Computational Commutative Algebra
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Lecture – 54
More on graded rings

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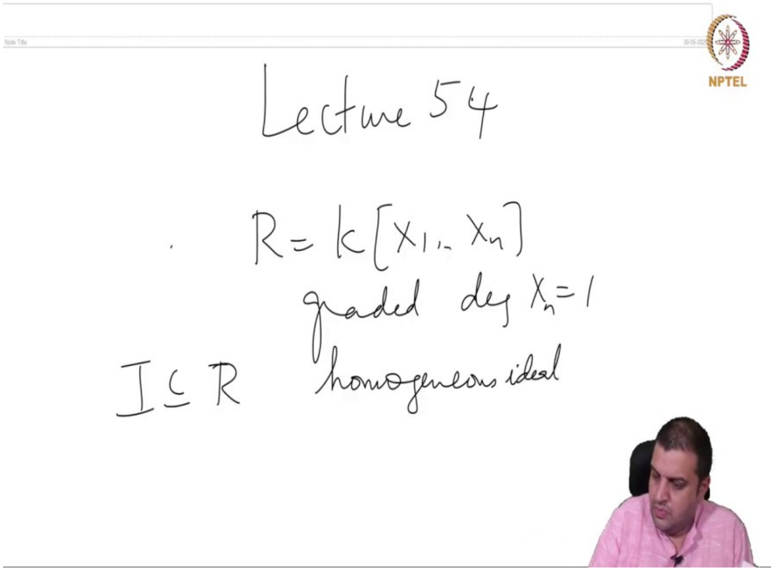

So, here is a proposition. So, now, we are just in this lecture we are always working with graded case. So, R is a polynomial ring may be I set up the notation R is a polynomial ring graded, and $deg\, X_n = 1$, and $I \subseteq R$ homogeneous ideal.

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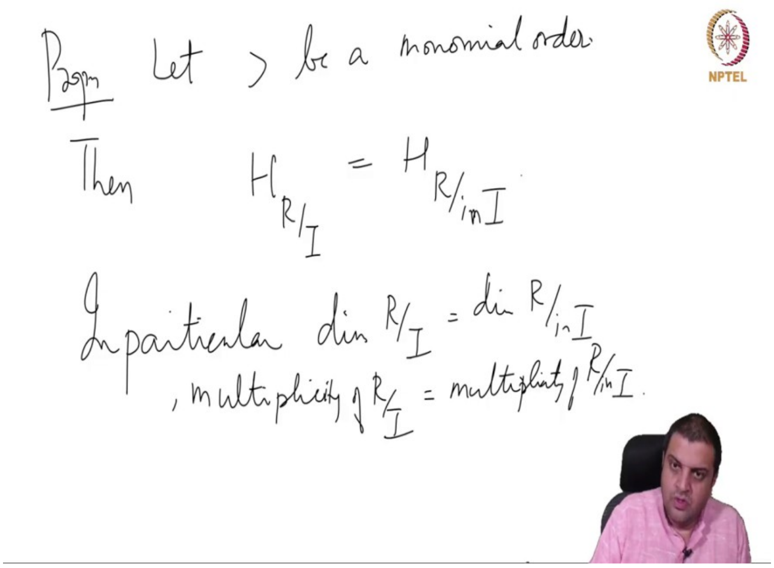

Proposition. Let this greater than ¿ be a monomial order, then the Hilbert series $H_{\frac{R}{I}} = H_{\frac{R}{¿I}}$. We cannot talk about Hilbert series which were not graded. The initial ideal is always graded, it is a ideal generated by monomial, so it is always graded. But the if I is not homogeneous, the left side is not a meaningful object.

So, in particular, $dim\frac{R}{I} = dim\frac{R}{¿I}$. So, any quantity that can be recovered from the Hilbert series should agree for both. So, this one is the dimension which is given by the exponent in the denominator 1-t to the something, so that determines the dimension. So, dimensions are the same, multiplicities are the same.

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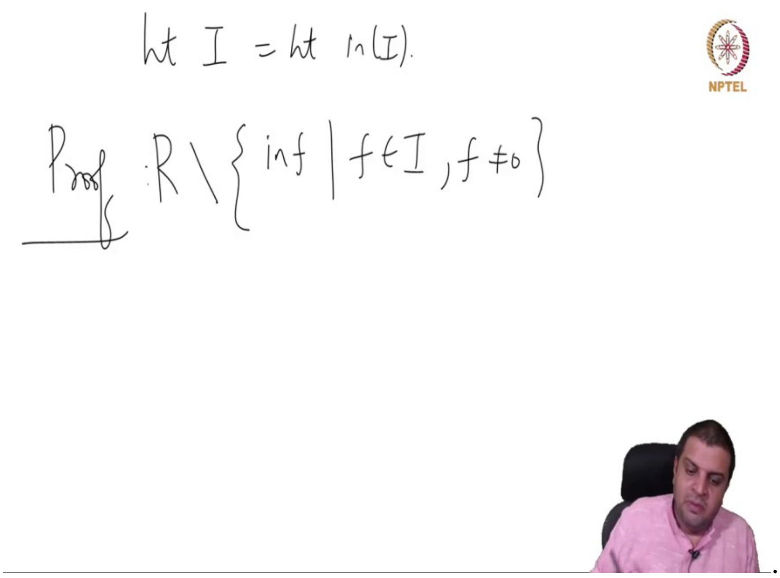

And because of this, height is the same. So, sorry this in is always with respect to the given. And the proof is the same which is ok. If we take $R \backslash \{\in f \vee f \in I, f \neq 0\}$.

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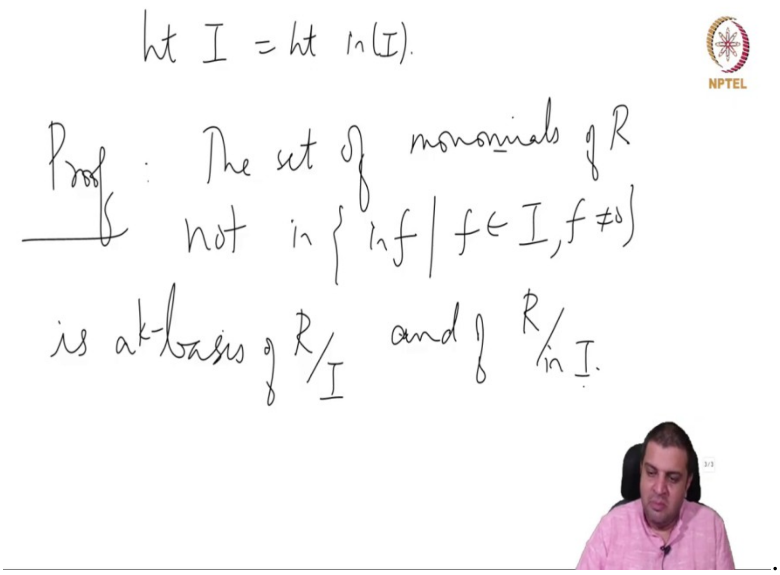

The set of monomials of R not in this set $\{¿f, | f \in I, f \neq 0\}$ is a k-basis of $\frac{R}{I}$, and for the same reason. Remember they are not the same elements. The monomials modulo I in this case and monomials modulo in I in this case, but it is the same I mean it is a bijective correspondence. So, these both have the same dimension, but that is not the point.

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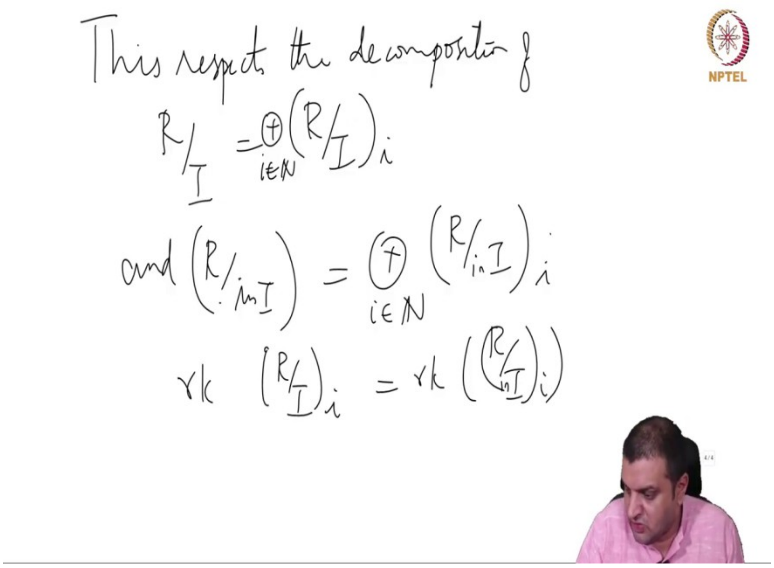

The point is that this respects the decomposition of $\frac{R}{I}$ as $\frac{R}{I} = ¿ i \in N \left(\frac{R}{I}\right)_i$ and $\frac{R}{¿ I} = ¿ i \in N \left(\frac{R}{¿ I}\right)_i$. So, this decomposition remember is k modules. So, you can break up the set of monomials in this list, monomials of R not in this set by degree and of a that set in a fixed degree will be the basis for $\frac{R}{I}$ in that degree and also for $\frac{R}{¿ I}$ in that degree. Yeah, sorry is one should say gives, because it is a residue class modulo I.

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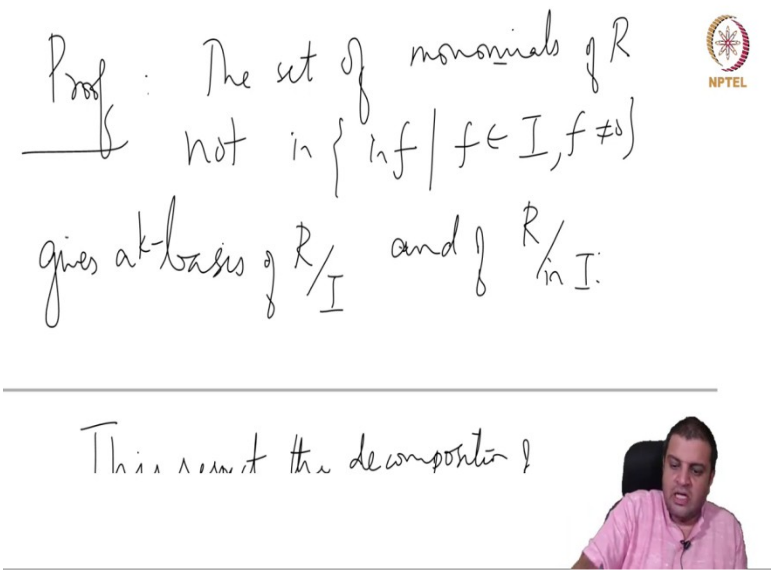

So, therefore, in each degree the dimensions are the same. So, therefore, $rk\left(\frac{R}{I}\right)_i = rk\left(\frac{R}{¿I}\right)_i$. And this multiply this by t with the I, then take the sum you will get the same function on both sides. So, the Hilbert series are the same.

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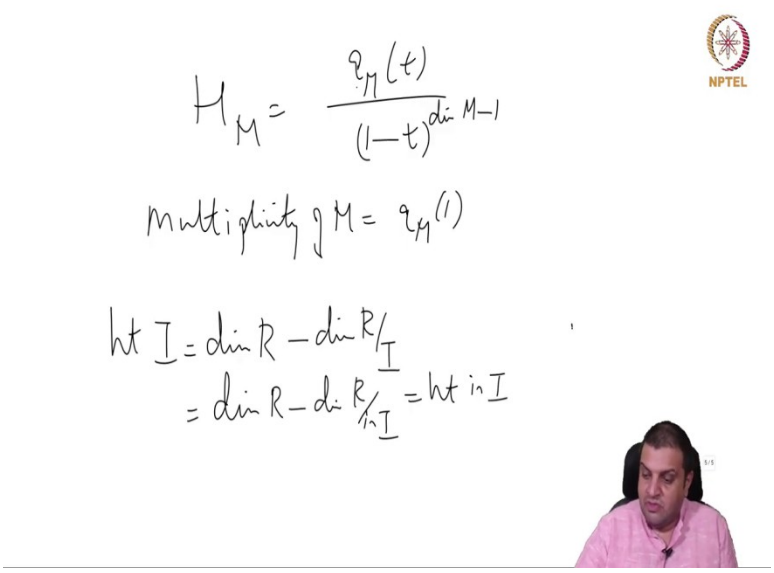

And dimension is given by remember $H_M = \frac{q_m(t)}{(1-t)^{dim M-1}}$. So, if two modules have the same Hilbert function, they will have the same dimension, and the same q_M and multiplicity of M is $q_M(1)$. So, therefore, things that depend if two modules have the same Hilbert function, they have the same dimension, and same multiplicity.

What about height? So, this is from the previous lectures, $ht\, I = dim\, R - dim\frac{R}{I}$. This is true not in arbitrary Noetherian rings, it is true in a polynomial ring which we did not completely prove it. We needed a stronger version of Noetherian normalization, and we needed a going down theorem to prove the statement, or to prove a statement which would imply this.

So, therefore, but if dimensions are the same, this is I mean this is by the observation. This is by the theorem which we only sketched and not proved and the same thing here. I and $¿I$ have the same height. Let us do a quick example to use this idea. And we will come back to this example about computing what is called syzygies, and being able to compute them.

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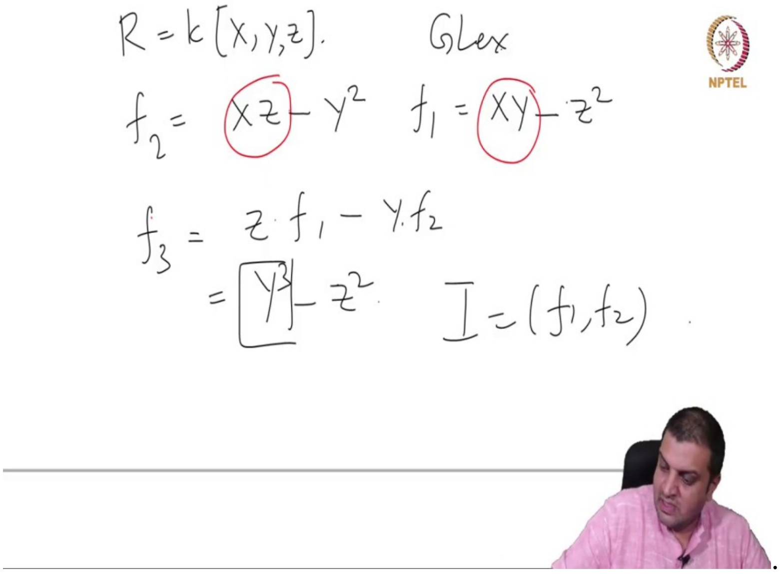

So, R is a polynomial ring in three variables, $R = k[x, y, z]$. And let us say we are considering graded Lex. So, let $f_1 = xy - z^2$, $f_2 = xz - y^2$. And in graded Lex, this is a leading term. So, the leading terms do have a, I mean some common factor. So, let us cancel that thing.

So, let us take $f_3 = z f_1 - y f_2$ and this is $y^3 - z^2$. And its leading term is this. So, if you just took the I mean we have seen this, if you just took the leading terms of the generator that may not give all the initial terms, image a minimal generating set need not be a Grobner basis.

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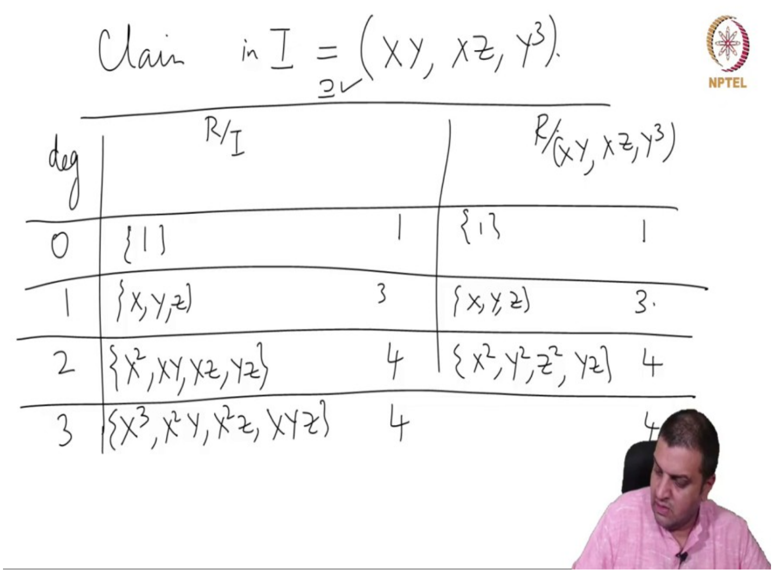

So, now, we want to claim that the initial ideal of I is generated by these three things xy, xz, y^3, there is nothing else. So, we will use ideas of the previous result to prove this. So, we will check this by looking at Grobner basis. So, let us put degrees here. Let us put $\frac{R}{I}$, and then let us put R mod this ideal. This is the claim xy, xz, y^3. So, we know this statement right it is the other direction that we want to prove.

So, here in degree 0, $\frac{R}{I}$ basis is just 1. So, I will write down the basis and the Hilbert function. Similarly, basis is just 1 and Hilbert function is just 1. In degree 2, degree 1, we have only killed a higher degree polynomial. So, here it is just $\{x, y, z\}$; 3. In degree 2, so what is left over? So, these are the this is what we go went modulo y. Each time we see, so degree 2 has quadratic polynomials.

Each time we see a y^2 modulo f_2 we can write it as xz. And similarly each time we say z^2 modulo of ideal we can write, so sorry I did not clarify what I is, I is the ideal generated by f_1 and f_2.

So, modulo I, y^2 can be written as xz, and z^2 can be written as xy. Therefore, there is x^2, then there is xy, xz. And y^2 can be written as one of these, z^2 can be written as one of these, yz cannot be written as one of these, so that is actually independent. And then, we can check what is in degree 2 what are the monomials not in this not divisible by that. So, that is also x^2, y^2, z^2 and yz; 4. So, both Hilbert functions are 4.

So, I will just do one more degree, and then we will see that it is actually 1,3,4,4,…. So, in degree 3, we have x^3, when we have $x^2 y$, these are leftover things $x^2 y$. Each time we see, so $x^2 y, x y^2$. If you see $x y^2$, then using this it can be rewritten as $x^2 z$. So, we do not need to take $x y^2$; same reason we do not need to take $x z^2$, I mean similar reason using this.

So, now, let us look at things that are not divisible by x. So, there is y^3 which can be written in terms of this xyz. And xyz is new, I mean xyz cannot be written using these relation. It can be changed to some anything relations.

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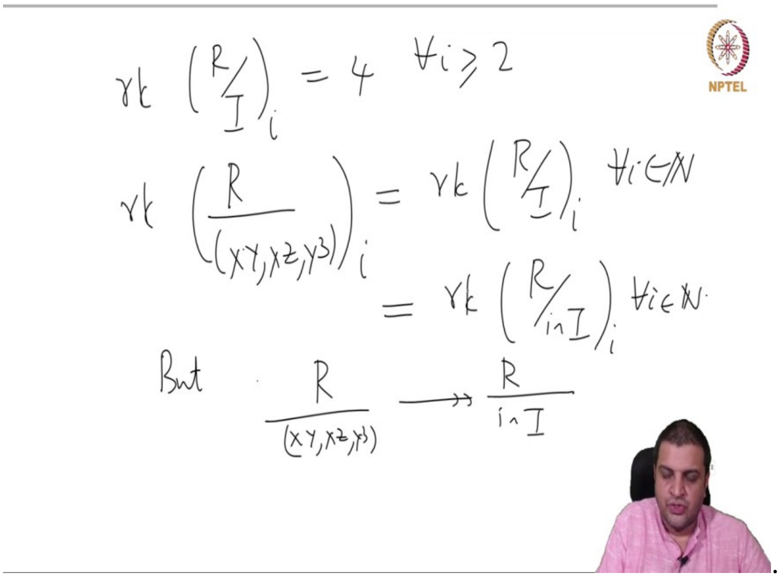

So, there is xyz. And this is all that there is there is 4, and similarly if you do this there is 4. And then the conclusion is that the rank of $\frac{R}{I}$ in degree 2 onwards $i=4$ for all $i\geq 2$. Rank of $\frac{R}{(xy, xz, y^3)}$ in degree i is equal to the same rank as this for all i. And this is equal to by the theorem this is equal to rank of $\frac{R}{¿I}$ in degree i for all $i\in N$.

But remember that this ideal is inside the initial ideal. But $\frac{R}{(xy, xz, y^3)}$ surjects onto $\frac{R}{¿I}$. What is the kernel? Well, because of this relation, kernel will have Hilbert functions constant 0. So, let us just go back here because of this inequality here. So, this, thing here this now implies that kernel is 0.

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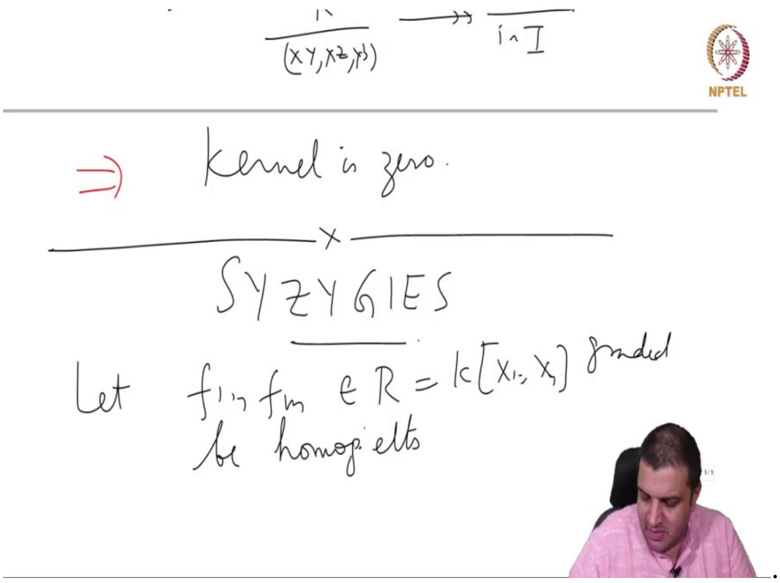

If, there is a kernel in some degree j, then this module will have more in degree j than this module in degree j this module will be bigger and will have bigger rank than this one, but that is not what the calculation shows they are all the same thing. So, therefore, the kernel is 0. So, this is a way to use these ideas to say that some we have already constructed the initial ideal, so comparing Hilbert functions.

So, now, I want to introduce a new topic this is going to be the related to the next few lectures which is the following called syzygies. So, what is that? So, we will only worry about homogeneous ideals now, it can be done in any way ok. So, let $f_1,\ldots,f_m \in R = k[x_1,..,x_n]$ be homogeneous elements. We can do it in any case I mean more generally for any Noetherian ring etc.

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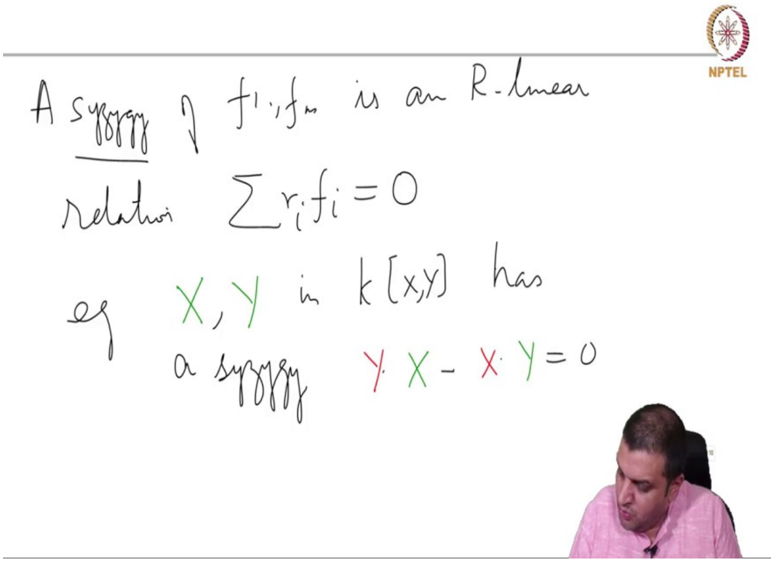

A syzygies of these things is an R linear relation $\sum r_i f_i = 0$. So, for example, X and Y in $k[X,Y]$ has a syzygies; $YX - XY = 0$. So, this is the maybe let me just use a different letter. So, just to say what is f and what is r in this notation? So, let us put the f's in green and the so this X here. So, these things in red are the coefficients r_i, and these things in green are these f_i. So, this is the syzygies of these two elements. So, how would we find the syzygies that is what we want, but what is the I mean ok?

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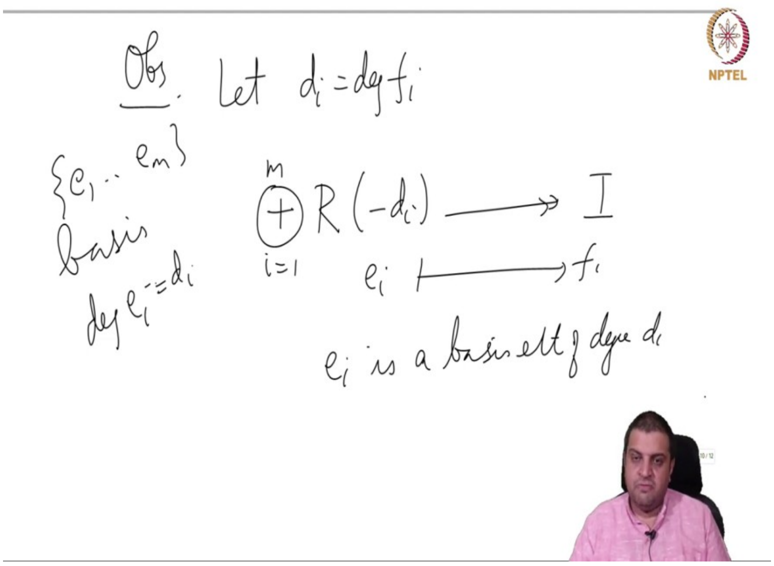

So, we are given this $f_1, \ldots, f_m$. Let d_i be the degree of f_i, because we already assumed that is homogeneous. Then we have the following free module R twisted by $-d_i$, so ¿$i=1R(-d_i)$ onto I in which basis element here of degree d_i maps to f_i.

So, e_i maps to f_i where e_i is a basis element of degree d_i. So, technically we should say choose a basis element with like this, give it degrees d_i as above, and then consider the free module it is graded this one, and then e_i maps to f_i, this is a subjective map.

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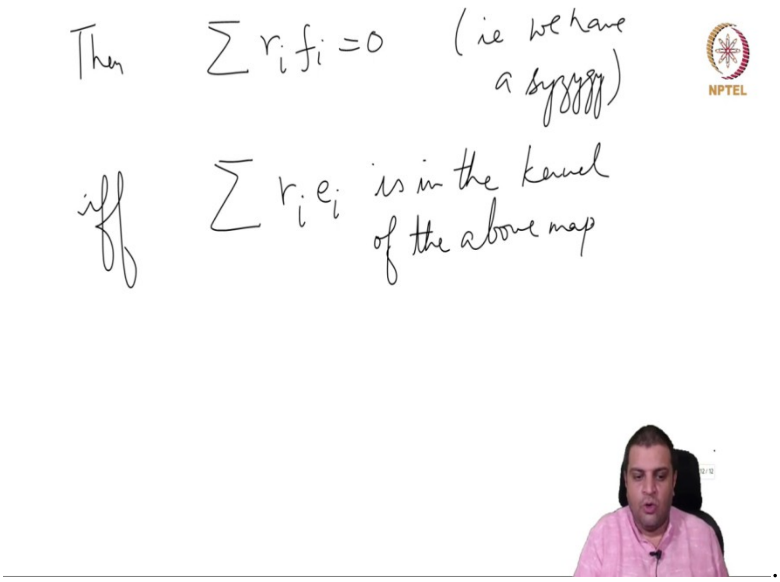

So, where do syzygies come? Then $\sum r_i f_i = 0$, that means, that is we have a syzygies if and only if the element $\sum r_i e_i$ in that free module. So, this free module is in the kernel of the above map. So, in some sense finding asking for syzygies is same thing as asking for a kernel of this this map.

So, then if you know if you have algorithms to find the kernels of group module homomorphisms, then we can find all syzygies. So, this is what we will I mean and building this up into what is called a free resolution is what we will do be doing in the next few lectures. But before that I want to show an example which tells us that if you know to compute Grobner basis, then we can compute syzygies.

So, we will do it for an ideal now, later I will mention that it can also be done for modules without explaining how it is done, but and that is how we will start I mean free resolution can be constructed. So, these things will be explained in the next few lectures. So, right now let us just look at how to use the same calculation as we do in the Grobner basis to come up with syzygiess.

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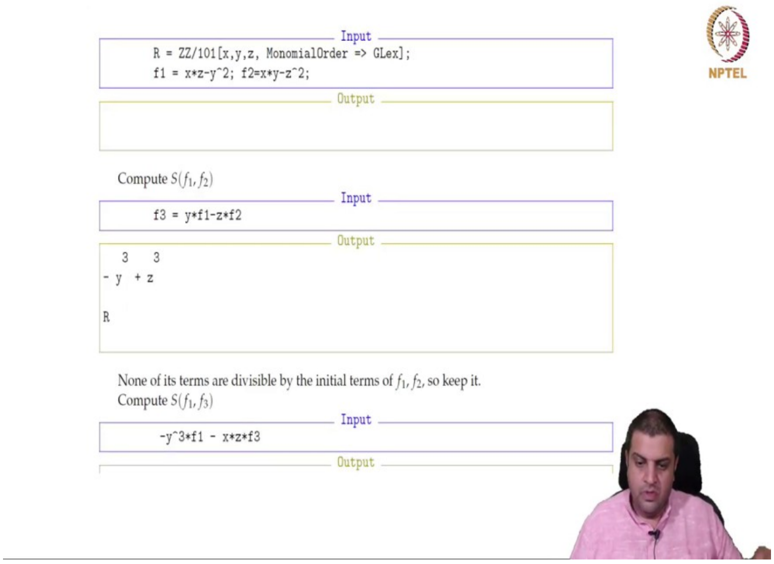

So, we will just do this in Macaulay. So, here are it is the same example from the last time f_1 is $xz-y^2$ and f_2 is $xy-z^2$ in GLex. And then we want to compute the S polynomial $S(f_1,f_2)$. So, we use different convention from what we did in that example.

But so the leading term is xz leading term here is y, so multiply this by y and this by x, and then take by z and then take the difference. So, $S(f_1,f_2)$, we call it f_3. So, this is $-y^3+z^2$. When we did it earlier we had swapped the order, so we got the negative of this, but let us keep this convention.

So, we look at this. So, none of terms are divisible by the initial terms of f_1 and f_2. So, this is our reasoning is what is going on in Buchberger algorithm. So, we will keep it. There is nothing to simplify, no further division algorithm remainder to be taken. So, we now have f_1,f_2,f_3. And we actually proved that this is a Grobner basis.

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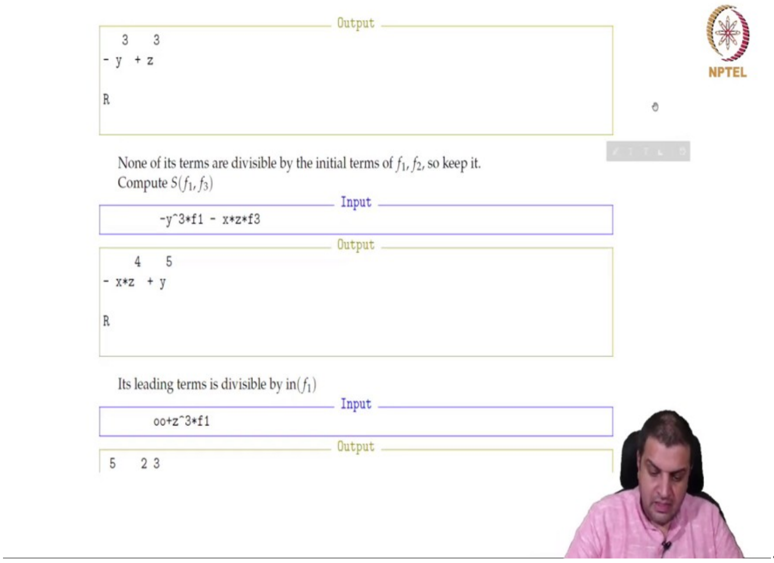

But let us we do not know it. So, well ask $S(f_1, f_3)$. So you have to multiply f_1 by $-y^3$ and f_2 by xz, and then take the difference. So, we got this one so some term. Now, if you look at this, this ones leading term is divisible by the initial term of f_1 right xz. So, we try to simplify it. So, what we did was, we had to multiply by z^3. So, oo plus z^3 times f_1.

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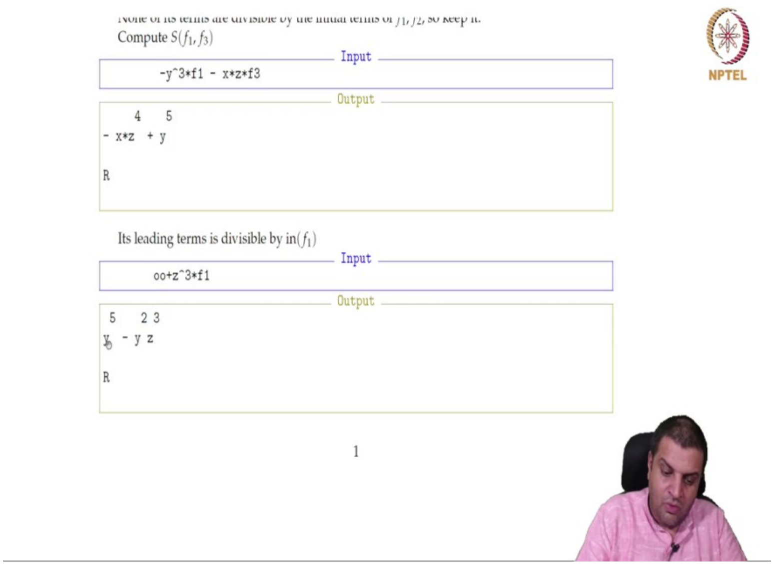

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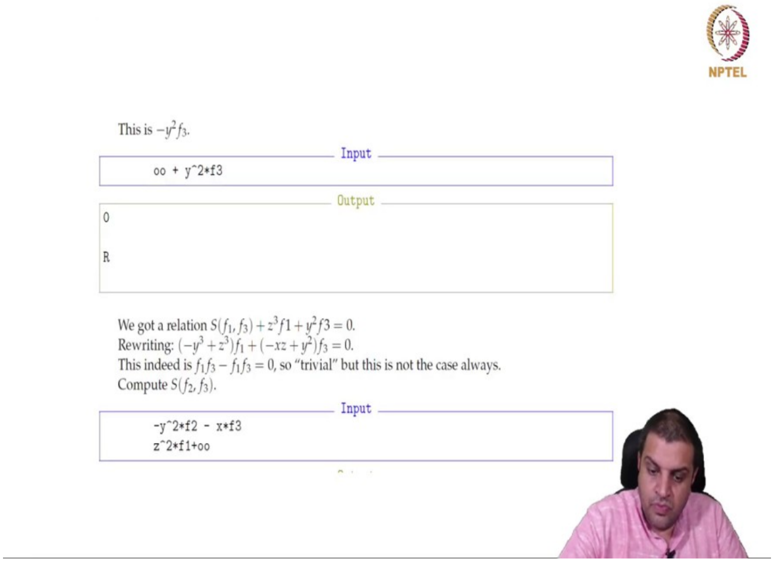

So, we got $y^5 - y^2 z^3$ which is actually $-y^2 f_3$ and we just check $oo + y^2 f_3$ and this is 0. So, this is indeed true. So, what is the conclusion that we have got? $s(f_1, f_3) + z^3 f_1 + y^2 f_3 = 0$.

We can rewrite it to look at like this. But then it just here it just says $f_1 f_3 - f_1 f_3 = 0$ like that $xy - yx = 0$. There is so in this case it is sort of an obvious trivial syzygies, but let us check this is not always the case.

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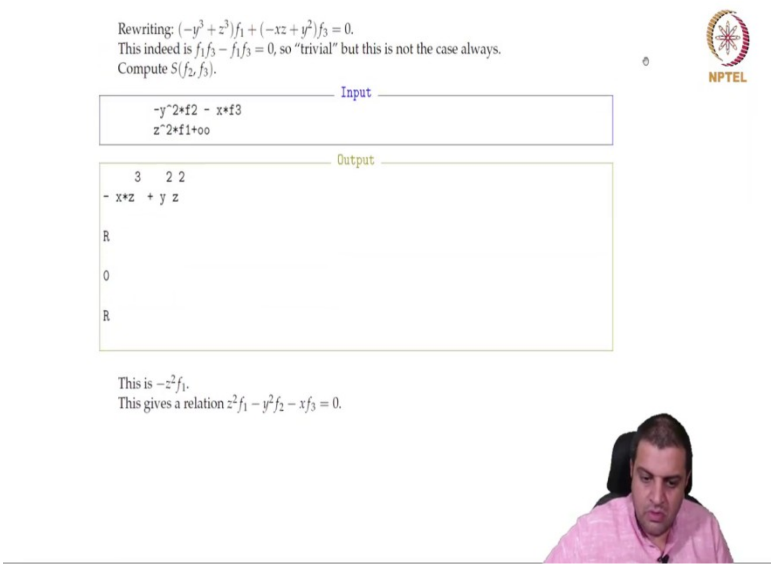

So, let us check f_1. Let us now compute $S(f_2, f_3)$. So, we ask this is the computation. We ask $-y^2 f_2 - x f_3$ there is a common factor. So, we get this polynomial. This polynomial it is immediately clear that this is $f_1 z^2$. So, we just ask I mean $-f_1 z^2$.

So, we just ask $z^2 f_1 + oo$ and it says it is 0. So, therefore, what the conclusion is that we got a syzygies. Now this is not one of those trivial syzygies $z^2 f_1 - y^2 f_2 - x f_3 = 0$.

So, the same procedure which gives the Grobner basis also if you rewrite it nicely and just work through it, it gives us the relations among the generators. So, this will be used to for us to build what is called a free resolution and try to learn I mean use that information in some context.

So, this is the end of this lecture. So, the next few lectures would be devoted to a preliminary understanding of free resolutions, some invariants coming about free resolution for homogeneous ideals and how to use it.