Computational Commutative Algebra
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Lecture – 55
Free resolutions

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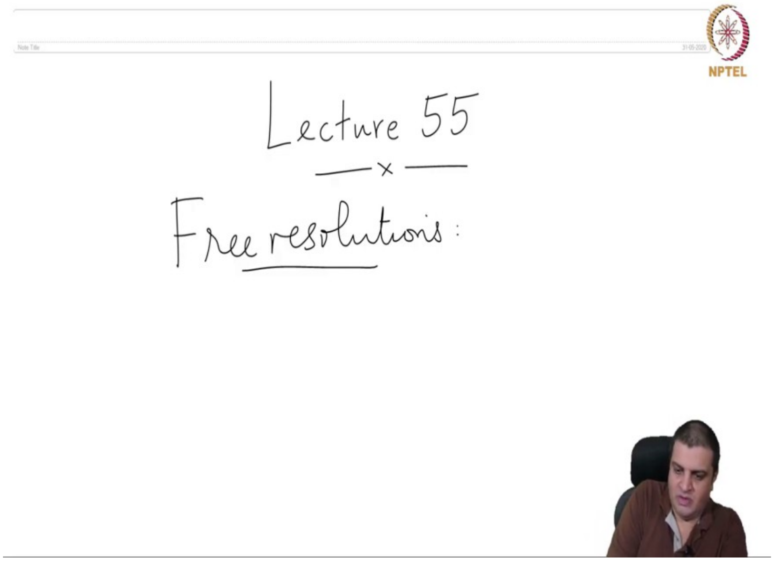

Welcome this is lecture 55. So, in the next this lecture and the next 5 we will look at syzygies called Free resolutions and what information can be cleaned out of computing free resolutions. Sorry some information that can be cleaned out of there are many other things that can be obtained by looking at a free resolution we will see an introduction to that.

So, we are discussing going to start free resolutions. So, we will not be able to prove many things properly, because to discuss these things one needs to spend some time developing basic notions about complexes and homological algebra which we have not done and we do not have the time now to do this. So, there will be a few results that we will assume and then we will see how to use them or how what.

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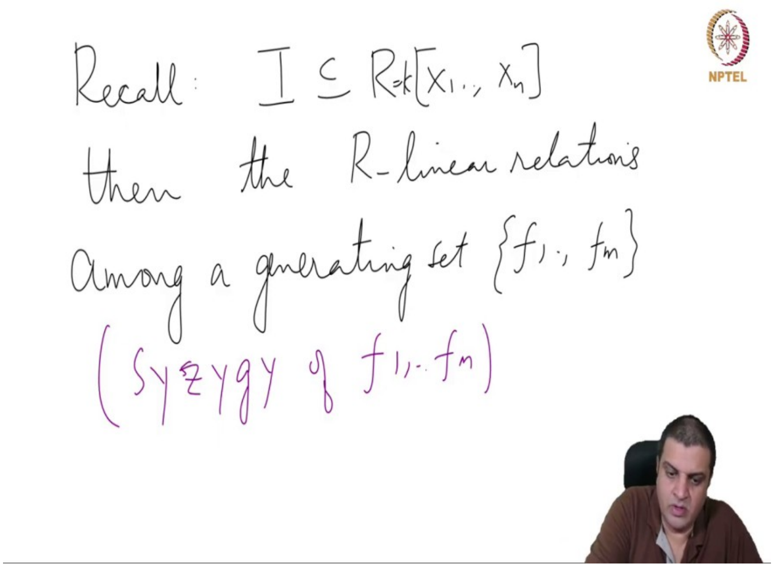

So, recall, so I have not yet defined this, but let us so recall that. So, if you have an ideal I inside a polynomial ring $R=k[X_1,..,X_n]$. Then when we only looked at this case but this is true more generally, then the relations among the R linear relations among a generating set let us say $\{f_1,\ldots,f_m\}$. So, this is called syzygy of this is what we looked at in the last lecture syzygy of $f_1,\ldots,f_m$.

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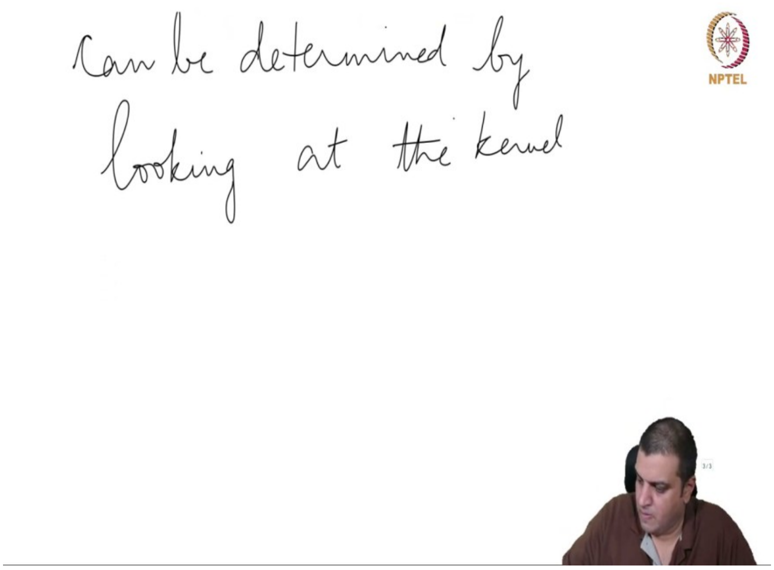

Can be determined by looking at the kernel.

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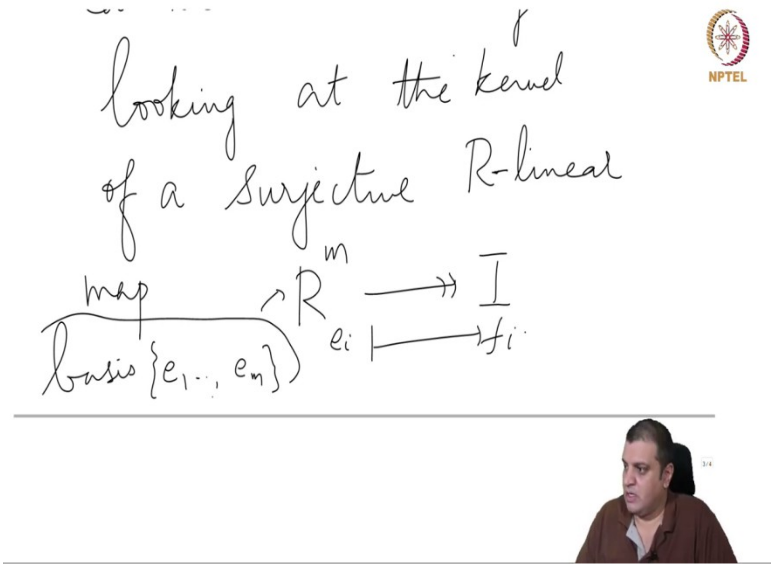

Can be determined by looking at the kernel of a surjective map R linear map $R^m \to I$, a basis element here goes to f_i. So, this is $\{e_1, \ldots, e_m\}$ is a basis. So, this is the basis for this module and if we map e_i to f_i then the syzygies of f_i are precisely the elements of this kernel.

So, this is what we saw in the last lecture. But notice that this is the same for modules also, because all that we are using here is an R linear map from a free module to I we are not using the fact that this is an ideal or that we are using is that it is an R module.

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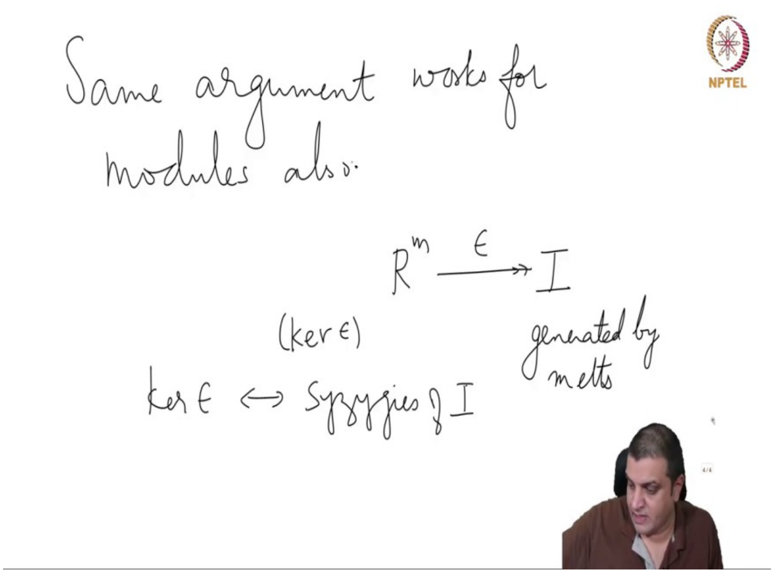

So, same argument works for modules also, again we will only be concerned about finitely generated modules that is the I mean much better theorems in that context than otherwise. So, how would we and why do we have to work understand do this for modules also?

So, one of the things that we would want to do is so given even if you are given an ideal let us say generated by m elements, we can get a free module of rank m to surject onto this would give the kernel of this map. So, let us call this map ϵ, kernel of ϵ is actually gives the syzygies of I of really the generating set, but we will see that it is actually I.

Now, so let us write that thing here I will write it slightly below here for a purpose. So now this is also finitely generated, but it is no more an ideal.

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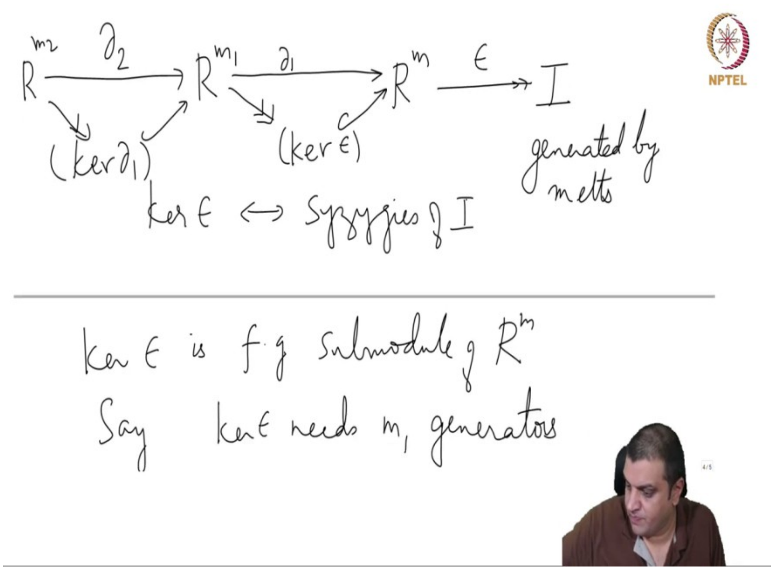

So, kernel of ϵ is finitely generated sub module of R^m and typically m has to be greater than 1. So, this is not an ideal at all. But we can repeat the same procedure that is say kernel of ϵ needs m_1 generators.

So, then we take a free module of rank m_1 map it surjectively on to kernel of ϵ and kernel of ϵ of course it is inside here injectively and the composite map we call it let us call it ∂_1 So, what have we done? We have constructed a presentation of I this we have done earlier.

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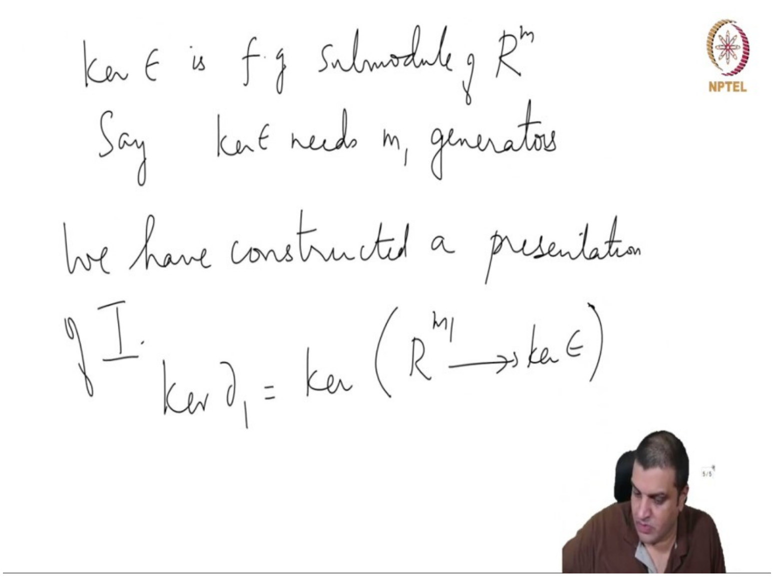

So, we have constructed a presentation of I, where we talked about finitely presented and all those things. So, this we have seen this earlier. Now, we can repeat this is a surjective map it is image here and δ is a composite. So, the kernel of the surjection from $R^{m_1} \rightarrow ker\,\epsilon$ is same as a kernel of δ_1.

So, kernel ∂_1 is kernel of the map from R^{m_1} to kernel of ϵ that is the way δ was defined. So, therefore, we can now consider here kernel of ∂_1 and then some R^{m_2} surjecting on to this and this is anyway. So, we what we are doing is we are getting a presentation matrix for kernel of epsilon using this map not just some.

Because we are using this map in constructing this, we are looking at the kernel of either of these maps ∂_1 or the subjective map onto kernel of ϵ. That we are using to get a presentation matrix here and so then let us call this thing ∂_2 and this we can repeat.

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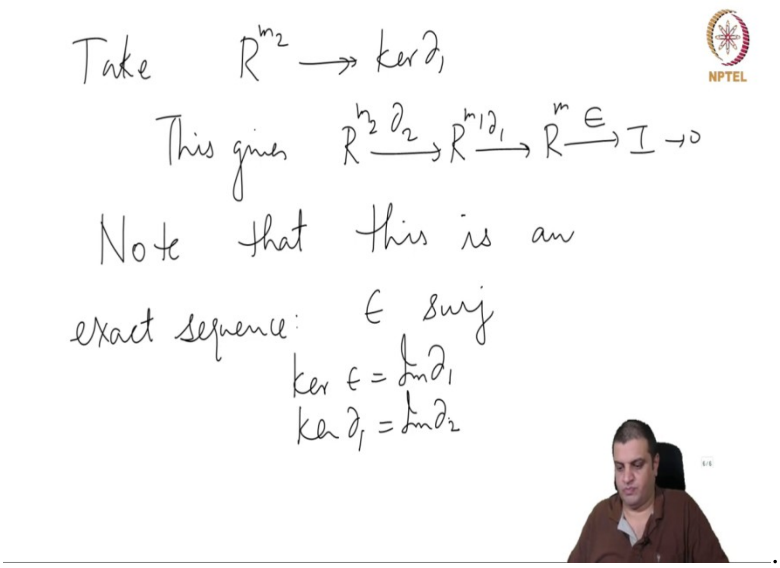

So, then take R^{m_2} surjecting onto kernel of ∂_1 this gives R^{m_2} to R^{m_1} which we called ∂_2. The map from R^m to I is ϵ, R^{m_1} to R^m is ∂_1 and R^{m_2} to R^{m_1} is ∂_2. So, note that this is an exact sequence.

So, what does what do we have? We have ϵ is surjective, kernel ϵ is image of ∂_1 and kernel of ∂_1 is image of ∂_2 and in fact we can continue this.

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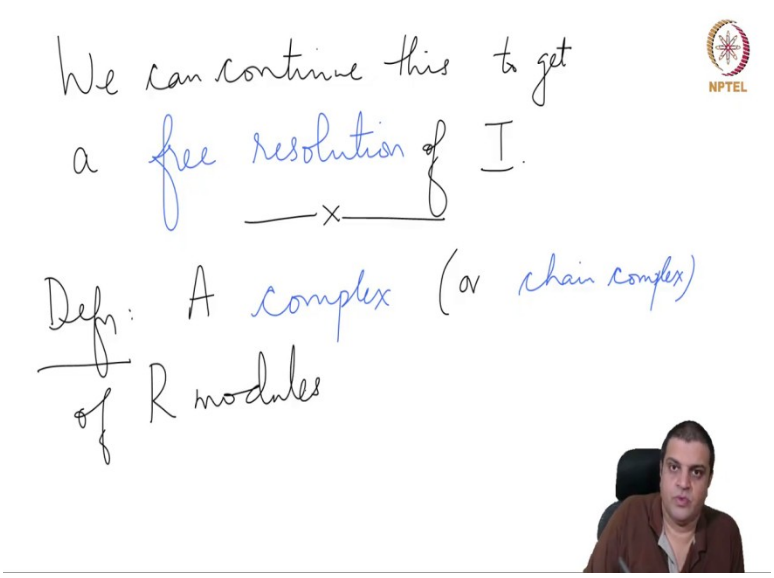

So, we can continue this to get what is called a free resolution of I and we could have done this for a and notice that in this process we had to learn to present not just ideals, but also modules. So, we may start with an ideal, but immediately after that we have to start worrying about modules.

So, we may as we will learn about modules and as we mentioned earlier nothing that this is an ideal is used all that we have used is that this is in R module. So, let us make a definition. So, before we define we need to, not for the purpose of this course, but in general when we want to do this we need to get we need to discuss something more general than exact sequences.

So, definitions this is only for us this is just a language that is all, we will not going to study this or worry about these issues in this course. So but let us wherever we start we understand what this means the words mean. Complex I mean technically a chain complex, but we will just always refer to as a complex or more precisely chain complex of R modules. It is a sequence of R modules and R linear maps.

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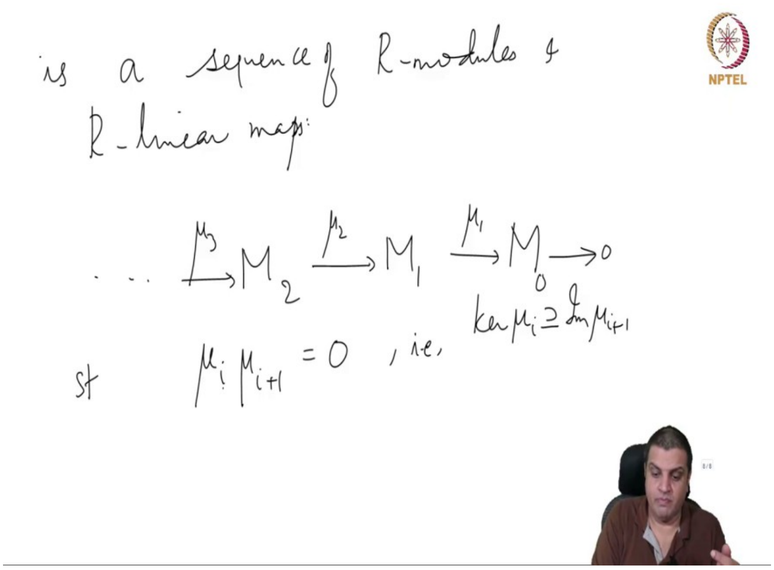

So, again we will only consider special cases. So, we will assume that the complex ends somewhere on the right of the this kind. So, these are all linear maps and such that, so let us call these maps let us say μ_2 going from M_2 to M_1; μ_1 going from M_1 to M_0 and so on.

So, μ_3 here such that $\mu_i \mu_{i+1} = 0$. So, that is kernel of μ_i contains the image of μ_{i+1}, notice this is how the composition will work. We apply μ_{i+1} first and then μ_i this is a composite map from M_{i+1} to M_{i-1}.

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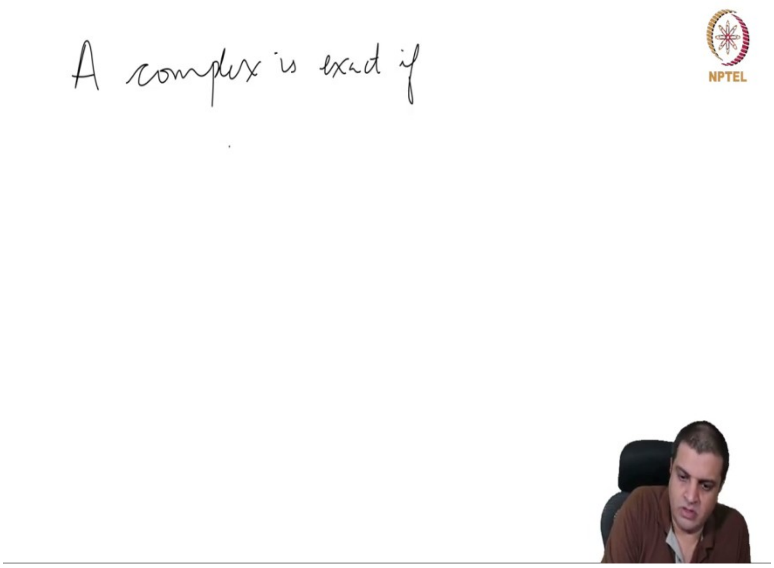

So, this is what is called a complex.

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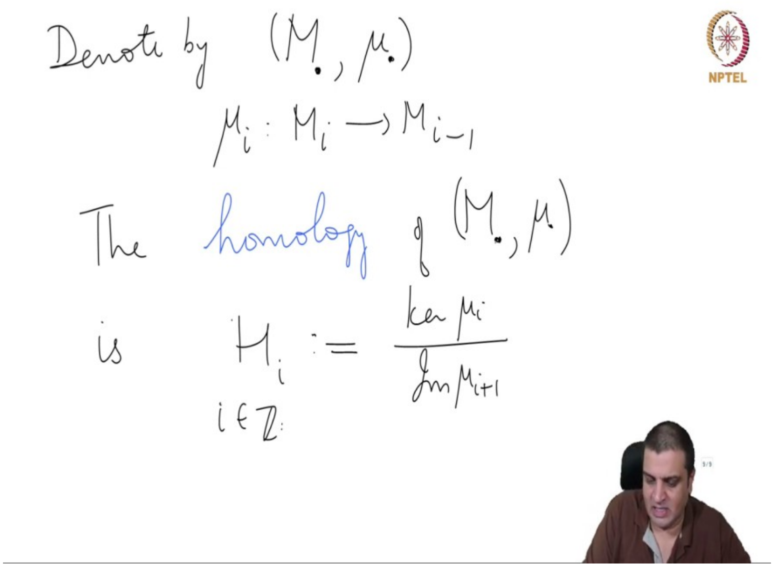

So, we will denote this by $(M_{\circ}\cdot, \mu_{.})$ and if we need to specify the maps sometimes we would not need we will also just like this. If we need otherwise we will just say M and remember that our convention is that $\mu_i$is the map from M_i to M_{i-1} and these are R linear maps. So, then the homology of M, again it is it depends on the maps not just the modules is H_i and this is for all i; is $H_i = \frac{Ker\,\mu_i}{\Im\,\mu_{i+1}}$.

Kernel of μ_i contains the image of μ_{i+1}, so the quotient is what we are interested in and this is for $i \in Z$, but often the modules that the complexes that we are concerned will be 0 after a while. It may not be 0 at M_0 itself, it could be like some M_{-500} but it would still be 0 after a while and so this is ok, but nonetheless we can still define it for all Z.

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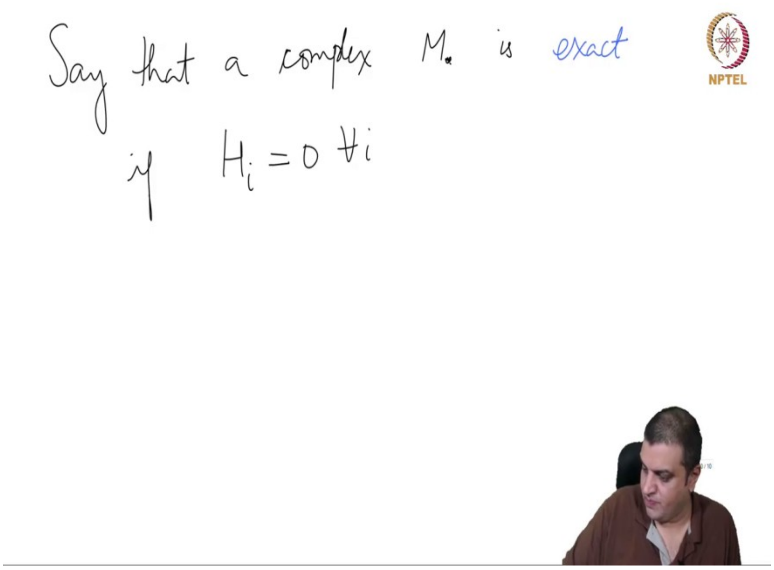

So, then say that a complex $M_{\circ \cdot}$ is exact if all the homologies are 0. So for example, so then we can make the definition of a free resolution.

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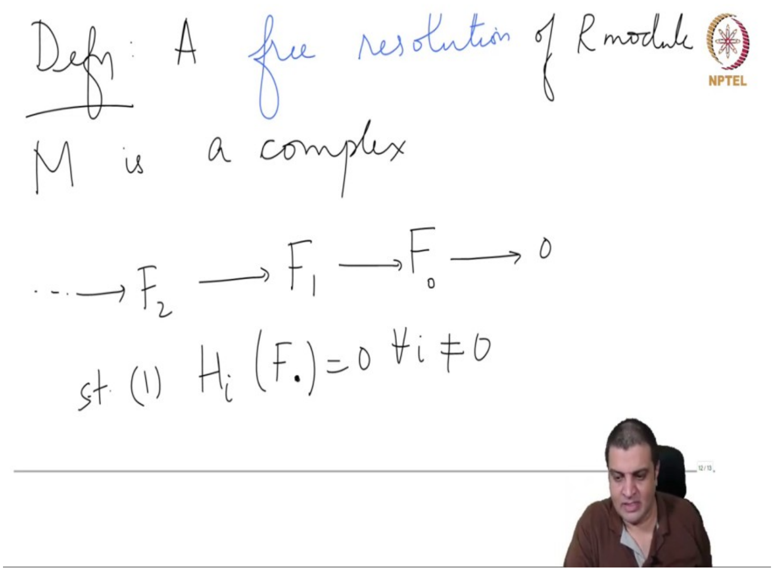

So, definition I mean we can make a proper definition of free resolution, free resolution of an R module M is a complex. $\ldots \rightarrow F_2 \rightarrow F_1 \rightarrow F_0 \rightarrow 0$ So, this will end at some F_0 and it may go infinite to this side, but we usually put this thing here I mean ok.

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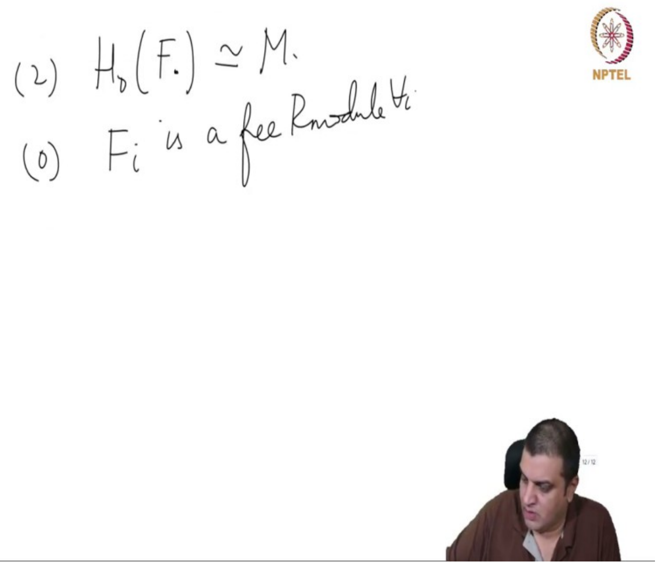

Such that $H_i(F_{\circ}.) = 0$ for all $i \neq 0$ and $H_0(F.) = M$ sorry that is not enough that does not make it a free resolution such that. And F_i is a free R module for all i.

So, that should have been if you have stated that first F_i is a free R module for all i that is why it is called a free resolution. And at in this complex at these places there is no homology, there is homology here which is isomorphic to M.

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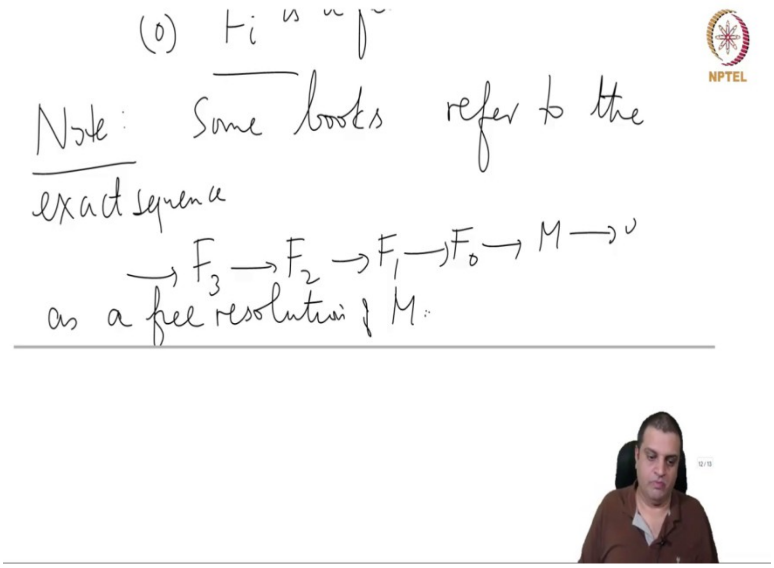

So, just a note. So, some books might I mean not might they some books refer to the. So, this F as we defined is not exact it has exactly one it is not exact. It has one nonzero homology and that is isomorphic to M and it is at the right most end.

So, this is what how our convention is this is what we will say some books refer to the exact sequence. So, we put everything as about $\ldots \rightarrow F_2 \rightarrow F_1 \rightarrow F_0 \rightarrow 0$, now what does it mean to say that the 0-th homology? It is M.

So, it means that is isomorphic to M, it means that the 0-th homology here is everything goes to 0 here this is the 0 map and the image of this map is the image of this. So, H_0 is the co kernel of this last map $F_1 \rightarrow F_0$.

So then if you write M here exact to $M \rightarrow 0$; it becomes exact as a free resolution of M. So, we will of course we will refer to this condition as free resolution; some books if you read it might actually write this as a free resolution. I mean, but it would be clear from the context whether the book is using this convention or the other convention. So, this is just ok. So, now, we want to define what is called a minimal free resolution.

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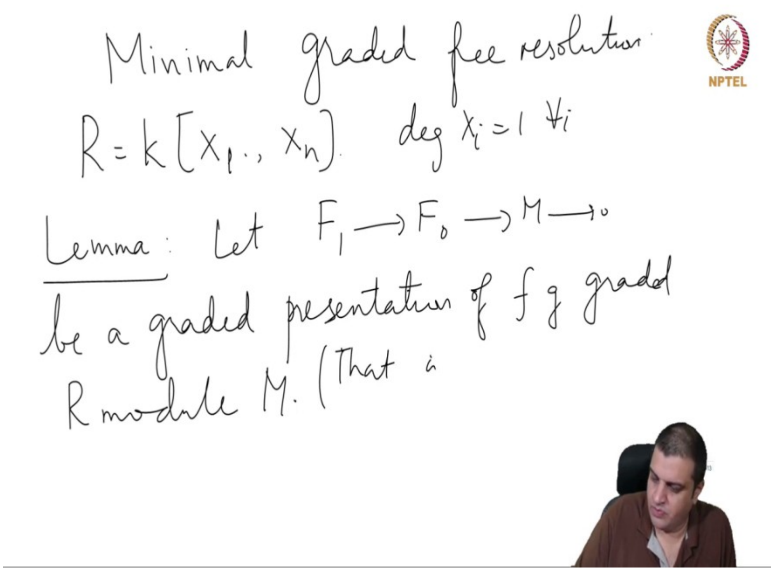

And so far whatever I have said just works for any Noetherian ring and finitely generated module. In fact, we have given use finitely generated anyway. So, this is some large generality, but now we so graded it.

So, two things that we want to emphasize minimal graded free resolution. So, now we will restrict ourselves to a graded ring, I mean we will restrict ourselves to the polynomial ring, but some most of it will again work over rings in which graded rings in which the 0-th piece is a field and finitely many generators all of which have positive degree.

So it will, so but we will restrict ourselves to the specific situation. So, here is a Lemma. So, let $\ldots \to F_2 \to F_1 \to F_0 \to M \to 0$ be a graded presentation of finitely generated graded R module M. So, we are given a finitely generated graded R module and we take a graded presentation.

So, we take a homogeneous generating set for M and then we map that many generators onto M, look at the kernel then we take homogeneous generators for the kernel and then correspondingly build F_1. So, these things are the free modules are graded and the maps preserve degrees. So, that is all the two non maps here this is anyway 0 so ok.

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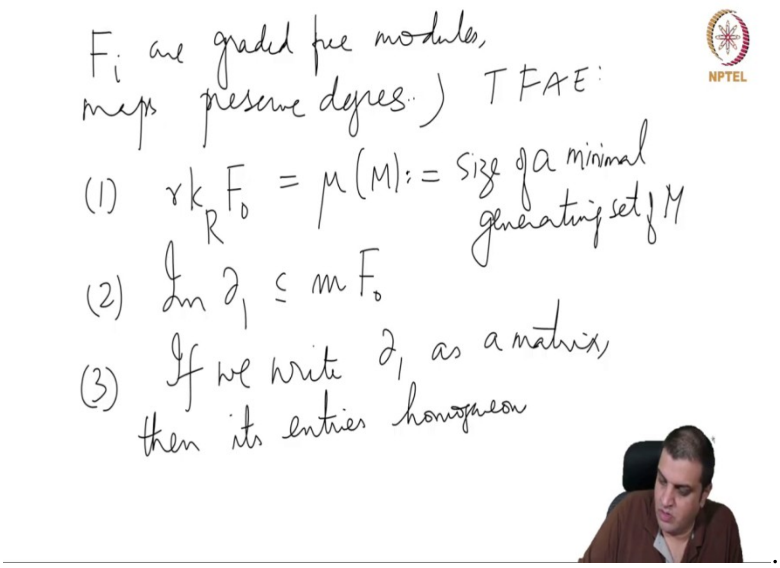

So, this means that is F_i are graded free modules and maps preserve degrees. This is what we mean by a graded presentation then the following are equivalent. 1, this is a free module. Rank of $F_0 = \mu(M)$ which is the minimum number of generators for M. The size of a minimal generating set of M it is it is finite. So, this is the definition.

So, this is the first hypothesis $rk_R F_0 = \mu(M)$. 2 let us call this map ∂_1 the presentation map; $\Im \partial_1 \subseteq m F_0$ and 3 if we write ∂_1 as a matrix, then it is entries are homogeneous elements of the maximal ideal. In other words it does not contain any units.

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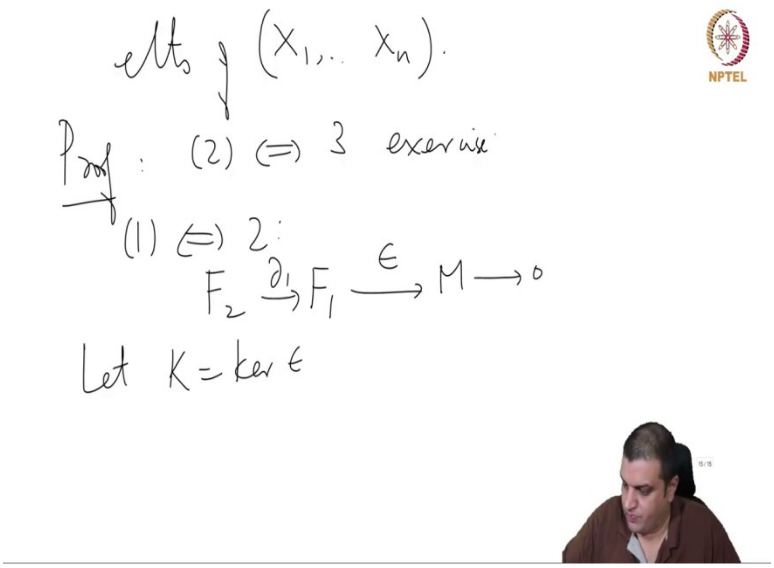

So, 2 if and only 3 I leave as an exercise it is just rewriting what means to say that I mean what it just trying to understand what it means to say a map lands inside $m F_0$.

So, proof 2 if and only if 3 exercise and so 1 if and only if 2. So, let k be the so sorry let us label this map also. So, let us us just rewrite the map. So, the map is $F_2 \to F_1$is $\partial_1$and this map let us call ϵ. So, let Kbe the kernel of ϵ. So, then 2 let K be the kernel of ϵ.

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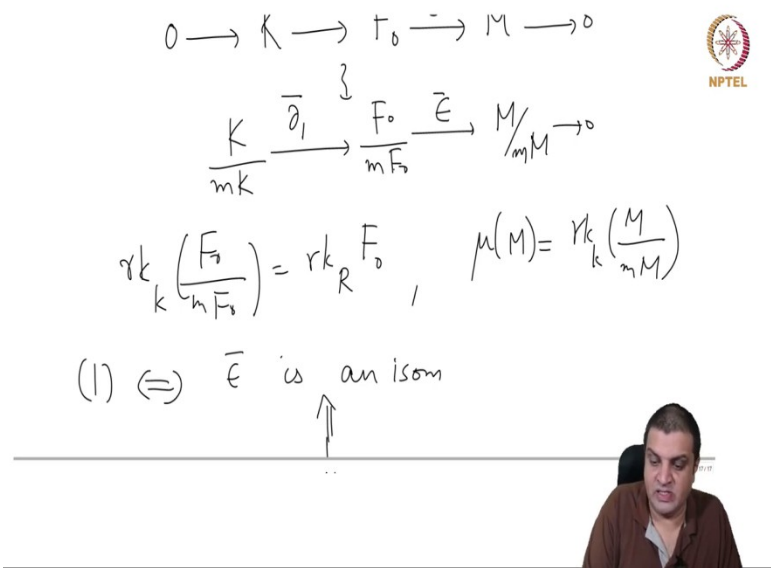

So, then we have a map an exact sequence $0 \to K \to F_0 \to M \to 0$ this is the map epsilon, this gives $\frac{M}{mM}$ this was we have seen this earlier. If $\frac{F_0}{m F_0}$ and then $\frac{K}{mK}$ this may not be injective, but we get such a map,. So, we cannot fill this back inside here, but so much of this is exact.

But the hypothesis is that so now what is the hypothesis? Rank before we get into the hypothesis $r k_k\left(\frac{F_0}{m F_0}\right)=r k_R$. If it is a free module then just use the same basis you will get the quotient module and $\mu(M)=r k_k\left(\frac{M}{mM}\right)$ by Nakyamas Lemma. So, the hypothesis is that these two numbers are the same in other words we are saying that this map is I mean this is an R linear map it is also $\frac{R}{m}$ linear map.

So, what we are saying is. So, let us call these things let us just call that map $\bar{\epsilon}$. So, then what we are saying is that 1 if and only if. So, 1 says these 2 numbers are the same, so 1 if and only if $\bar{\epsilon}$ is an isomorphism. So, that is one place, so well we have already used for this statement we have already used Nakayamas lemma, so M is finitely generated. So, we have already used it, but here is another place where it is used that these are finite dimensional vector space.

So, if there is a surjective map, then it must also be injective, finite dimensional vector space of the same rank so surjective map must also be injective. So therefore, this is an isomorphism.

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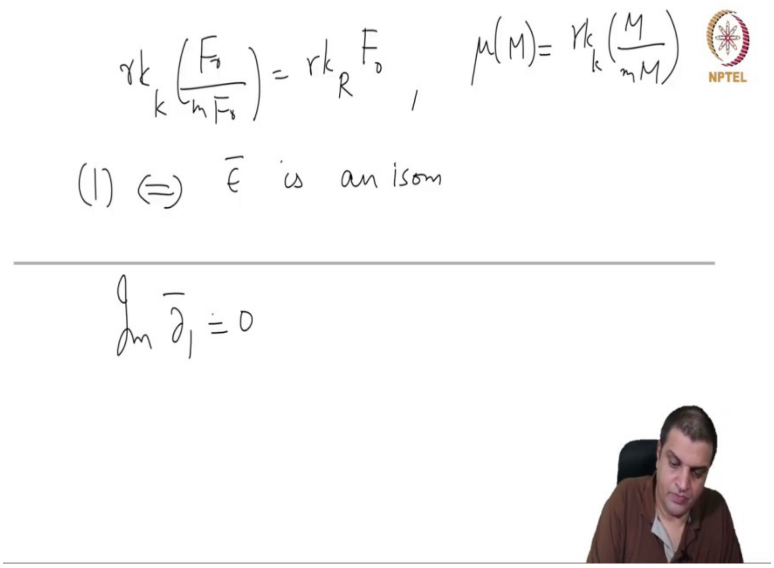

So, in other words the image of this map. So, let us call this thing $\overline{\delta_1}$ image of $\overline{\delta_1}$ is 0, so this is an isomorphism.

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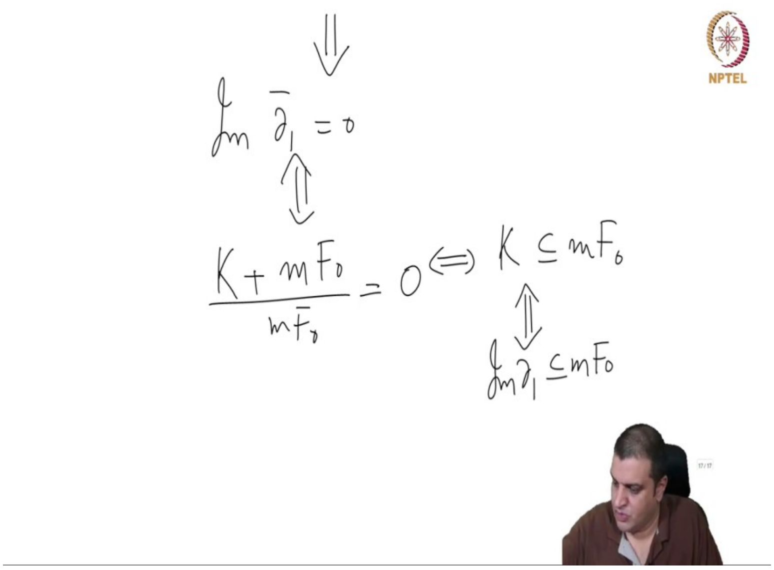

If and only if image of $\overline{\delta_1}$ is 0, but what does that mean? Image of $\overline{\delta_1}$ is well it is just image of k inside the quotient. So, it is just $\frac{K+mF_0}{mF_0}$. So, this is true if and only if $\frac{K+mF_0}{mF_0}=0$, but that is the same as k is inside mF_0. But which is what we wanted to prove. So, this is, but remember δ_1 is a surjective map onto k, so this means that image of δ_1 is inside mF_0 so this is the proof.

So therefore if we if you construct a free resolution picking minimal number of generators at every stage we would get a minimal free resolution.

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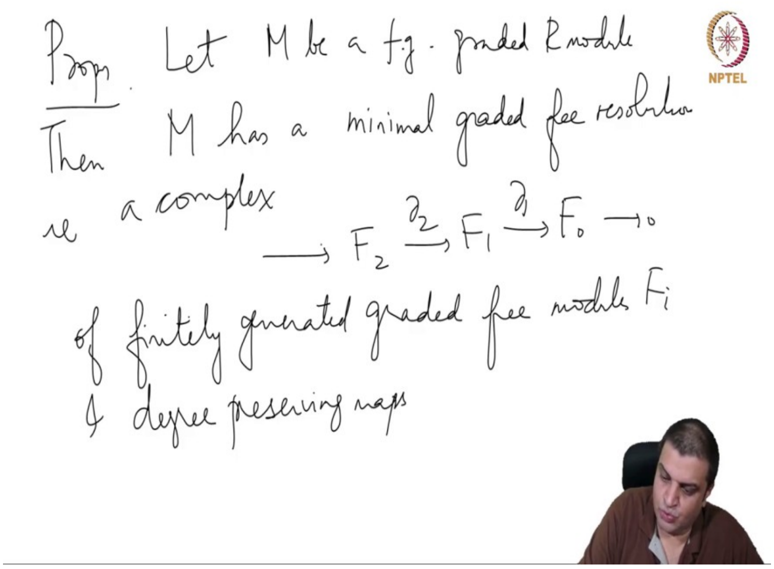

So, then let us summarize these things in a proposition. Let M be a finitely generated graded R module, then M has a minimal graded free resolution and what does that mean? It means $\ldots \rightarrow F_2 \rightarrow F_1 \rightarrow F_0 \rightarrow 0$ that is so what is that is a complex. This such that a complex of finitely generated graded free modules F_i and degree preserving maps such that ok.

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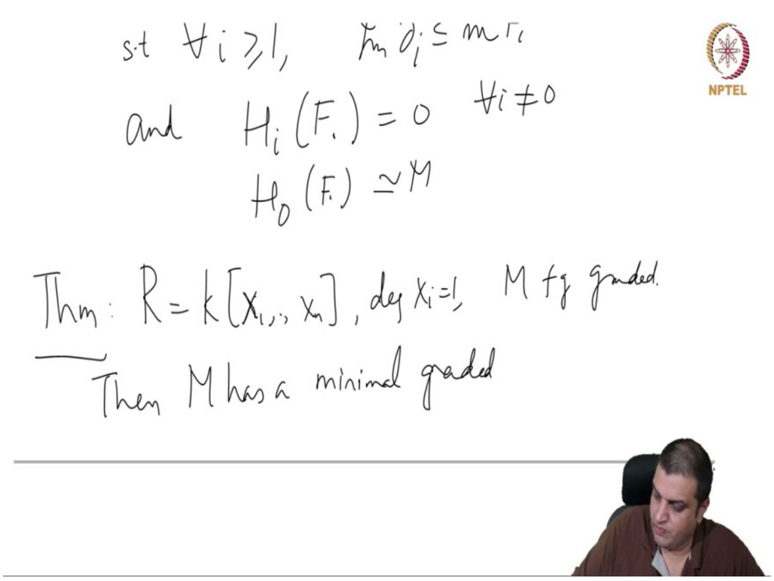

So, let us also we also need to label the maps ∂_1, ∂_2 and so on, such that for all $i \geq 1$ i image of $\partial_i \subseteq m F_i$.

So, this is the minimal part and $H_i(F.) = 0$. So, we defined it as a complex is 0 for all i different from 0 and $H_0(F.) \cong M$. So, this is just saying it is the free resolution.

So, all the ranks of these are finite the maps are minimal in this sense and there is no homology all the way till here and the homology here homology there is no homology in these red parts and the homology in the green part is M ok. So, such a thing exists. So, this is what we want.

And so we want to define 2 things, so first is a theorem we will not prove. So now we do want R to be in so far this works even in the generality of graded rings you know finitely generated algebras over a field. Not necessarily polynomial rings graded degree of X_i equals 1, M finitely generated graded.

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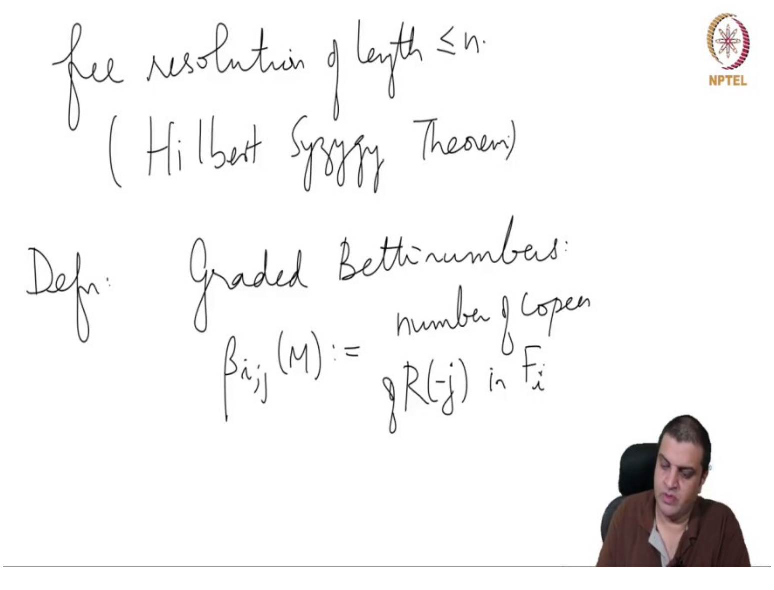

Then M has a minimal graded free resolution of length at most the number of variables I mean the rank of n the dimension of R.

So, this is and we will not so this is called Hilbert syzygy theorem. So, in the next lecture we will go back to computational ideas behind compute finding resolutions and there and then we will try to sketch a proof of how one could prove Hilbert syzygy theorem. Although this is typically proved using some after first course you know I mean after some understanding of homological algebra not in a computational way.

I mean although Hilbert's original proof itself was non homological, because homological algebra was not understood at that time. So, just one more definition and then we will look at we define the graded Betti numbers $\beta_{i,j}(M)$ to be the number of copies of $R(-j)$ this is the rank 1 free module with a generator of degree j. So, number of copies of in F_i in a minimal graded free resolution and you might wonder this there are so many choices made in the construction earlier. So, why is this even well defined a fact is that.

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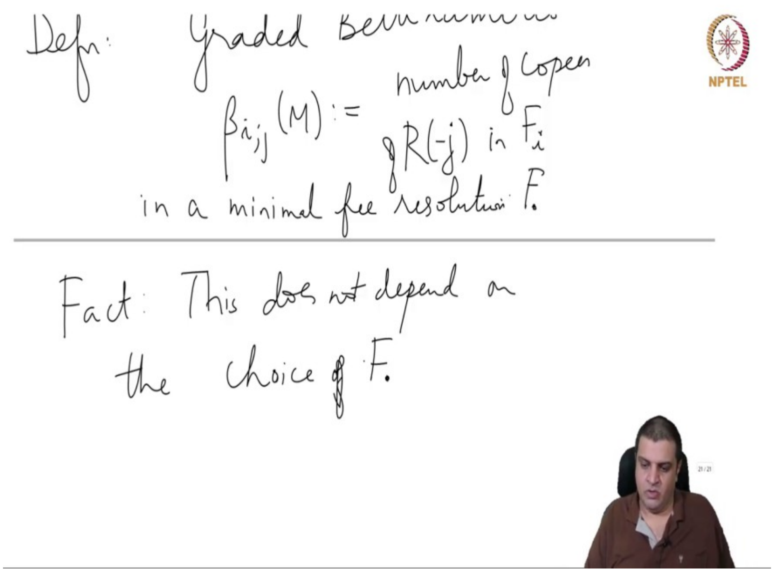

And the fact is that this does not depend on the choice of minimal graded free resolution $F.$, choice of F again we cannot we are not in a position to prove it, but we will use this thing for now.

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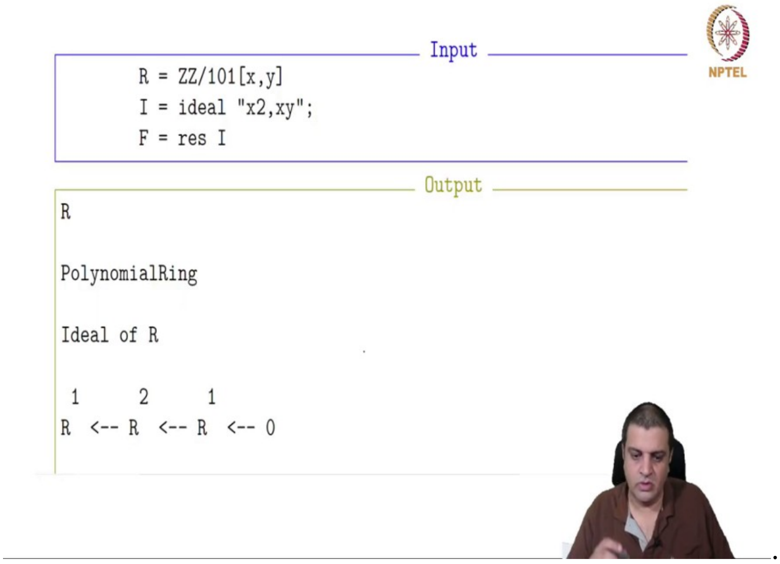

So, now let us quickly look at a Macaulay example. So, here is a polynomial ring in 2 variables x and y, ideal we want to take (x^2, xy), so then we asked. So, then we asked the command is res which is an abbreviation for resolution we asked for the resolution of I. So, note that so it produces something called chain complex.

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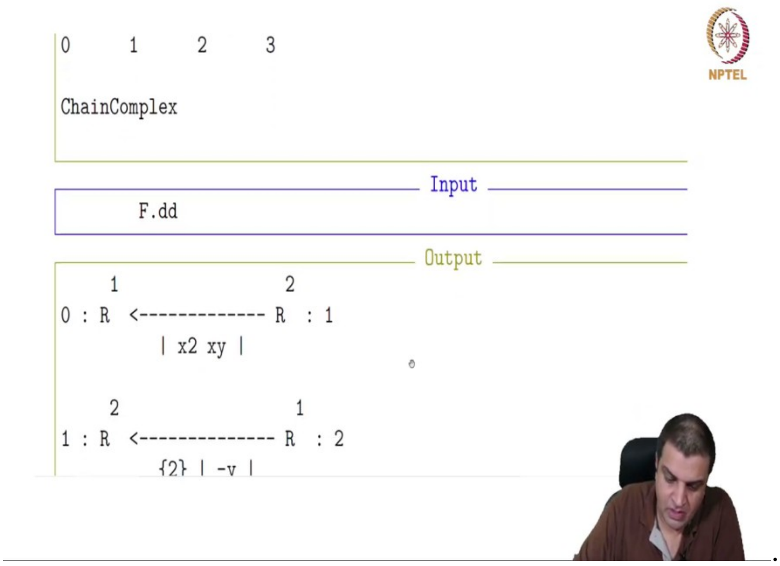

But note here that it is not a resolution of I it is really the resolution of $\frac{R}{I}$. So, please keep that in mind. So, that so we call that thing F, if you want to see the maps in F there is a command called I mean there is a key for a chain complex called dd. So, you can ask $F.dd$ there is no space it is just $F.dd$.

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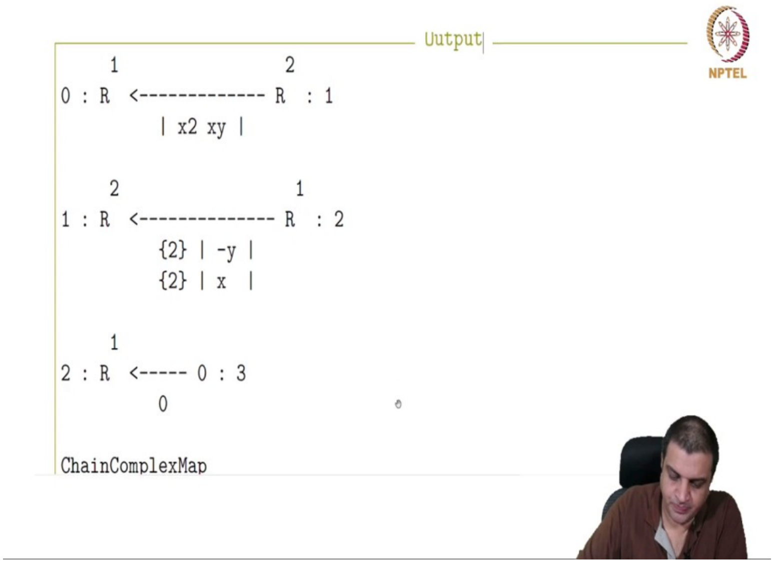

So, it will put out the maps the first map is so as I said this is a resolution of $\frac{R}{I}$ and not of I. So, the map is $R^2 \rightarrow R^1$ in which base generator here the first basis element gets multiplied to x^2 times the basis element here, second one gets multiplied to xy times the basis element here in order to make this degree preserving. If the generator here is degree 0 the generator here must have degree 2.

So, that is what this thing says. So, in the second map this is the map from $F_2 \rightarrow F_1$ the map is this and it says the target of the map the basis elements have these degrees 2 2. So, this just says the basis I mean a basis element here gets multiplied to $-y$ times the first basis element plus x times second basis element and the both basis elements have degree 2 and hence the basis element here will have degree 3.

Only then you can make this map degree preserving and the final map is 0 and it shows these things that ok. So, do not worry about chain complex map.

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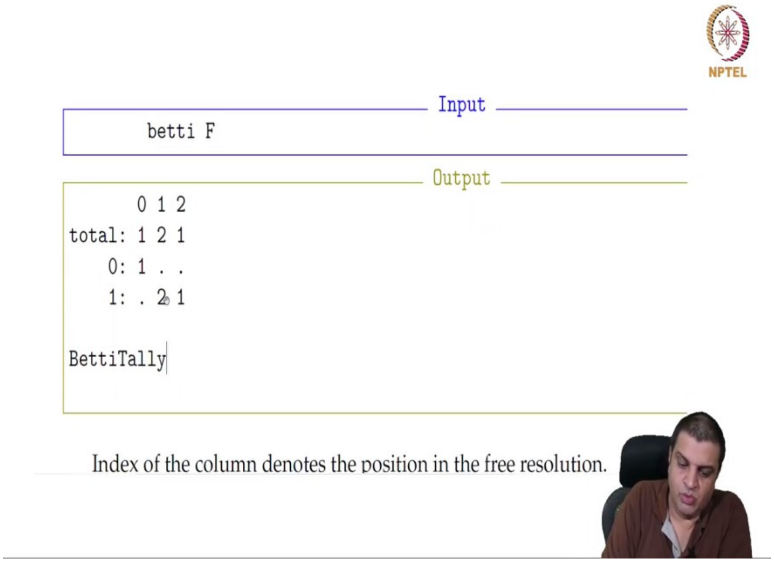

Now, we ask for Betti of the free resolution. So, it produces this and this is what we are discussed.

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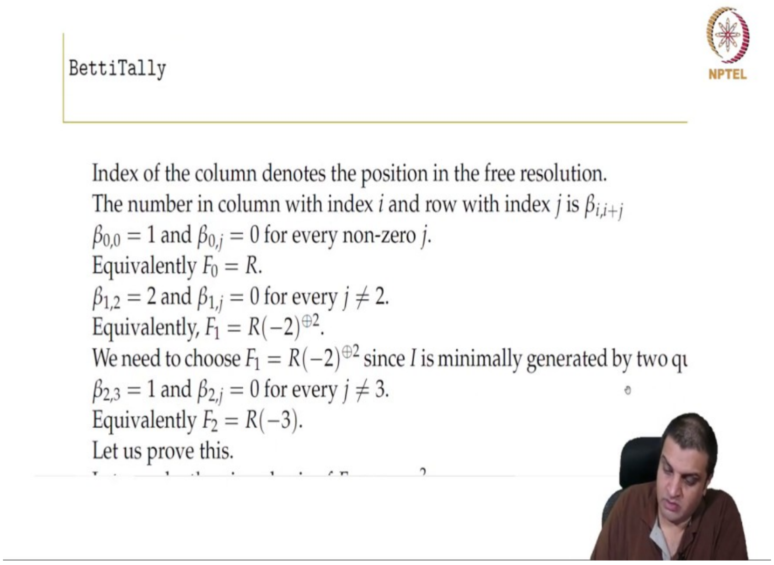

So, let us we need a brief explanation of what this means. So, what is in position? So, column index indices are the position in the free resolution. So, think of this as being written from the right to left the arrows are going from the right to left, so exactly how we saw it in the displays earlier. So, the 0-th the rank of F_0 is 1, F_1 is rank 2 and F_2 is rank 3 that is what this total row means, in the so the column indices are where in the free resolution we are looking at.

The row indices has the following the this is how it is done, the number in column with index i and row with index j is $\beta_{i,i+j}$. So, one should keep that in mind. So, here it says $\beta_{0,0}$ is 1 and $\beta_{0,j}=0$ for every nonzero j and that is another way of saying F_0 is R. Next $\beta_{1,2}=2$ and $\beta_{1,j}=0$ for every j different from 2 and that is just another way of saying $F_1=R(-2)^{\oplus 2}$ with a direct sum of 2 copies of itself.

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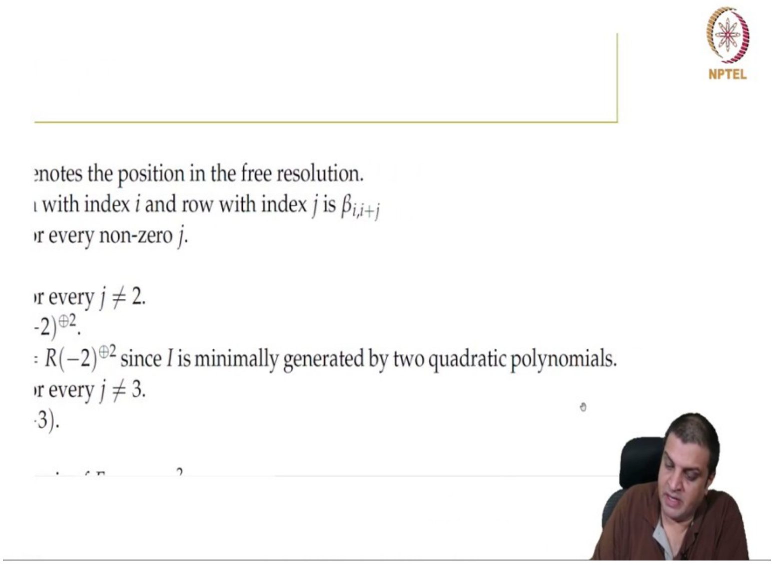

So, then this is what 2 quarter that is because I is minimally generated by 2 quadratic polynomials we needed to choose this and there is not much choice and $\beta_{2,3}=1$ and for other j, $\beta_{2,j}=0$.

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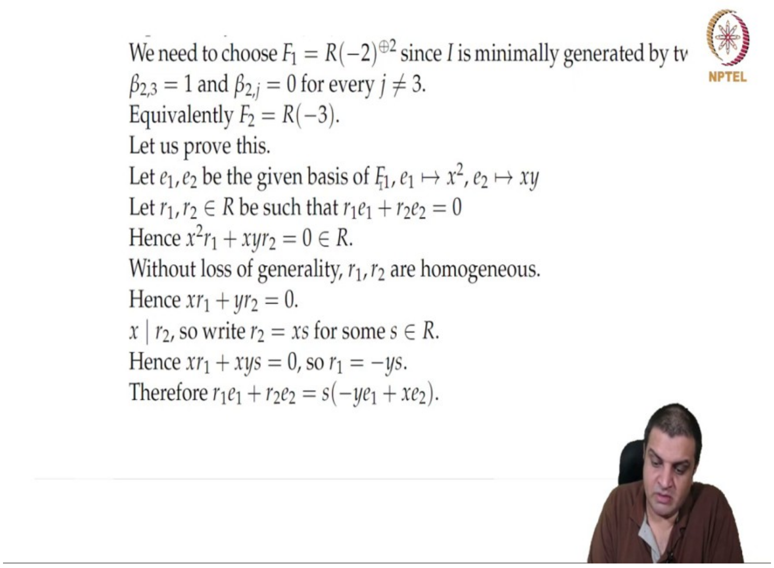

So, equivalently $F_2 = R(-3)$. So, here is a proof which I will anyway explain give it as an exercise you should work this out. So, if you write the basis for F_1 as e_1, e_2, then e_1 goes to x^2, e_2 goes to xy, then r_1, r_2be syzygy of the e_i mean relation among them. But then we apply the map ∂_1 going from F_1 to F_2. So, then we get this x_1. So, this is just applying the map e_1 goes to x^2 and e_2 goes to xy, so we get this relation.

We can assume that they are homogeneous but because of this now this is a relation inside r, so you can divide by x. So, then you get $x r_1 + y r_2 = 0$. So, from this $y r_2$ is divisible by x. This is a UFD. So, x must divide r_2 because it does not divide y the x is irreducible.

So, we can write r_2 as x times s for some $s \in R$, then we put that back we will set r_1 is $-ys$ this is just putting that thing back. And therefore an element of the kernel which was $r_1 e_1 + r_2 e_2$ remember it is the kernel of that thing is $sy e_1 + x e_2$ and that is exactly what sorry that is exactly what; that is exactly what this map says.

The kernel of this map is generated by the image of this and image of this is $-y_1 e_1 + x e_2$. So, this is the end of this lecture and in the next lecture we will look at computing syzygies etc, then we will try to learn some I mean do some more examples.