Computational Commutative Algebra
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Lecture – 57
Koszul complex

Welcome, this is lecture 57. In this lecture, we look at Koszul complexes, how they are constructed using a mapping cone. We will start with looking at a mapping cone then we will define Koszul complexes as a mapping cone.

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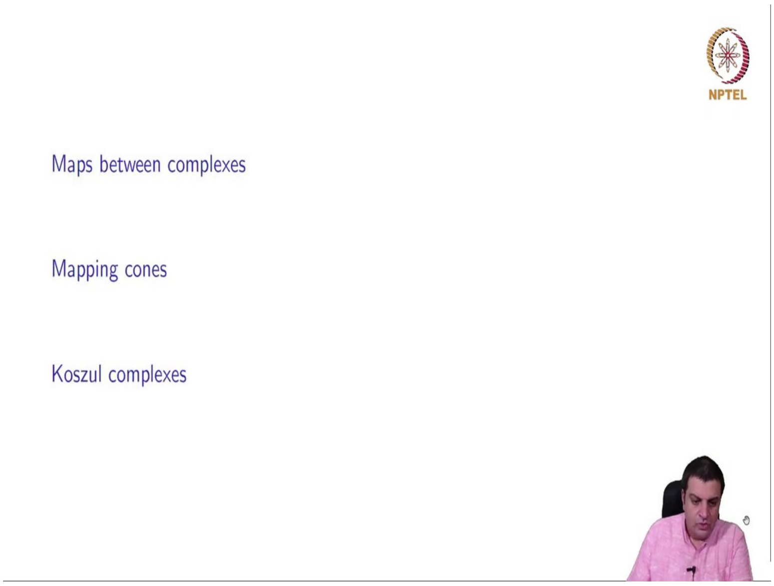

So, what we will quickly review what we, how we defined complexes and then, we will now define maps between complexes and then, we will talk about mapping cones and then so, that is the plan for this lecture.

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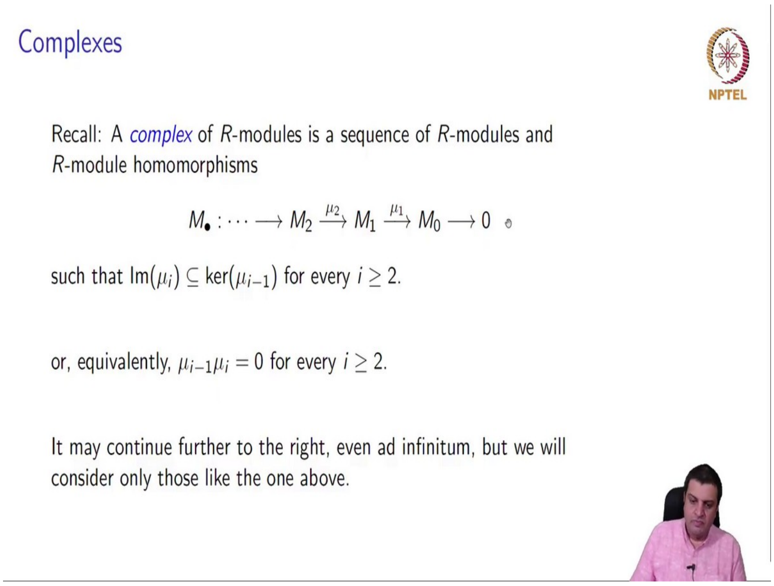

So, recall that a complex of R-modules is a sequence of R-modules and R-module homomorphisms, so, $M. : .... \to M_2 \to M_1 \to M_0 \to 0$ and sometimes you will have to refer to the maps so, we will just label them.

So, why is it a complex? So, that image of $\mu_i : M_i \to M_{i-1}$ $\Im(\mu_i) \subset ker(\mu_{i-1})$equivalently composition of two consecutive maps is 0. Remember the composition is μ_i first and then μ_{i-1} .

in principle one does not have to have a 0 here, you could go even to infinity on the right side, but the complexes that we will consider in this course will always end somewhere and then, it is 0 afterwards. So, we will just consider only such complexes.

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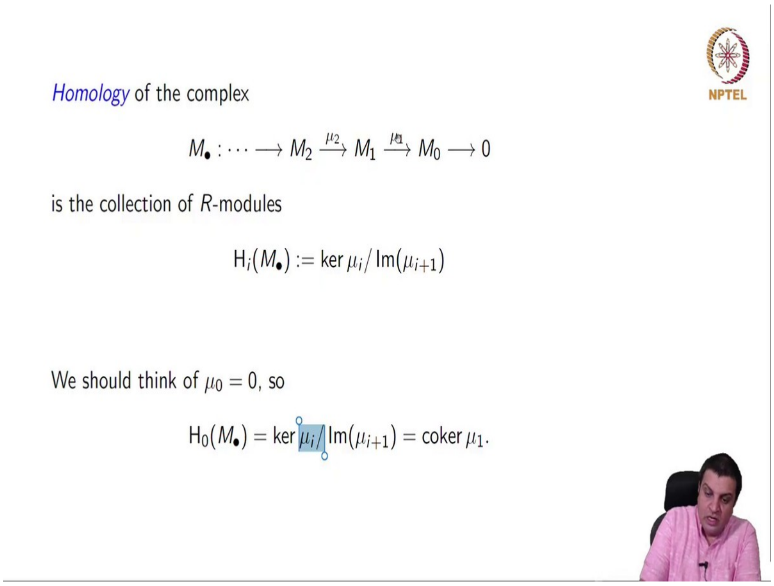

The homology of a complex is the ith homology $H_i(M.) = \frac{ker(\mu_i)}{im(\mu_{i+1})}$. So, remember that image of μ_{i+1} is always inside the kernel of μ_i. So, the quotient, the module is called the homology of this complex, the ith homology of this complex.

And we can at this point, we can think of the last map from $M_0 \rightarrow 0$ you can think of it as a map μ_0 and as a 0 map. And hence, $H_0(M.) = \frac{ker(\mu_0)}{im(\mu_1)} = \frac{M_0}{im(\mu_1)}$.

. which is what is called, what we call the $coker(\mu_1)$. So, remember that the last the H_0 of such a complex is a co-kernel of this last map $M_1 \rightarrow M_0$.

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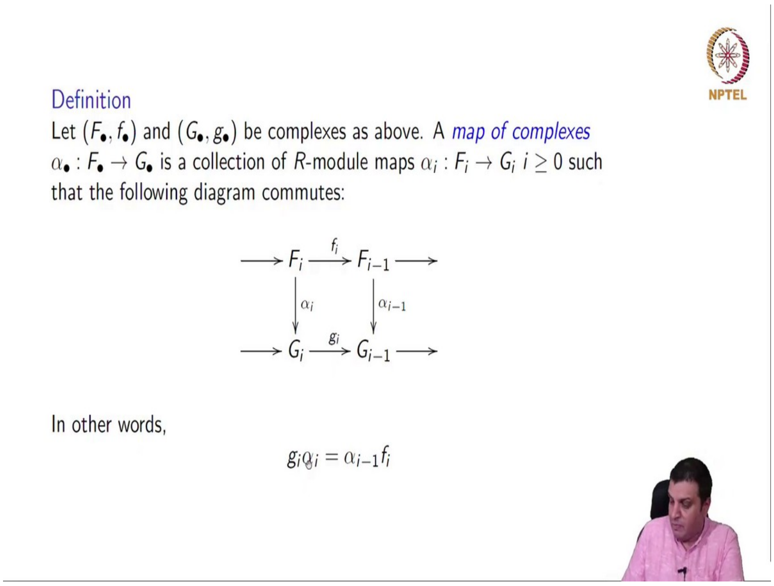

If we have two complexes (F. ,f.) with where we denote the maps by small f. and (G.,g.) where we denote the maps by small g., if their complexes as above so, ending at some 0 and then 0 modulus of 0 afterwards. A map of complexes is a collection of R-module homomorphism from $\alpha_i : F_i \rightarrow G_i$ for each i so, such that the following diagram commutes.

What does that mean to say diagram commutes? It means that between any pair of points all the arrows going from one to the other are either give the same function. So, in this case, it is only one to check which is from $F_i \rightarrow G_{i-1}$, there are two paths $F_i \rightarrow F_{i-1} \rightarrow G_{i-1}$.

And $F_i \rightarrow G_i \rightarrow G_{i-1}$ so, along the sides of this rectangle there are two different paths and they those two functions agree. So, in other words, $g_i \alpha_i$ which is the path down first and then to the right that is the same as right first and then down $\alpha_{i-1} f_i$. So, these two composites are the same that is what diagram commutes means and that is what it means in this context of a map of complexes.

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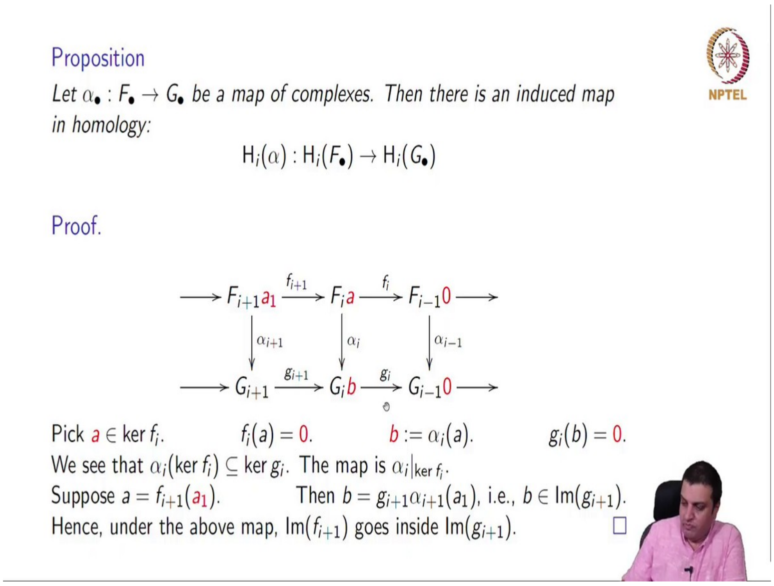

So, if you have a map of complexes, then it induces a map in homology that is a map $\alpha : F \to G$, induces a map which we will denote by $H_i(\alpha) : H_i(F.) \to H_i(G)$. So, we will so, these proofs are done using what is called a diagram chasing. So, we will just do the diagram chasing on this diagram itself

So, we need to find an element, we need to look for elements in the kernel here that is $\frac{ker(f_i)}{im(f_{i+1})}$ and we need to get them to an element in $\frac{ker(g_i)}{im(g_{i+1})}$. So, from the kernel in the top row at F_i to the to the homology of the bottom row that is inside G_i . So, we will do this by doing what is called a diagram chasing.

So, let us pick an $a \in ker(f_i)$. Remember elements of homology are kernel mod image so, they are represented by the residue classes of elements of the kernel going modulo image of the previous map. So, we will just take an element in the kernel. So, it is in kernel. So, it just means that $f_i(a) = 0$. So, the things that I mark in red are the elements that we pick and yes, we will see how the elements progress in this diagram chase .

So, $f_i(a) = 0$. Now define $b := \alpha_i(a)$. Now, because the diagram commutes, $g_{i-1}(b) = 0$. So, we chose $a \in ker(f_i)$ so, it went to 0 here and then came to 0 here so, along top and right, we have we get 0 inside this G_{i-1}. Therefore, on the left and bottom also one should get this. So,

$b \in ker(g_i)$.

So, in other words, what we just observe is that $\alpha_i(ker(f_i)) \subseteq ker(g_i)$. So, we can think of this as a map α_i restricted to $ker(f_i)$. Now, let us check what happens under this map to $im(f_{i-1})$. Remember that, $im(f_{i+1}) \subseteq ker(f_i)$. So, under this map, what would happen to image of f_{i-1}.

So, now let us take, let now let us assume that a is in the image of the previous map. So, in other words, $a = f_{i+1}(a_1)$. So, this a is a image of this a_1, then because of the commutativity of the diagram one can immediately see that this $b = g_{i+1}\alpha_{i+1}(a_1)$. In other words, $b \in im(g_{i+1})$. So, if a is the image of the top row map, then so, is this b that is the conclusion.

In other words, under this map α_i restricted to this kernel, $\alpha_i(im(f_{i+1})) \subseteq im(g_{i+1})$. So, now what do we have? We have a map from the kernel in the top row to the kernel in the bottom row under which the image in the top row goes inside the image of the bottom row when you this is.

So, now, this gives a map from kernel mod image on the top row to kernel mod image in the bottom row and that is exactly what $H_i(F) \to H_i(G)$ is. So, in so, what the that is the end of the proof. So, the conclusion is that any map of complexes through this diagram chases chasing show gives a map in homology.

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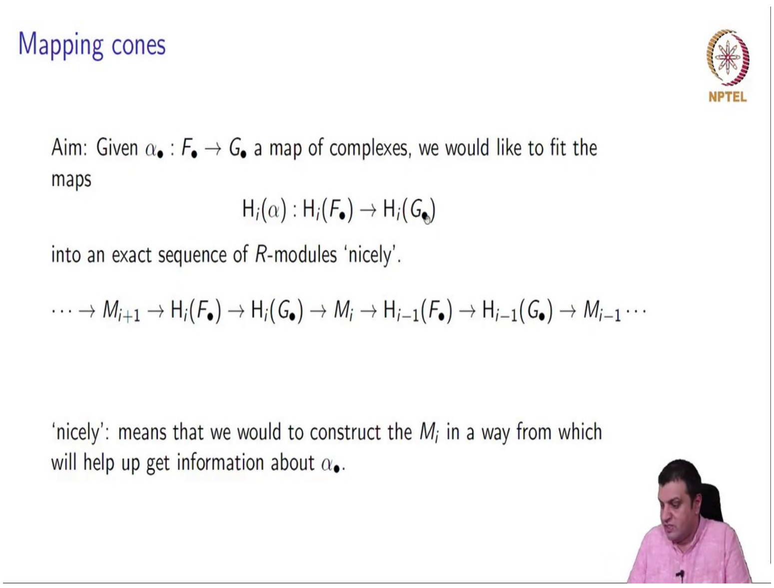

So, now we wanted to look at this thing called mapping cones which is a way. So, here is a motivation or our aim. So, we have given a map of complexes. So, we have these maps now. We would like to fit them into some exact sequence like this. So, this is the map; this is the map in homology at the ith level so, this is at the ith level.

This is the map in homology at the i - 1 th level and we would like to put them in such a long exact sequence because it would help us conclude various things. So, this is again I mean post facto justification, we would like to put them like this in a nice way and what does what do we exactly mean by nicely?

So, we would like to describe these M_i in such a way that we can say something about the maps α_i or they are somehow in some constructed in some natural way we, there is no definition I mean is to what exactly we should I mean how, what exactly the natural way is, but we will describe one such thing called mapping cones.

So, the aim is to understand the induced map in homology, how does it fit together, what is its kernel and co kernel. So, these things would be clarified with if you put them in a nice exact sequence like this. So, the aim is to construct such a nice exact sequence.

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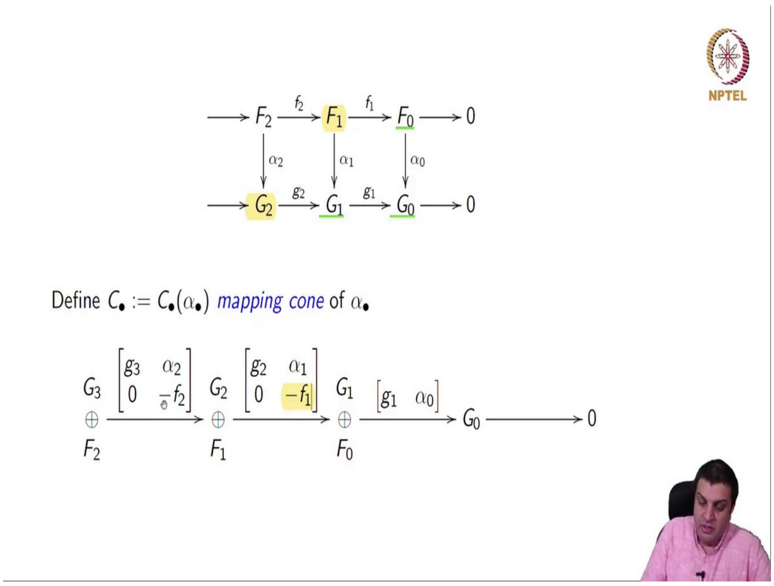

So, this is what we start with. So, we have to now going to describe the construction of the mapping cone. We have a complex F on the top row, complex G in the bottom row and I have drawn it from the rightmost end where $F_0 \rightarrow G_0$ and then towards to its right these are all

0 modules. So, we have a map like this.

So, now, we would like to describe what is called the mapping cone of this map of complexes. So, here is the thing let us see how this was constructed. So, now, let us look at G_0 . There are two maps coming to G_0, one $\alpha_0 : F_0 \to G_0$ and one $g_1 : G_1 \to G_0$.

So, F_0 and G_1 mapped to G_0. So, in this mapping cone, we put this here $G_1 \oplus F_0 \to G_0$ and the map $[g_1 \alpha_0]$from the first factor, we use the map g_1, for the second factor which is F_0 we use α_0. So, that just came directly from this.

So, here is G_0, we just map to all the modules that map into this which is from the F complex and the G complex, one module each so, that is what we have here. . Now, we have now in the second position of this complex this is new complex, we have $G_1 \oplus F_0$ so, the things on this diagonal like this.

And there are two modules behind them which map to these that is there is G_2 and then there is F_1. $\alpha_1 : F_1 \to G_1$ and $f_1 : F_1 \to F_0$ and $g_2 : G_1 \to G_1$.

So, let us put that thing together here. So, this is written as like a matrix, but it is not exactly a matrix, it is just a convenient way of writing; because it is a map from a direct sum. So, $G_2 \oplus F_1 \to G_1 \oplus F_0$. From the first part here to the first part here which is $G_2 \to G_1$, we have a map g_2.

Then, from the first part here to F_0, the second part here there is no map so, it is 0. From F_1 which is the second part here to the first summand which is G_1 there is a map α_1 so, we have this. And from F_1 to F_0, there is a map little f_1, but we just put it a minus sign and we will see why this is needed.

So, we so, we just put a only thing you keep in mind is that there is a minus sign there. Similarly, from we have $G_2 \oplus F_1$, the map would be from $G_3 \oplus F_2 \to G_2 \oplus F_1$, G_3 to G_2 there is a g_3 map,G_3 to F_1 there is no map, F_2 to G_2 there is an α_2 and F_2 to F_1 we have a map f_2, but we use that with a minus sign.

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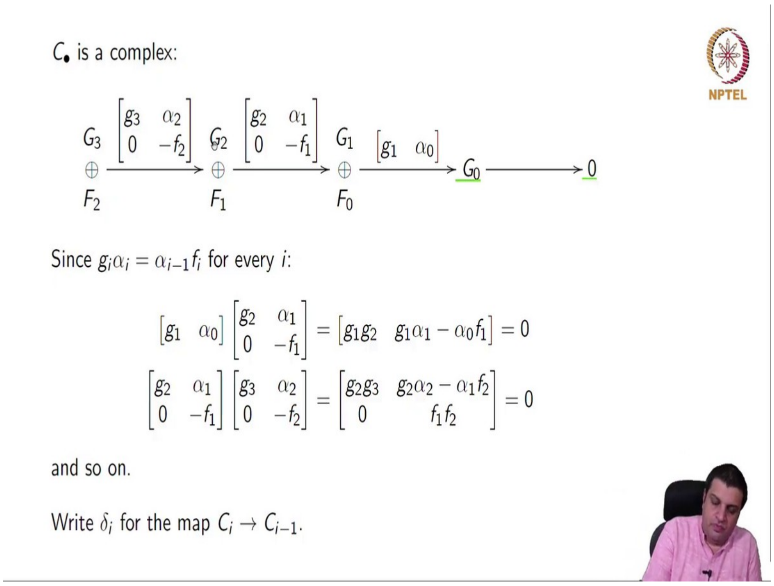

So, now we have this, and the point is that. So, this I have just copied it. This is the same complex that we just described in the previous page. The point is that this is a complex why? So, let us check this. So, which means we should check that this map here, this map this map here the square the two, I mean the $G_2 \oplus F_1 \to G_1 \oplus F_0$ this map here to the next map the composite should be 0.

So, remember the maps are applied in the opposite order, this is the one that supplied first and then, this 2 by 1 sorry this 1 by 2 looking matrix. So, this is the composition and one can immediately check that because of the; because of the diagram commutes $g_1\alpha_1 = \alpha_0 f_1$ and because we introduced a minus sign, they cancel each other and we get 0 map and similarly here also.

The diagram commutes so, $g_2\alpha_2 = \alpha_1 f_2$, but we introduced a minus sign precisely to cancel this and so, we get 0 in this in this part and $g_2 g_3$ is anyway 0 because it is a complex and $f_1 f_2$ is also 0 because it is a complex and so on to the left.

So, except at this at the rightmost place where you know you one has to multiply a matrix like this, rest of it is all like these two by these are not matrices, but you know two fact, two summons on both from each one of those complexes. So, we get this. So, this is a complex.

So, now, we will need to work with this complex a little bit. We will write δ_i for these maps. So, delta so, this is $G_0 = C_0$, this thing here is $C_1 = G_1 \oplus F_0$, $C_2 = G_2 \oplus F_1$; the index of G, the subscript for G is the one which describes the subscript for C. So, this is how. So, the map we will describe as δ_i.

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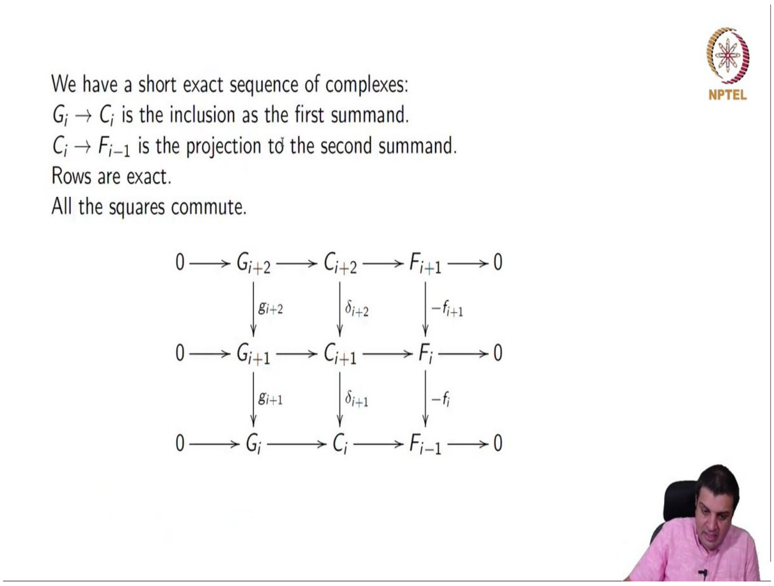

So, now, from what we have constructed, we have what is called a short exact sequence of complexes. So, I will not define it, I mean I will just I will describe these things in this context, but we will do an exercise which describes it in I mean in in full generality.

So, what exactly do we mean? Actually, what are these maps first of all? The vertical maps g and vertical maps f_i and here $-f_i$ they were given to us. These delta i's were constructed as I mean in the previous pages. So, the now, we have to describe the horizontal maps.

The, this is the inclusion in the first summand. Remember $G_i \to C_i = G_i \oplus F_{i-1}$ so, the inclusion from G_i to C_i is the inclusion as the first summand and $C_i = G_i \oplus F_{i-1} \to F_{i-1}$ is the projection to the second summand. So, therefore, the rows are exact. So, that is what here we mean.

So, this is important rows are exact and all the squares in these commute that is because in other word, we are saying is that we will see it in a minute. I mean rows commute one can check explicitly because the first factor if you go here and apply delta, it is same thing as applying g because nothing happens to the second and so, it is. So, one can write down from those matrices and check that the all those squares commute.

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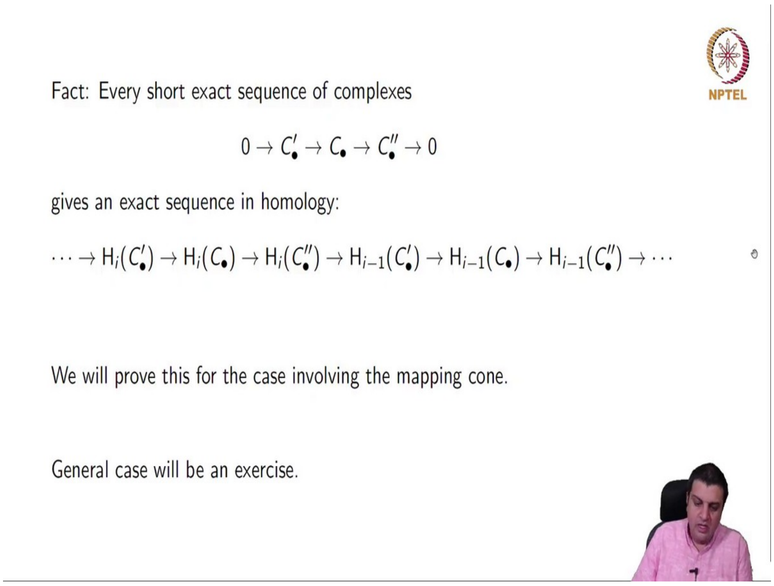

Now, here is a general fact about short exact sequence of complexes. So, we will make a proper definition of this etcetera and prove this fact in this generality in the exercises. So, we for every short exact sequence of complexes $0 \to C' \to C \to C'' \to 0$, there is an exact sequence in homology $\to H_i(C') \to H_i(C) \to H_i(C'') \to H_{i-1}(C') \to H_{i-1}(C) \to H_{i-1}(C'') \to$.

So, this is a general fact we will . So, we will prove this just for this mapping cone case explicitly general case as I said will be an exercise.

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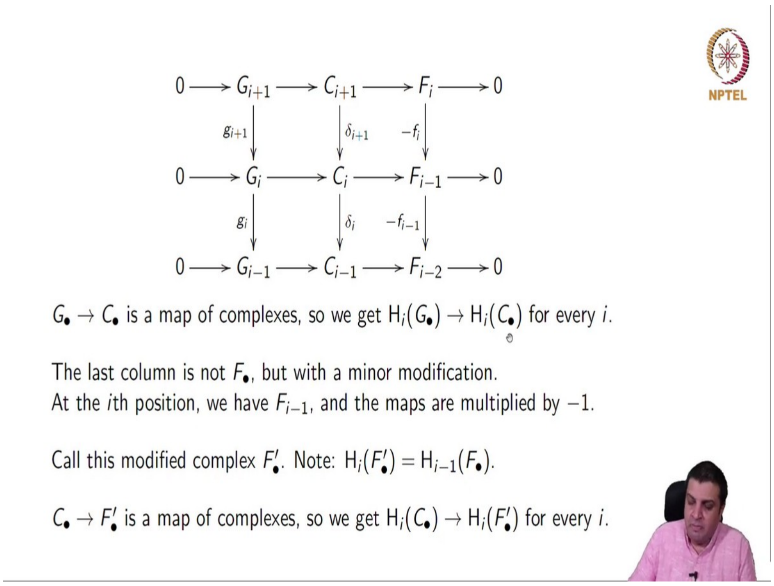

So, the squares here commute says that if you look at the vertical; vertical arrows, the collection form the complex G and the middle arrows here vertical things form the complex C and the arrows commute is the same thing as saying the left from the complex G to the complex C is a map of complexes. So, we get this we observed earlier, there is a map of homology from $H_{i-1}(G) \to H_i(C)$ there is a natural map which came from the map of complexes.

Now, just observe that the last column is not the complex F, but there is a minor modification which is that its indices have been shifted by 1. At the ith position of the new complex , we have F_{i-1} and the maps are multiplied by minus 1. So, there is a shift in the indices and the multiplication by a minus 1 in the maps. So, we will call this new complex F' just a notation.

The observation that we need to make is $H_i(F') = H_{i-1}(F)$. The kernel and image do not change if you multiply by -1 and all that we have done I mean for while computing homology only thing that we have done is just changing the indices so, what homology we would have obtained at i -1 position if we are treating it as F, we would now get it at i th position. So, now, $C \to F'$is a map of complexes and so, we get a map H i of C to H i of F prime, but remember that is H i minus 1 of F for every i ok.

So, then one needs to check so, this is just a sort of preliminary discussion, one has to check that if you take this map, I mean $H_i(G) \to H_i(C) \to H_i(F')$, it is exact at the middle part which is $H_i(C)$ that we would not describe, it will come from the general principle about short exact sequence of complexes giving long exact sequence on homology. So, I, we would not explicitly check in this lecture, but let us see how the mapping, what specific thing happens in the mapping cone.

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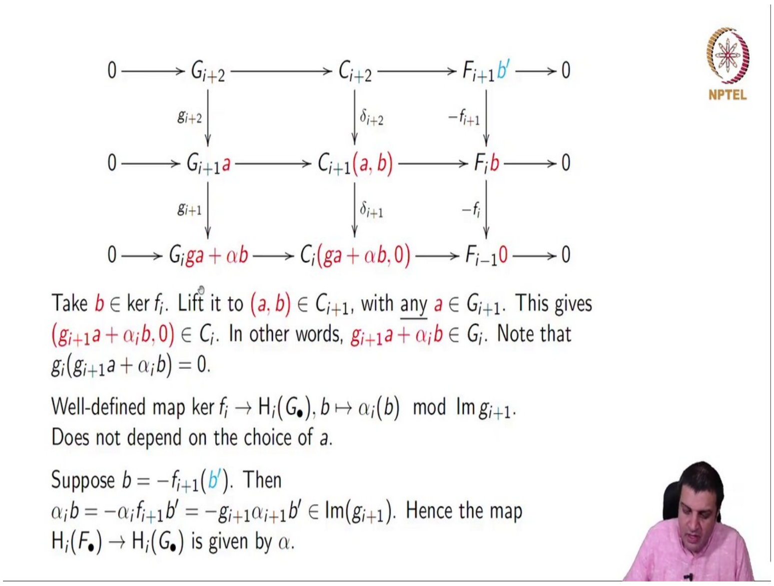

So, I have left some gaps to fill in elements here, we are going to do a diagram chase. So, now so, let us go back. Composing these two, $H_i(G) \to H_i(C) \to H_i(F')$ prime we get something like this.

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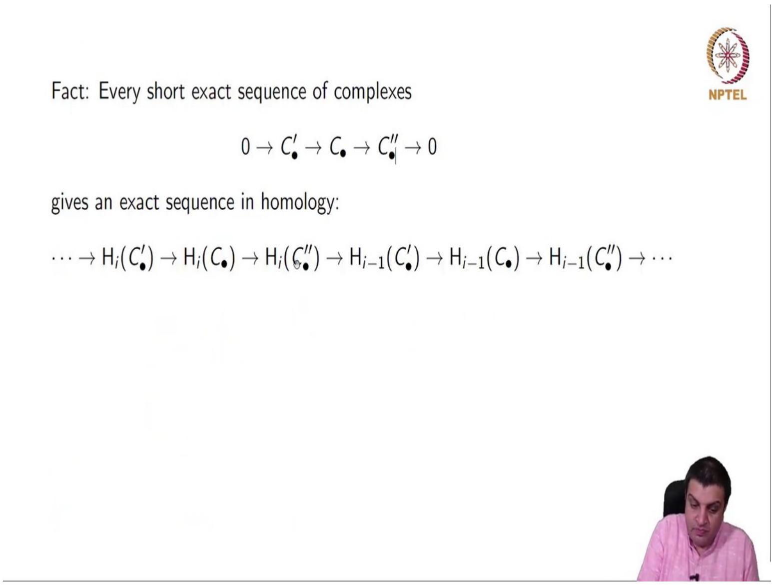

The middle part is C, left one is G and right side is F’ so, we get something like this. What we want is the switch from the ith stage to i - 1th stage. So, $H_i(F') \to H_{i-1}(G)$ so, it is this map that it is this map here that we are trying to construct in the what we will describe now so, yeah.

So, we need something to go from this homology here remember this $F_i = F'_{i+1}$. So, from this point to this I mean the homology of the right most complex at this point; at this point to the homology of the left most complex at this point that is what we will need to construct.

So, let us take a b. So, well do a diagram chase like earlier. So, we will take a let us take a $b \in ker(f_i)$. $ker(f_i) = ker(-f_i)$. So, this b here goes to 0. Now, lift it to any (a, b). So, remember $C_{i+1} = G_{i+1} \oplus F_i$. So, lift it to any a, b, where $a \in G_{i+1}$. So, it does not matter what a is lift it here.

Let us apply delta, let us come down to this map. Well, when you come down to this map, g gets applied to a because a came from this, α gets applied to b because alpha came from here, b came from here and then for the second summand I mean the second component f gets applied to b ok. So, what we have is $(g_{i+1} a + \alpha_i b, 0) \in C_i$.

So, I have just in this thing, I have just to conserve some space, I will just call it $ga + \alpha b$, but usually after a while keeping track of these indices unless there is a pressing need for it one does not, I mean 1 understands that this is after applying the correct g in the correct degree . So, we get an element here and remember that the map from I mean in the second factor the map is $b \to f(b)$, but $b \in ker(f_i)$ so, it is 0 . So, one has this.

In other words, this element here came from G_i . So, this is the element that we . So, from here we lift it arbitrarily, then we apply this map and then, we got an element inside G. If you apply; if you apply the g map to this $ga + \alpha b$, we would get 0 because this one can just check this is just check map two consecutive maps in a complex and for this factor, one has to use that the diagram commutes $g_i \alpha_i = \alpha f b$, but f(b)= 0 so, one gets 0 here.

So, in other words, what we have is we have a well-defined map from the $ker(f_i) \to H_i(G)$ not to the kernel because the actual map depends on the choice of a . So, there is no well-defined map from $ker(f_i) \to ker(g_i)$, there is a map $ker(f_i) \to \frac{ker(g_i)}{im(g_{i+1})}$

. So, so, one has to make it a well-defined map, R linear map, one has to go modulo the $im(g_{i+1})$so, this is the map, $ker(f_i) \to H_i(G)$ where $b \to alpha_i(b)$.

And this map does not depend on the choice of a because whatever if you chose an a prime here, if you do the same thing calculation here, you will get a different element here, but that different element will depend will be inside the $im(g_{i+1})$. So, in the homology it would be the same element.

So, one has to check that this is R- linear etcetera in which it is. So, now; we got we get a map from $ker(f_i) \rightarrow H_i(G)$ which came through the map of complexes α , it is an element in the kernel to which the map α_i is applied so, this is what we have.

Now, what we had wanted was from $H_i(F') \rightarrow H_{i-1}(G)$, in order to do this, we need to understand what happens to the image of the previous map. So, now, suppose that b is actually the image of the previous map so, $b=-f_{i+1}(b')$, then one can just immediately check this $\alpha_i(b)=-\alpha_i f_{i+1}(b')$ so, now one immediately sees that if b is in the image of this, then $\alpha_i(b) \in im(g_{i+1})$.

In other words, the image $im(f_{i+1})$ goes to the image of this thing so, inside the homology it is 0. So, hence, we get a map from $H_i(F) \rightarrow H_{i-1}(G)$ given by α.

Remember this is how the mapping homology was defined. I mean if you have an element b in the kernel, we the map takes the $\bar{b}$ to $\overline{\alpha_i b}$ and that is exactly what this map does.

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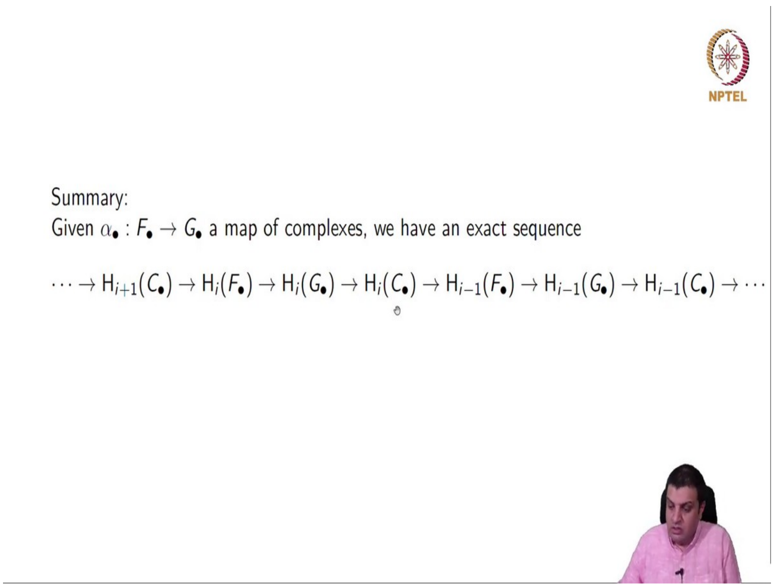

And now so, with this we have established so, let us just summarize. Given α map of complexes we have an exact sequence. So, let us just make sure we get this thing. So, this is F_i, the homology here is $H_i(F)$, homology here is $H_i(G)$ so, the map that we have constructed is from here to here and then, we have this. So, $\rightarrow H_{i+1}(C) \rightarrow H_i(F) \rightarrow H_i(G) \rightarrow H_i(C) \rightarrow H_{i-1}(F) \rightarrow$

. So, this is how this thing comes.

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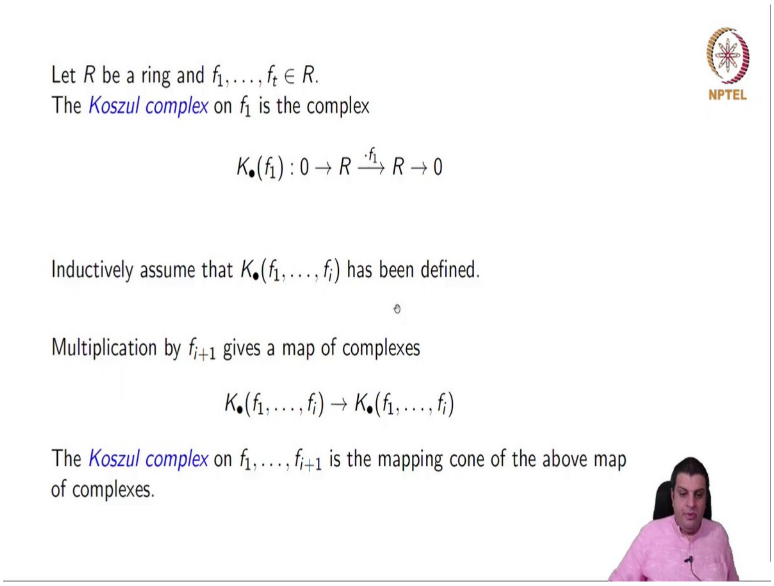

Now, we come to this point of defining Koszul complexes. So, we have a ring and $f_{1,} \ldots, f_t$ elements in R. The Koszul complex on f_1, a single element is the complex $K.(f_1): 0 \rightarrow R \rightarrow^{f_1} R \rightarrow 0$multiplication by f_1. Inductively, let us assume that up to for i elements we have defined it.

Now, multiplication by f_{i+1} gives a map of complexes just a map of complexes and the $K.(f_{1,} \ldots, f_{i+1})$ is a mapping cone of the above map of complexes . So, if there are two and we will see in Macaulay, how these things are constructed?

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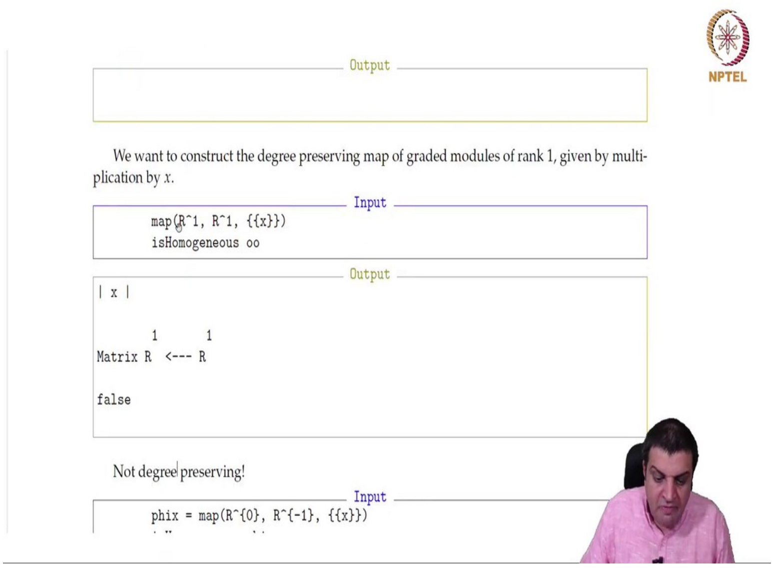

So, now we define a map from rank 1 R-module to rank 1 R-module multiplication by x and we ask isHomogeneous 00 and it says its false and the reason is both generators in both these modules basis elements have degree 0 and hence, multiplication by x would not preserve degree, it is just saying it is not a homogeneous map.

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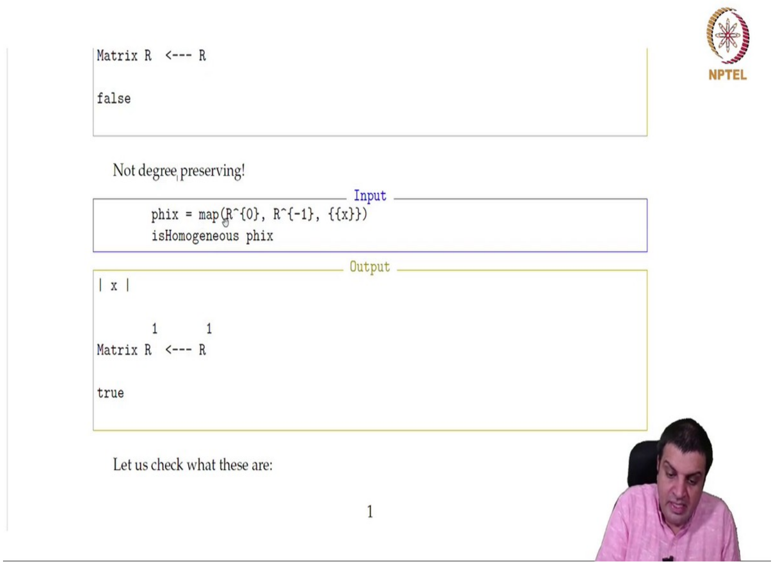

So, we can make it homogeneous by giving degrees like this. So, this is a rank 1 free module where a basis element has degree; where a basis element has degree 0, this basic this is R^{-1} so, basic there is a basis in which the element has degree, generator has degree 1 and multiplication by x now preserves degree.

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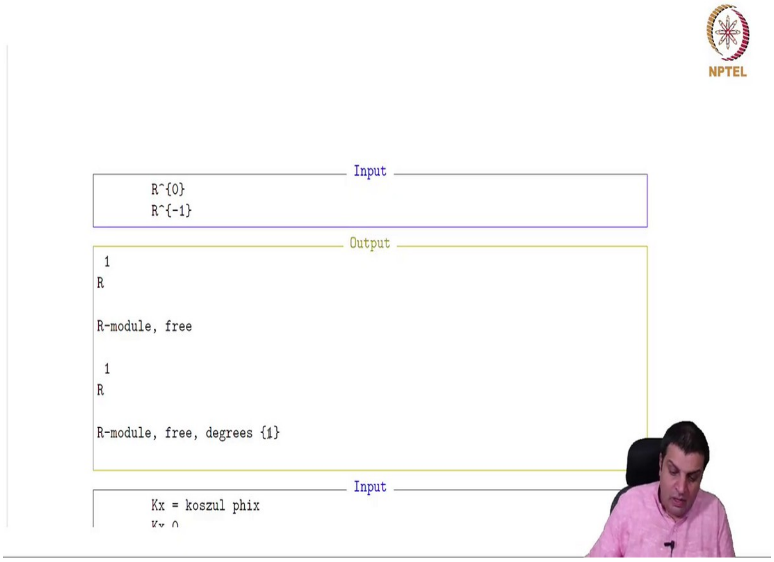

Yeah so, let us just ask what these things are. R^{0}, just a R-module free. R^{-1} is R-module free, degrees is 1, the generated degree is 1.

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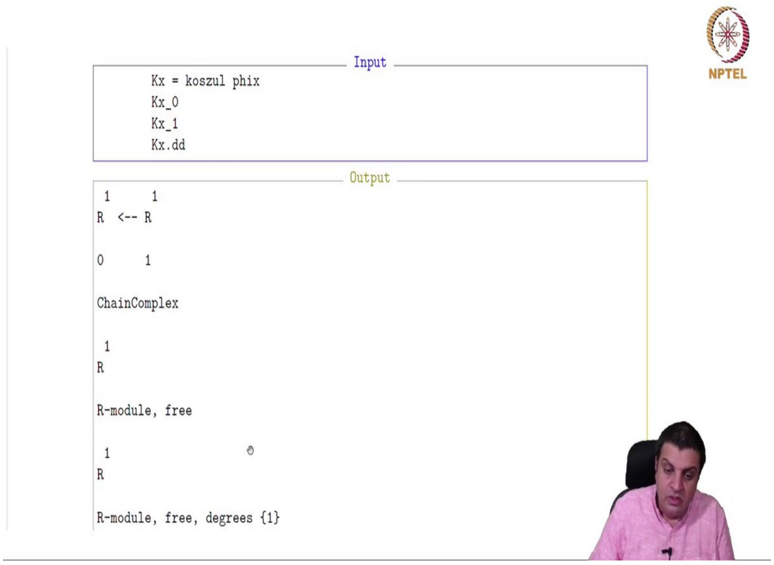

We ask Macaulay to construct the Koszul complex. So, Koszul complex is asked for a map. So, Koszul of phix was that multiplication by x. So, we call it Kx. Then, we ask so, this one these three commands will tell you the first element the module in the complex and the differentials in the complex, the maps in the complex

So, it says so, it showed us this complex 0 and 1 it says the chain complex. First module I mean 0th module here is R^1 free, second module is R^1 free, but the generated degree is 1.

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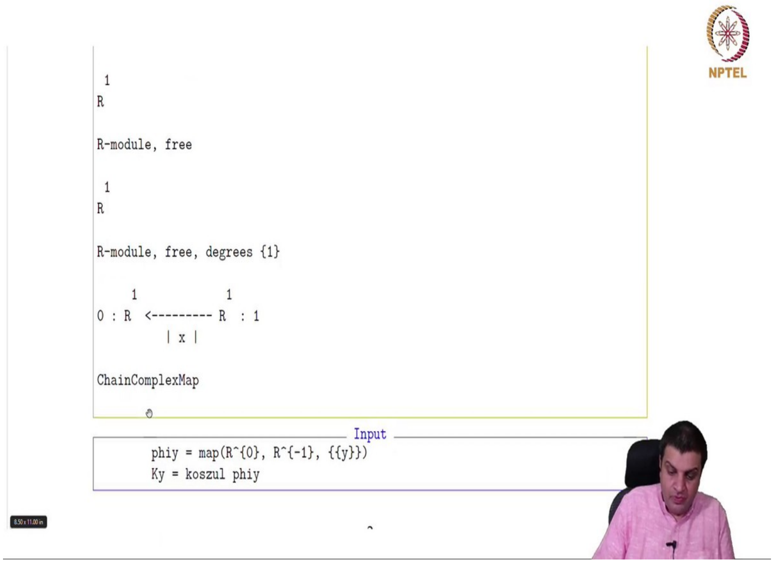

And if you ask for its map, it says this is the map. So, now we define similarly phiy and we ask for Koszul, we can ask for the Koszul complex.

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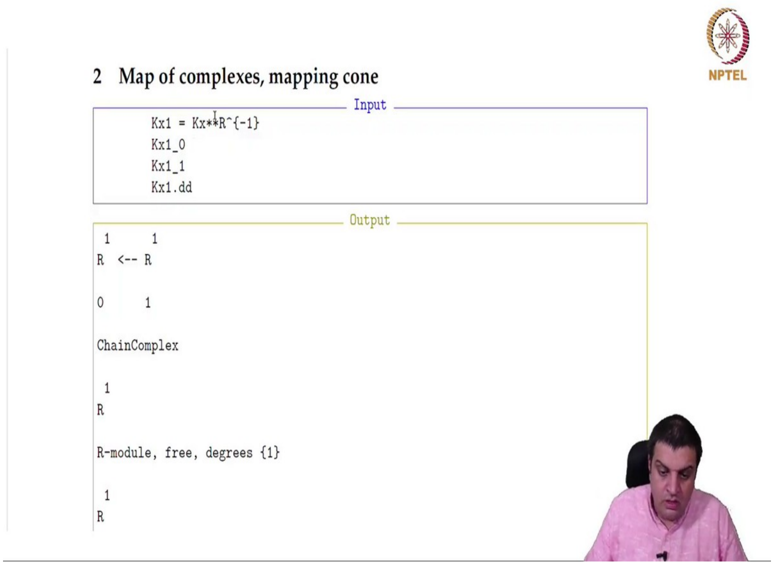

So, now, mapping cones. So, now, we need multiplication by y to be homogeneous. So, we cannot use Kx itself, we need to change its degrees. If multiplication by y has to be homogeneous, then it has to be, it has to come from R ^{-1} to rR^0.

So, we shift, we take Kx itself, but when we in order to be able to multiply by y, we shift the degree and call it. So, the Kx** R^{-1} it has a correct interpretation in terms of tensor products, but for us we will just treat it as shifting degrees.

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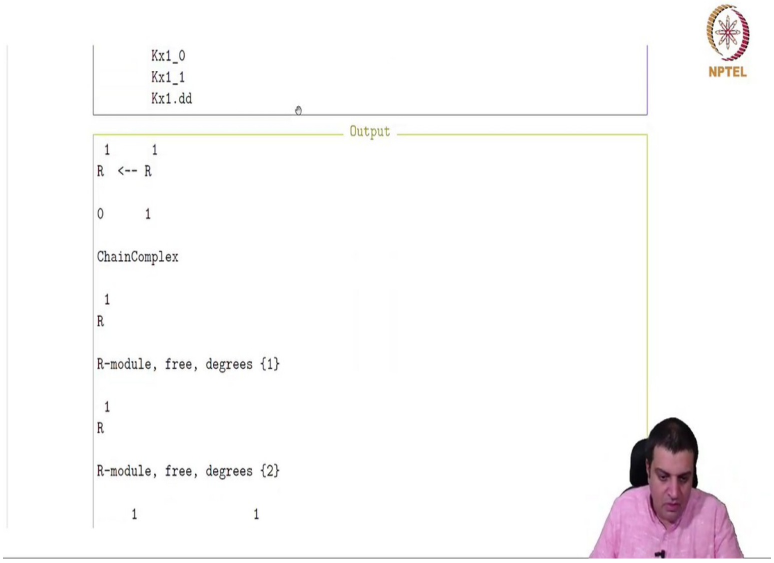

So, now we get. So, we asked for Kx1. So, the first the 0th module now is in degree 1 and the first module is in degree 2, generator is in degree 2.

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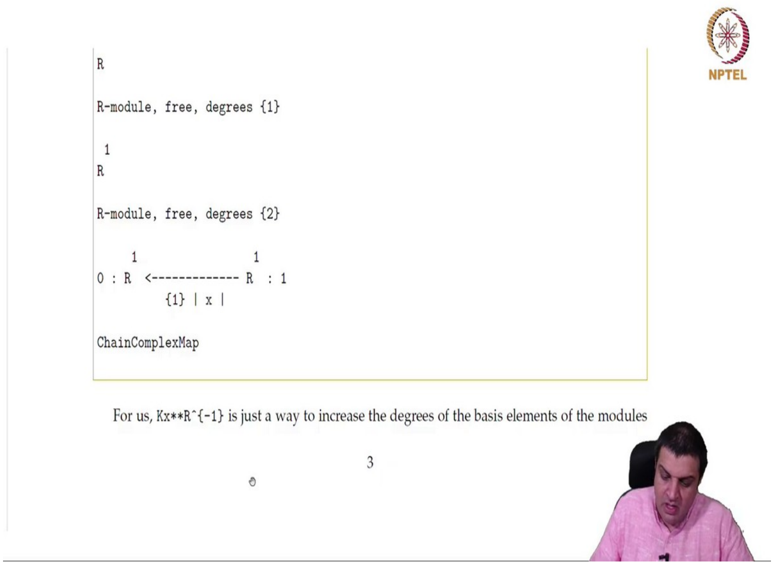

And the map is still multiplication by and as I said for us Kx ** R^{-1} is just a way to change the degrees of the basis element ok.

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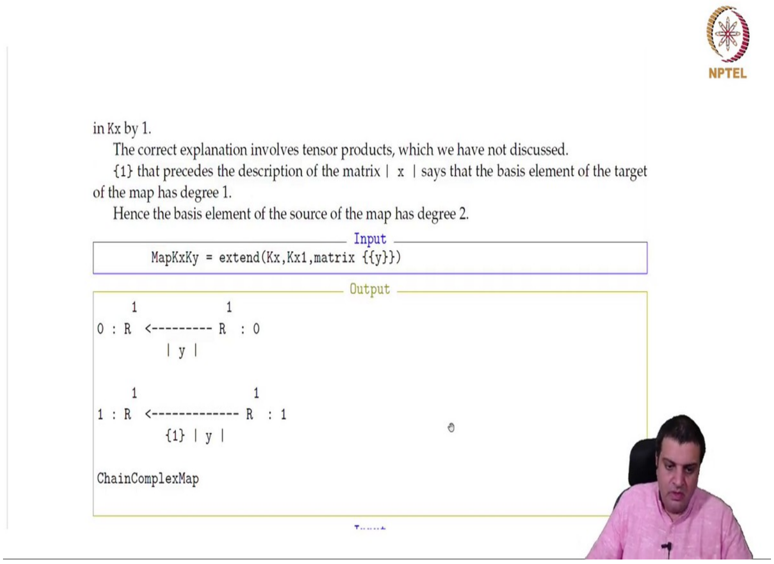

The 1 here that precedes x as the target has degree 1 so, the source has degree 2 yeah. So, that is just, I am just stating.

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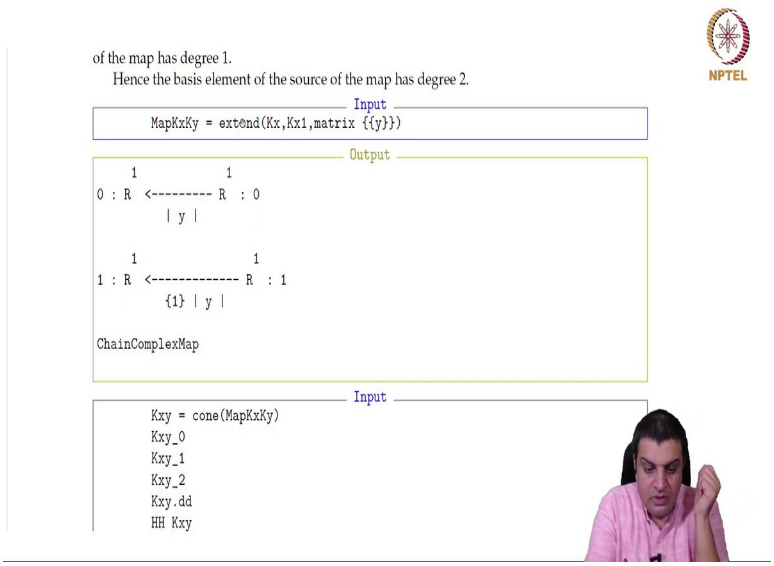

Now, let us so, now, this extend is a function which tries to extend a map ah, a matrix as a map of complexes. So, we want a map from $Kx1 \rightarrow Kx$ multiplication by y that is what this one does So, now, it is a map of complexes in both factors its multiplication by y and then, we ask Macaulay to construct its cone and we call the cone Kxy.

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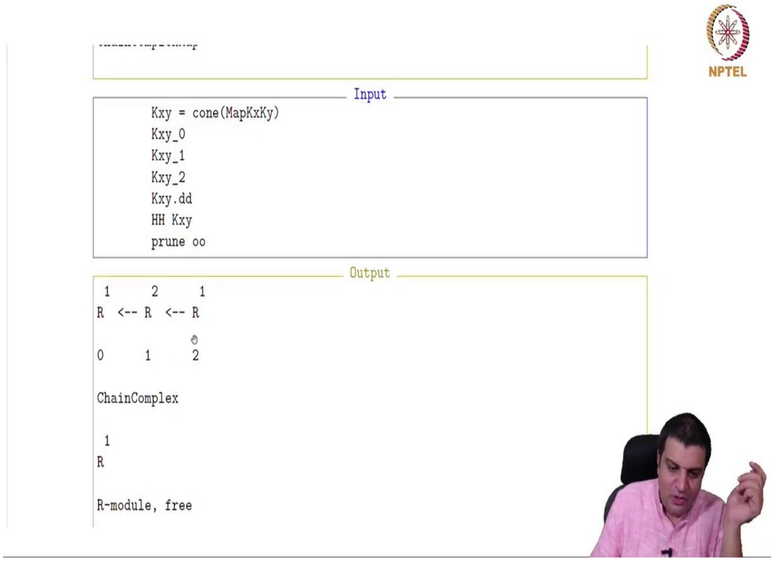

So, if you ask for Kxy now, its cone of 2 elements and you know one can inductively prove that the matrices in the I mean the modulus in the Koszul complex are free modules. 0th module is this 0, first module is 2 copies remember it is in the; it is in the diagonal there are 2 elements and after that both complexes end. So, there is only 1 in the next stage.

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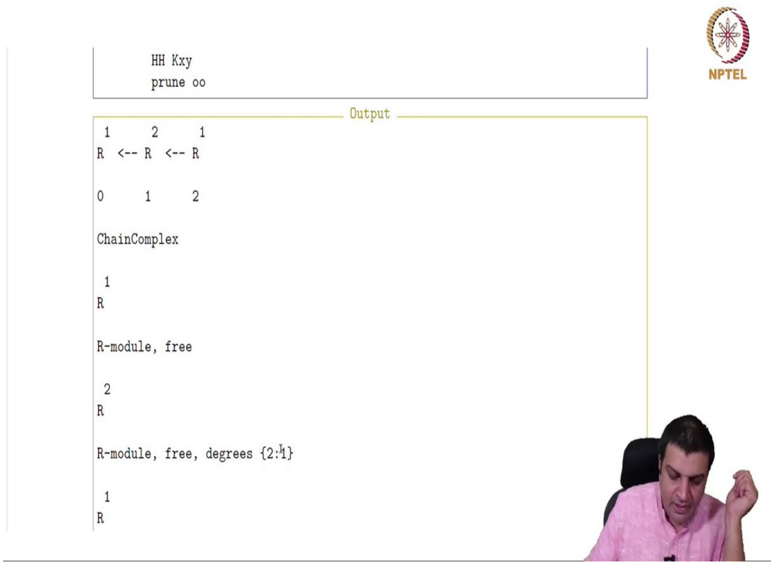

And so, here, it is degree 0, the first module is 2 : 1 which means 2 generators of degree 1.

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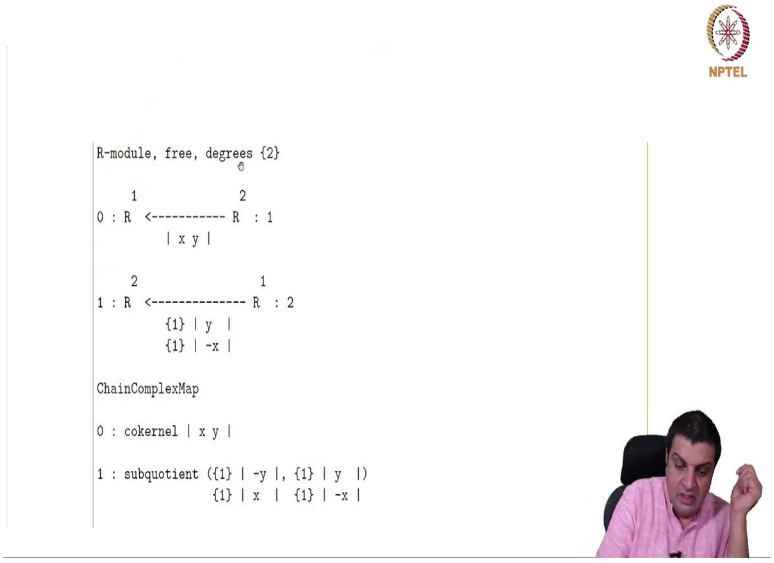

And the second one, the final one is degree 2 so, that is those are the commands and then, we ask for the maps and these are the maps. So, when we work out the chain complex maps using the mapping cone, one can see that these are the maps. We can ask so, we here we also ask for its homology and prune is to minimize the number of generators.

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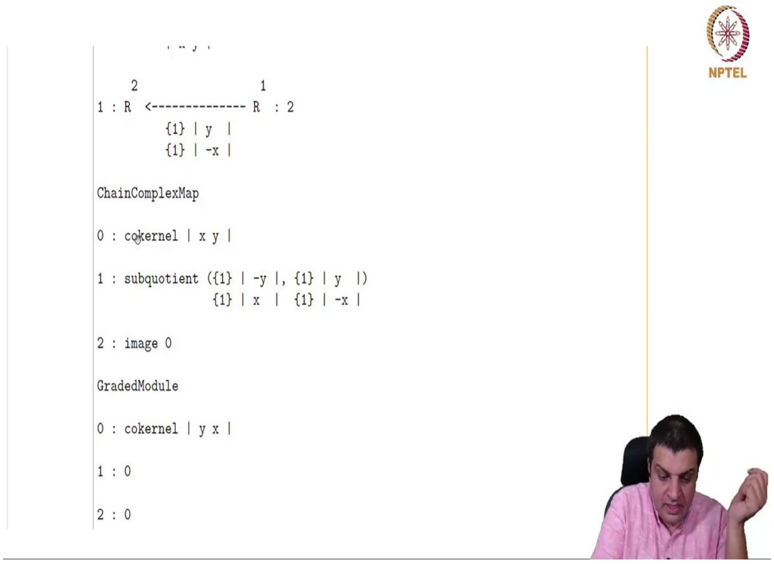

And so, when you ask for its homology, it said its a co kernel |x y|, the middle one it said it is a sub quotient of this so, in other words, it is the it is a submodule generated by this modulo the image of this one and when you ask for it to prune, this this the first homology is 0, so, one gets only 1 element and let us just quickly look at it for 3 elements.

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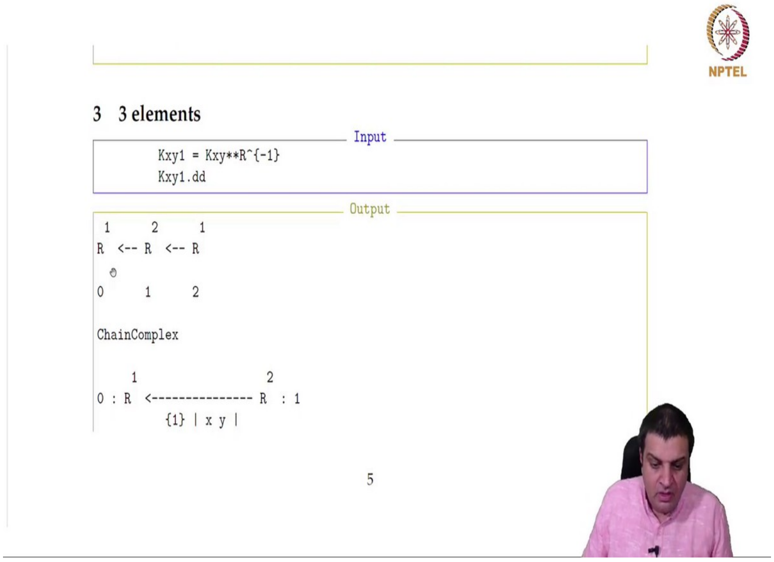

So, now, we do the same thing we take the complex Kxy and then, shift its degrees by 1 ok. So, all that we have done is just shift its degrees.

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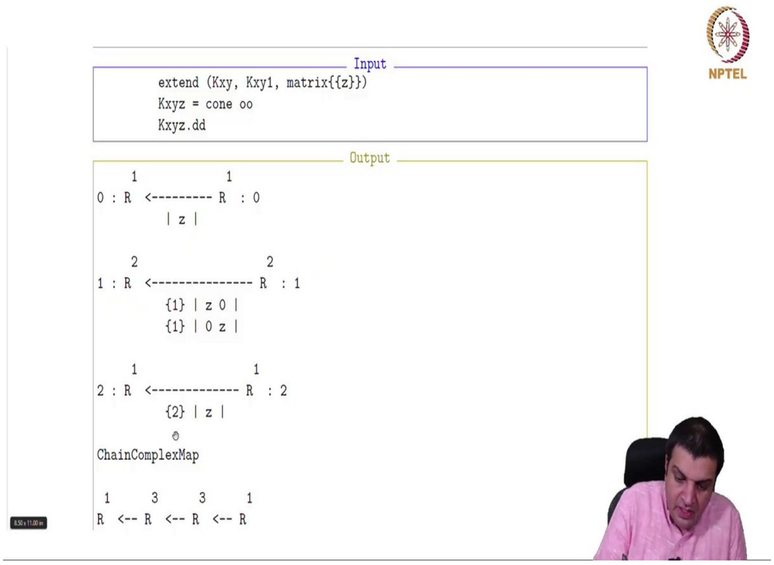

Now, take so, the new complex is called Kxy1. So, get a map from Kxy1 to Kxy multiplication by z, then ask for its cone and look at its look so, this is the; this is the extension. So, this is shifted and everywhere its multiplication by z. So, here there are two factors. So, it is a 2 by 2 matrix and then, we ask for its cone. So, we get something like this with 1, 3, 3, 1.

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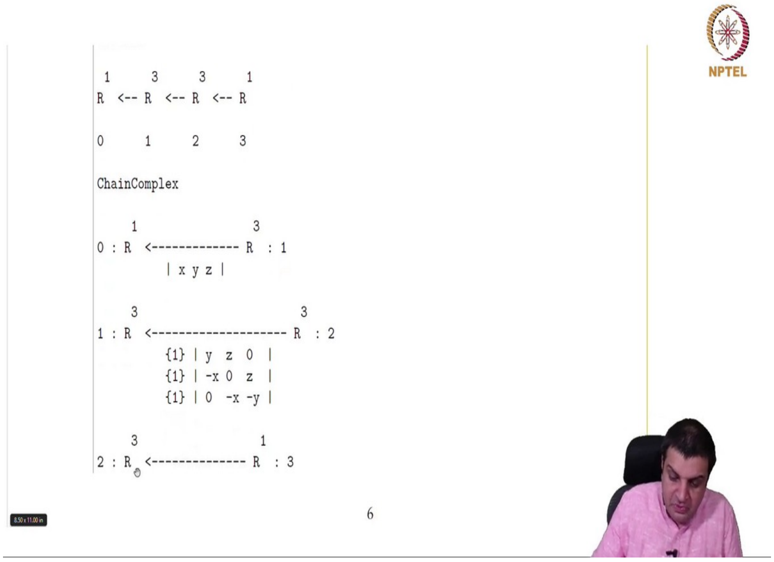

And we ask for its maps this is what we get. If you did this by hand, we would get I mean depending on the sign conventions, we might get a slightly different with different signs, but it would be isomorphic to this.

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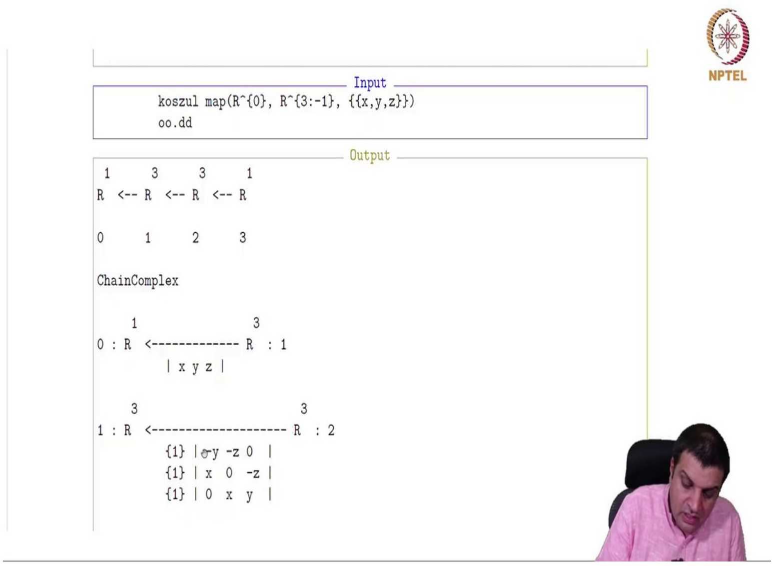

And so, we can ask. So, this we what we constructed by cone, we could have just asked Macaulay to construct the Koszul complex from the map to so, the generator here is in degree 0, 3 generators of degree -1 and they get mapped to x times this, y times this and z times this we ask for its chain complex. So, this this sign I mean it is a convention of how one is taking the tensor product. So, this matrix is almost the same as this matrix except for some changes in sign, but that is.

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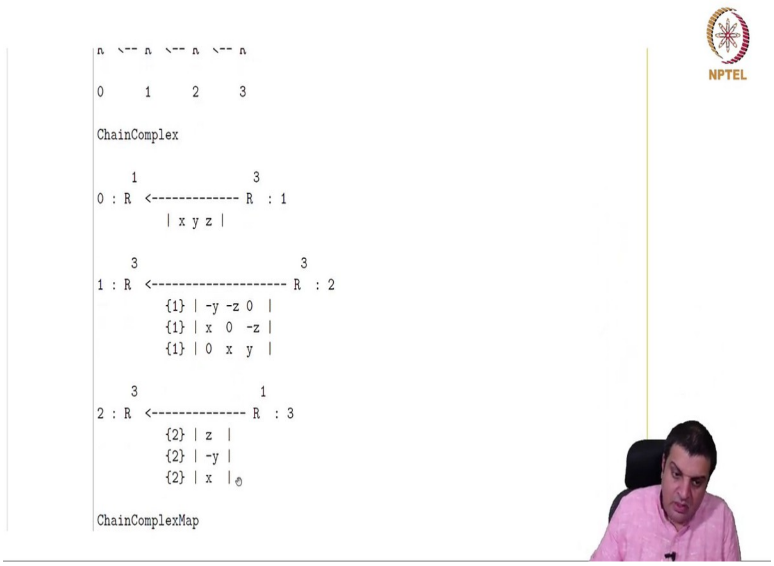

And then finally, one gets this. Yeah, that is the end of this lecture.