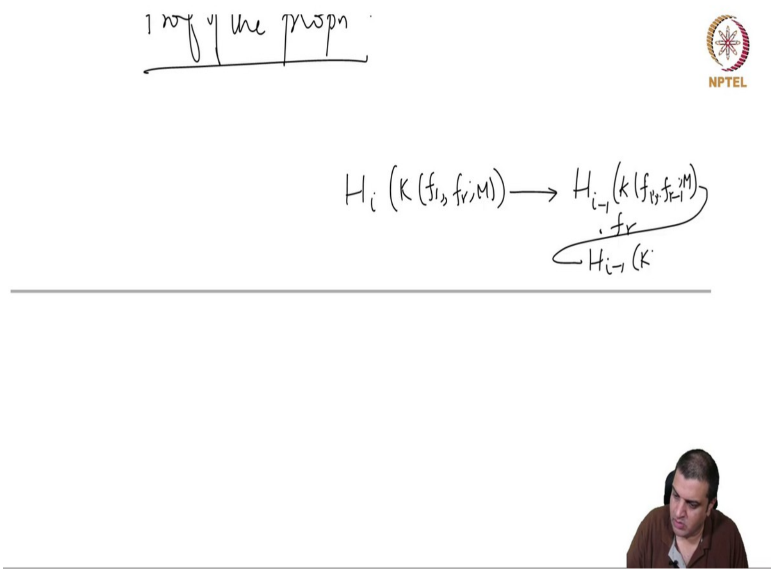

So, we have this $H_i\left(k\left(f_{1,}\dots,f_{r-1};M\right)\right) \xrightarrow{f_r} H_i\left(k\left(f_{1,}\dots,f_{r-1};M\right)\right) \to H_i\left(H_0\left(k\left(f_{1,}\dots,f_r;M\right)\right)\right) \to H_{i-1}\left(k\left(f_{1,}\dots,f_{r-1};M\right)\right) \to H_{i-1}\left(k\left(f_{1,}\dots,f_{r-1};M\right)\right)$

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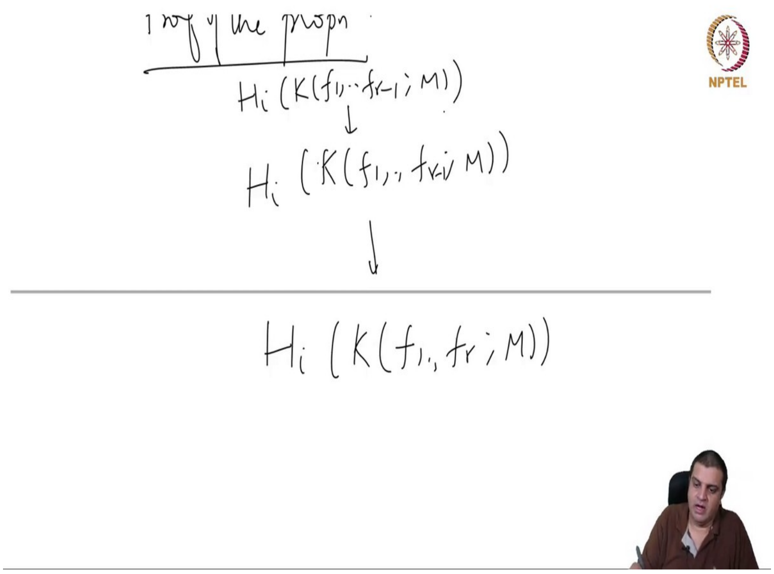

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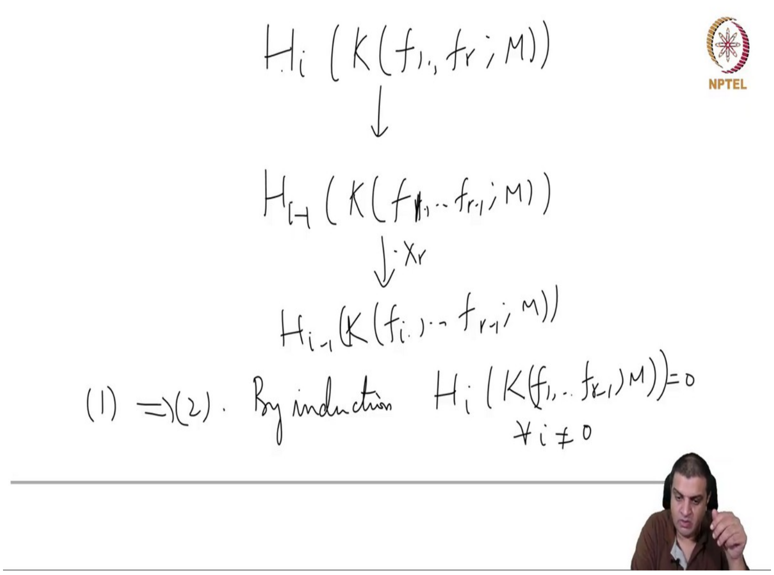

So, the proof will go by induction. In one direction, so assuming proving 1 implies 2. We just have to we will prove, so this will prove. So, in 1 implies 2 we can assume that the statement is true for sequence of r- 1 elements. So, 1 implies 2 by induction, . So, one has to consider two cases separately, because after $H_i(K)$ is H_{i-1}

, so where if i is 1, this is 0, but that is that does not vanish,.

So, by induction we can assume that $H_i(k(f_{1,},\dots,f_{r-1};M))=0$ for all $i\neq 0$, so then, . So, if $i\geq 2$, then we have H_2 here, H_1 here for fewer elements which is 0 by induction, and H_2 for fewer elements which is also 0 by induction.

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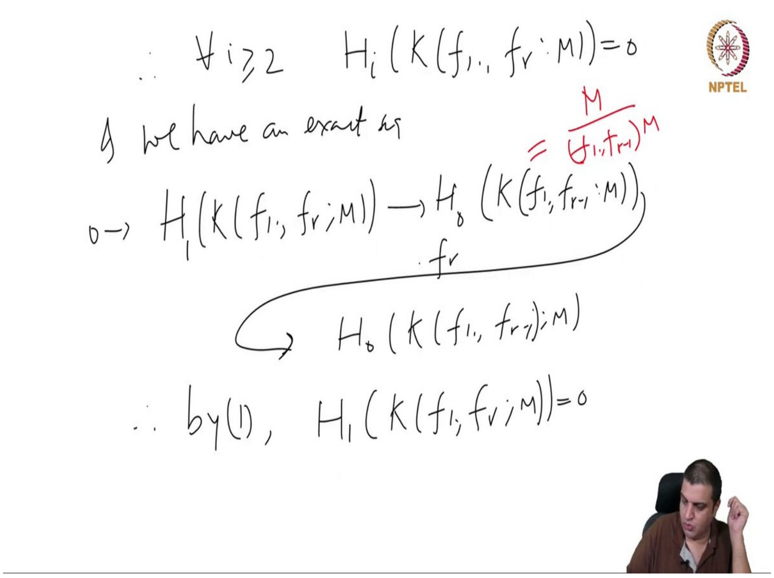

So, therefore, for all $i\geq 2$, $H_i(k(f_{1,},\dots,f_r;M))=0$, . And we have an exact sequence ; so, the last part. If i = 1, then this is 0 by induction, $H_1(k(f_{1,},\dots,f_r;M))\rightarrow H_0(k(f_{1,},\dots,f_{r-1};M))\rightarrow^{f_r} H_0(k(f_{1,},\dots,f_{r-1};M))$

.

But what are, what is this? This we observe that $H_0(k(f_{1,},\dots,f_{r-1};M))=\frac{M}{(f_{1,},\dots,f_{r-1})M}$

multiplication by f_r, but that is injective, that this is a nonzero divisor therefore, by 1 sorry. I mean this is I mean this is same as this this is injective therefore, by (1) $H_1(k(f_{1,},\dots,f_r;M))=0$. The other things we have already checked. So, this is the one direction of the proof. In the other direction we are assuming that for positive i this is 0. So, therefore, so this is 2, 2 implies 1.

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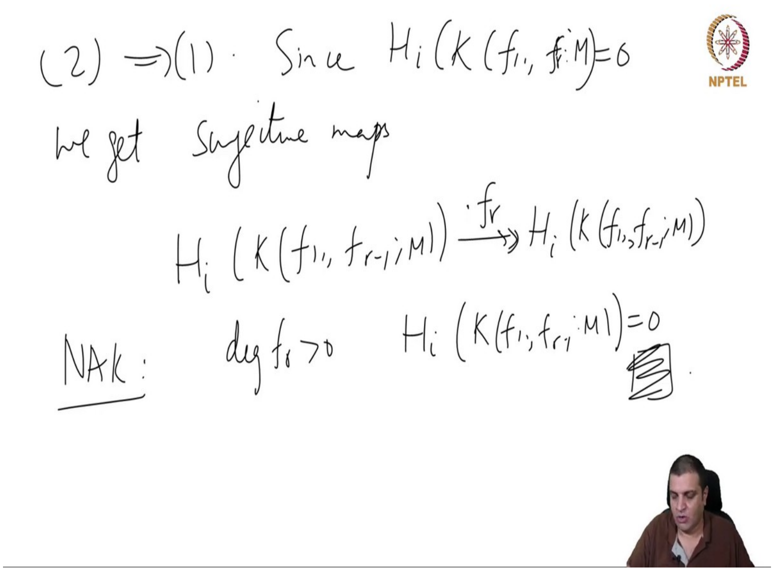

Since, $H_i\left(k\left(f_{1,} \ldots, f_r; M\right)\right)=0$. We get surjective maps from $H_i\left(k\left(f_{1,} \ldots, f_{r-1}; M\right)\right) \to^{f_r} H_i\left(k\left(f_{1,} \ldots, f_{r-1}; M\right)\right) \to 0$

. So, that is what the first map here is. This is 0, so we get a surjective map. I mean it could be it is also injective, but that is not; that is not really relevant or useful at this point.

This is multiplication by f_r, but this is subjective map. But again Nakayama lemma comes in, but it is the this is since we are handling only graded thing it is much easier with the greater Nakayama lemma, so which means that f_r is positive degree. So, a generator here is a multiple of something from here of previous degree, but in previous degree there is nothing.

So, this by Nakayama lemma this says that $H_i\left(k\left(f_{1,} \ldots, f_{r-1}; M\right)\right)=0$, but even if you are working in a local ring, all of this would have gone through even here, this would say that a module is something in maximal ideal times itself and hence it is 0.

So, this is in either case Nakayama lemma will be useful, either in the local case or in the graded case. It is just much easier to follow it in the graded case. So, this is the; so, this is one characterization of I mean of what does it mean to say that all Koszul homologies vanish. So, then we just want to quickly make some remark.

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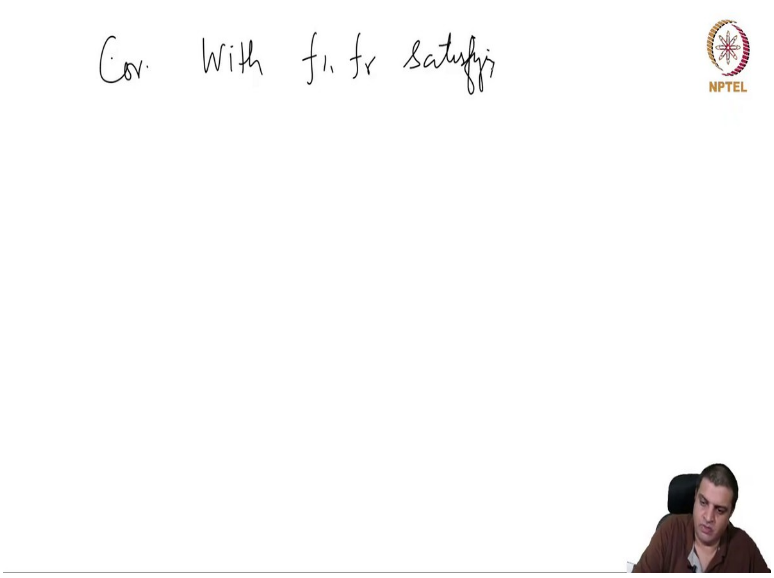

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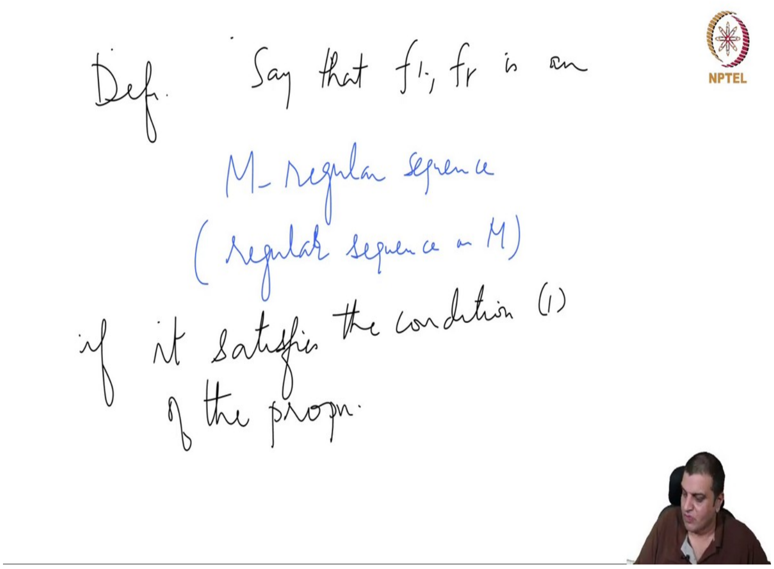

Let us just say definition we say that $f_{1,} \ldots , f_r$ is an M-regular sequence (or regular sequence on M). If it satisfies the condition, conditions of the proposition, let us just say condition 1 of the proposition, we will take that as a definition.

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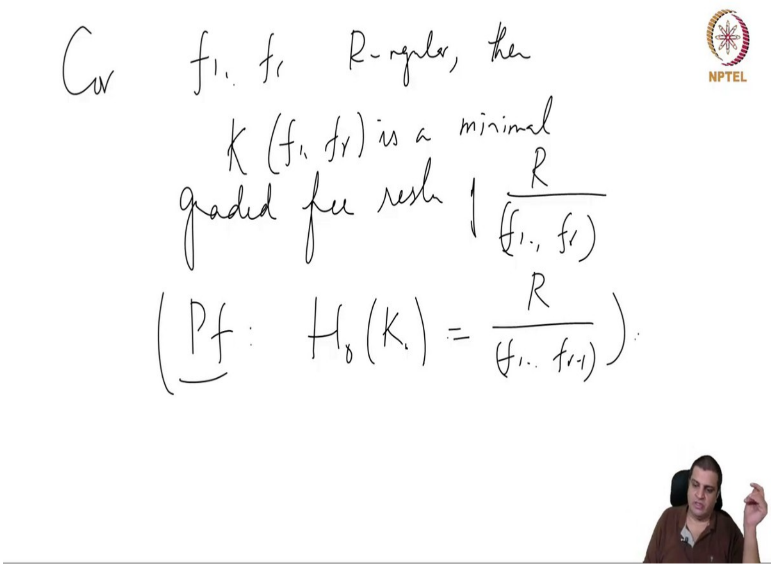

So, then corollary if $f_{1,}\ldots,f_r$is R-regular, then $k(f_{1,}\ldots,f_r;M)$ is a minimal graded free resolution of $\frac{R}{(f_{1,}\ldots,f_r)}$. So, this is because with this hypothesis the other things vanish and the only thing that we need to observe which we already done once is $H_0(k(f_{1,}\ldots,f_{r-1};M))=\frac{R}{(f_{1,}\ldots,f_r)}$ of the Koszul complex is this thing. So, we have a complex of free modules graded degree preserving all that is satisfied the homologies in position 1 to the left are all 0, at the 0th position this is the homology.

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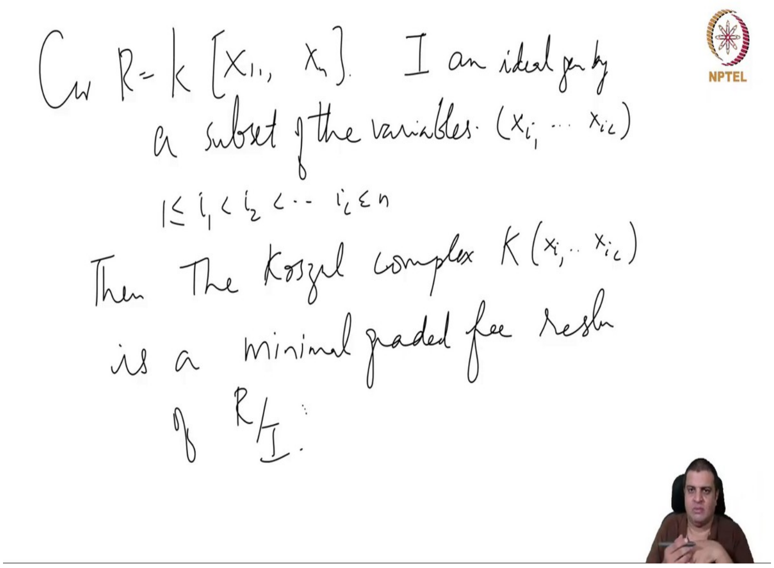

So, it is a resolution. And now we can go back we worry about the polynomial ring now, ok. I an ideal generated by a subset of the variables. Let us just say $(X_{i_1}, \ldots, X_{i_c})$, where $i_1 \leq i_2 \leq \ldots \leq i_c$. Then, the Koszul complex $k(X_{i_1}, \ldots, X_{i_c}; R)$ is a minimal graded free resolution of $\frac{R}{I}$, and it has length at most it has length exactly c.

And now therefore, we can prove it for all monomial ideals. Therefore, we can prove it for all monomial sub modules of free modules. Then, once we learn how to degenerate from an arbitrary sub module of a free module to a monomial submodule, we would have proved Hilbert's syzygy theorem. But there are other proofs, and most textbooks will discuss some other proofs.

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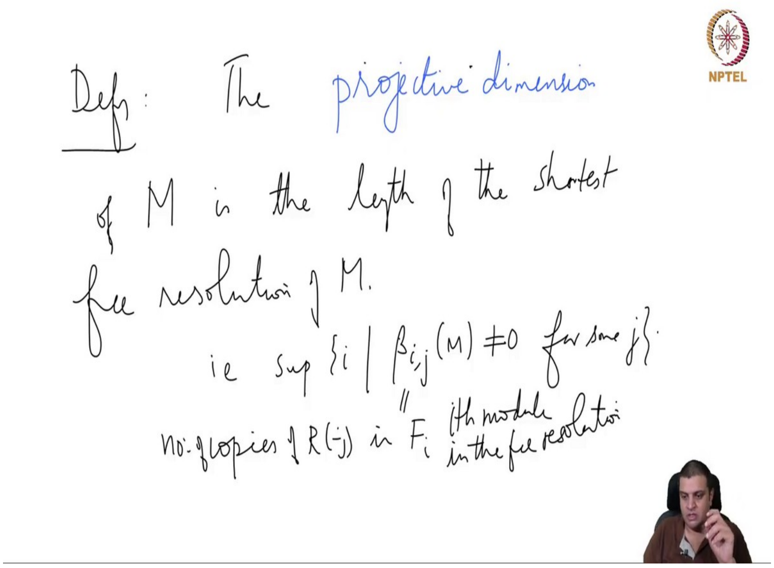

We just want to define just to become familiar, the projective dimension of M is the is the length of the shortest free resolution of M or in other words $\sup\{i : \beta_{ij}(M) \neq 0 \; for\ some\ j\}$.

Definition can go for any ring, any noetherian ring, non-noetherian ring, it does not really matter. But for arbitrary graded rings we can treat it as this; for we are we have we will have to allow for a supremum. For polynomial rings the supremum is actually can be replaced by a maximum because Hilbert's syzygy theorem says it cannot go beyond n. So, we can use this definition. And so, we will not do anything with this.

So, in the next two lectures, we want to discuss another invariant coming from the Graded Betty numbers, this, these things. Remember, this was just the number of copies of R(-j) in F_i. This is the free part ith module in the free resolution. . So, we will do some exercises with this, but we will not discuss this any further in this, I will not lecture any for any more on this.

So, in the last two lectures would be about what is called Castelnuovo Mumford regularity. And, we will see how it is used in the first lecture. So, in next lecture we will see how it is used to understand, how it is related to Hilbert functions. And, in the final lecture we will use it to understand a problem in geometry which stated without any reference to any of these techniques, but it will come useful at that point.