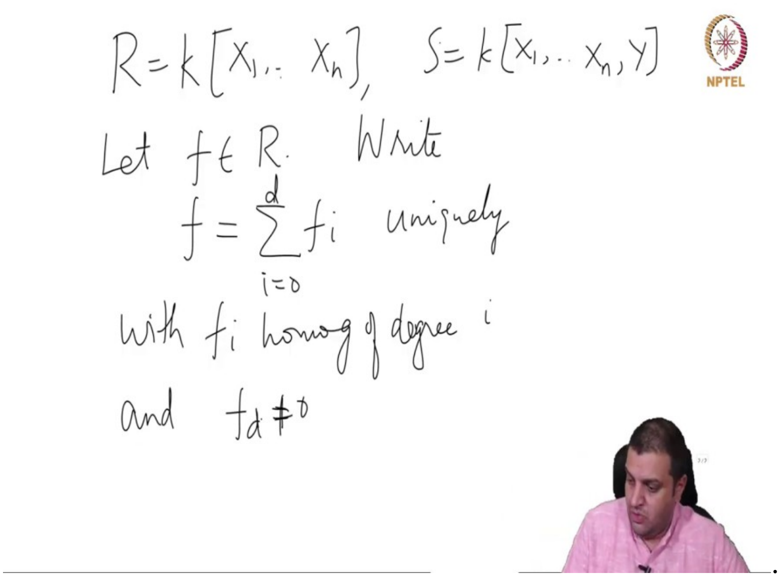
Computational Commutative Algebra
Prof. Manoj Kummini
Department of Mathematics
Chennai Mathematical Institute

Lecture – 53
Homogenization – Part 2

This is lecture 53 and we are this lecture is about Homogenisation, which is how would we convert a problem about non graded rings, in which we know from to a problem about graded rings and ideals. Which there are some techniques, that there are some advantages which we saw in the last lecture of why we should worry about projective space. So, this is the idea behind it ok, this is what we should do we do ok.

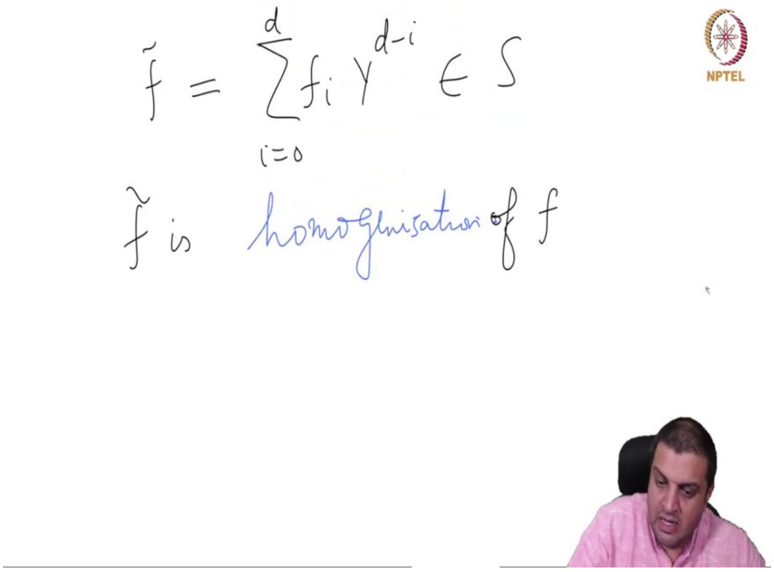
(Refer Slide Time: 01:00)



So, let us set up notation R would be a polynomial ring in n variables. So, this is where the inhomogeneous things will be there and k is a field and S is a polynomial ring in n+1 variables which I will denote as $X_1, \ldots, X_n, Y$. So, Y will be used to homogenize.

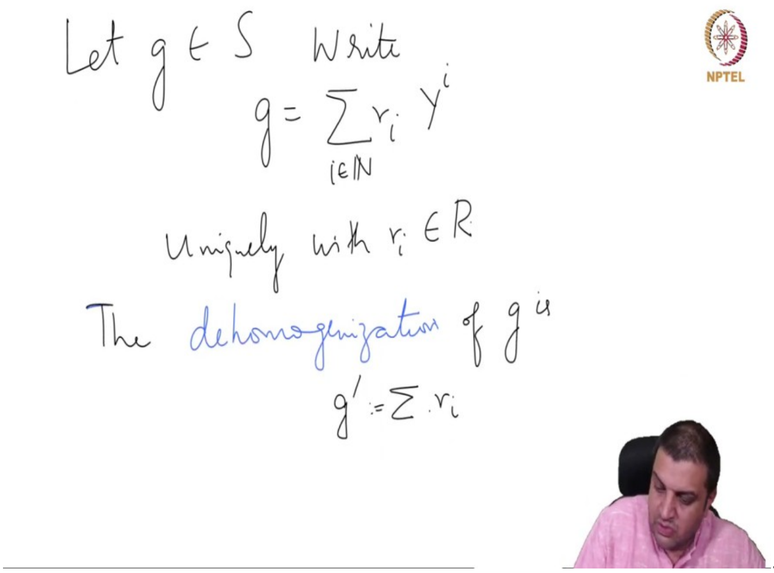
So, let $f \in R$, write f as $f = \sum_{i=0}^{d} f_i$ uniquely with f_i homogeneous of degree i and $f_d \neq 0$. So, d is the total degree of f write it uniquely like this.

(Refer Slide Time: 02:15)



Define $\tilde{f}=\sum_{i=0}^{d} f_i Y^{d-i} \in S$. So, we start with an element of R split it into its homogeneous components and where d is the degree now, because it is a last highest thing with this is non zero and write it like this. So, therefore, $\tilde{f}$ is called the homogenization of f with respect to Y. So, this is we wanted to sort of reverse this process also.

(Refer Slide Time: 03:18)

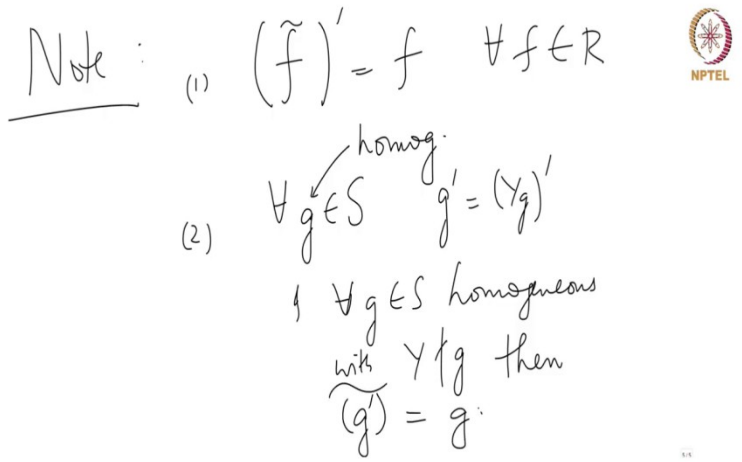


Let $g \in S$ and whenever we say g elements of S we mean homogeneous elements, but in this particular case to define it does not really matter. So, write just for the definition write

$g=\sum_{i\in N} r_i Y^i$, uniquely with $r_i \in R$ that is just saying S is a polynomial ring over R in one variable.

So, you can write it like this uniquely. So, write it like this define the dehomogenization of g is just this. So, we will denote this by g'. So, throughout in this lecture for an element g like this g' will denote this, this object and $\tilde{f}$ will be the homogenization. We will stick to this notation throughout this lecture.

(Refer Slide Time: 04:48)



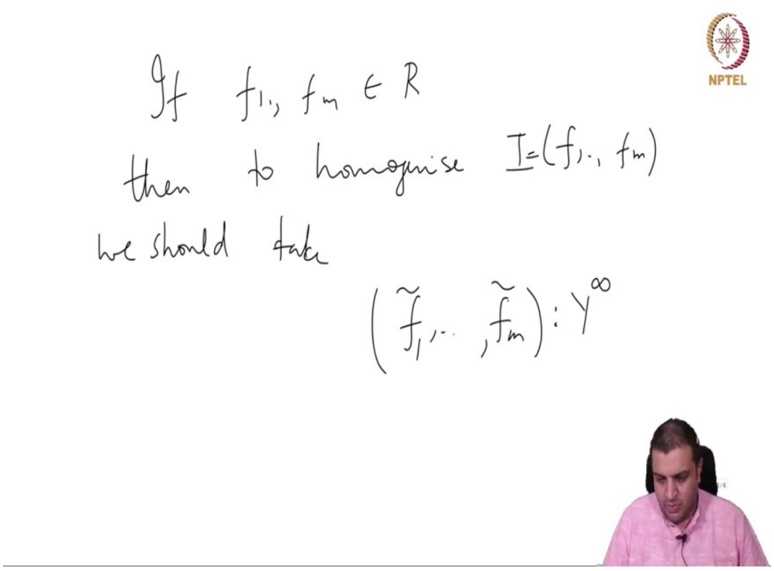
So, note if we take an $f \in R$ and we homogenize it. So, now, in the degree deficient parts, we are we are multiplying by enough copies of Y. And then we take the prime this is f, that is $(\tilde{f})' = f$ for all $f \in R$. So, first homogenize then dehomogenize it is identity.

On the other hand, for all $g \in S$, $g' = (Yg)'$. Because, if we notice this thing here, if we just multiply this whole thing by Y we would just get a Y plus 1 here for r_i. But, that is ignored when you take the sum.

So, $g'$and $(Yg)'$ are same. So, if you take g prime and then take tilde, we may not get g prime. And so, this is let us now stick with homogeneous it might be true more generality, but I have not thought about it; so, for all $g \in S$ homogeneous with Y not dividing g. So, what we see from here is if Y divides g, I mean Y divides some g its prime is same thing as g divided by

Y prime. If you have a g homogeneous g for which Y does not divide which means, the top degree part does not involve Y, then $\widetilde{g}' = g$.

(Refer Slide Time: 07:12)

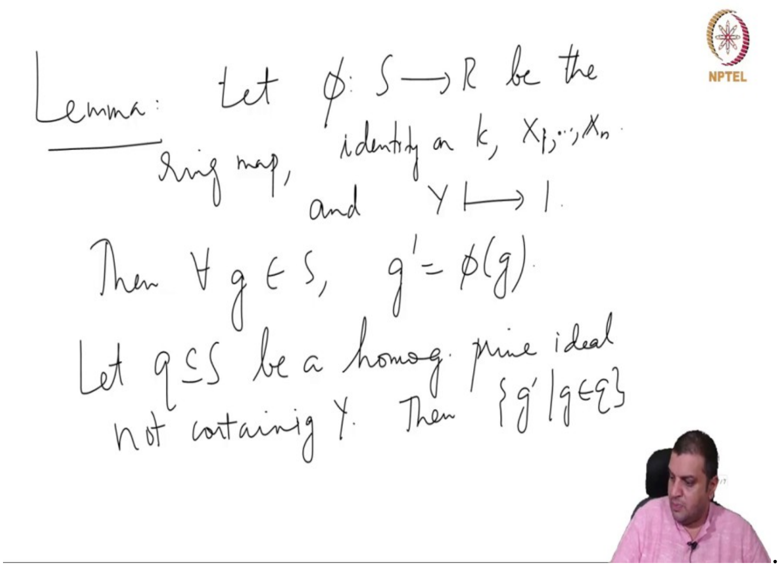


If $f_1, \ldots, f_n \in R$, then to homogenise the ideal generated by $(f_1, \ldots, f_m)$. We should take $(\widetilde{f_1}, \ldots, \widetilde{f_m})$ and then saturate out Y.

This is the correct ideal to look for is what we want to prove. So, it takes a little bit to prove the statement to make the precise statement, what we want the notion of what we will do is what is called projective closure.

So, we get to that point I will define those things. So, this is where we are headed; we want to understand that, if you want to homogenise the correct object to look for is homogenise a generating set. And then so now, this is an ideal inside S saturate out Y.

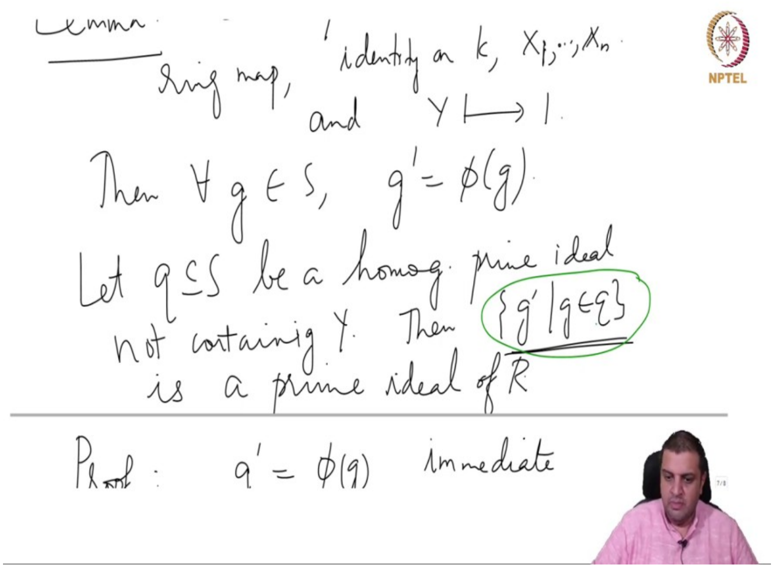
(Refer Slide Time: 08:25)



So, in this direction let us start. Let us get a there is a lemma. Let $\phi : R \rightarrow S$ be the ring map, with that is identity on $k, X_1, \ldots, X_n$ and Y maps to 1. So, X_i is mapped to themselves and Y goes to 1 and elements of k map identically.

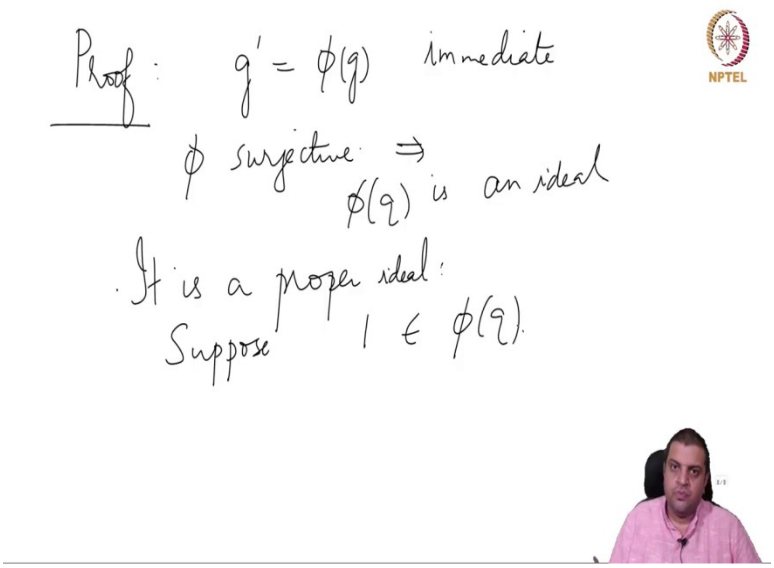
So, let ϕ be this map, then for every $g \in S$, $g^{'} = \phi(g)$. So, this dehomogenization is actually a ring homomorphism. So, we will use this let $q \subset S$ be a homogeneous prime ideal not containing Y.

(Refer Slide Time: 10:06)



Then if we just take the elements $\{g' | g \in q\}$ is a prime ideal of R. So the key observation is dehomogenization is actually a ring map. And then, we get this thing for a homogeneous prime ideal.

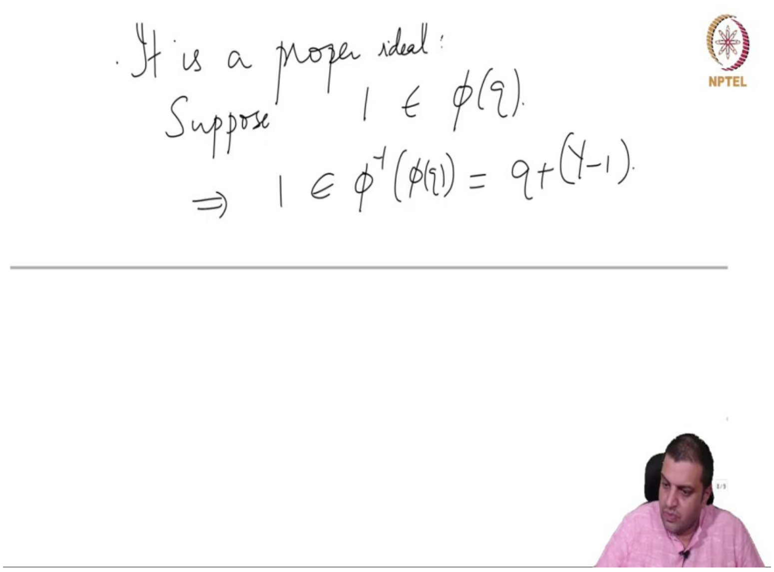
(Refer Slide Time: 10:29)



So, proof this is immediate, $g' = \phi(g)$ is immediate, ϕ is a surjective ring homomorphism. So, therefore, image of any ideal is an ideal this is not true in arbitrary ring homomorphisms.

So, ϕ is surjective. So, therefore, $\phi(q)$ is an ideal. This is just $\phi(g)$ for every $g \in q$ that is just the image of q under the function ϕ, $\phi(q)$ is an ideal now. So, we want to prove that this is a proper ideal. Suppose 1 is an image of ϕ. So, suppose 1 is in $\phi(q)$.

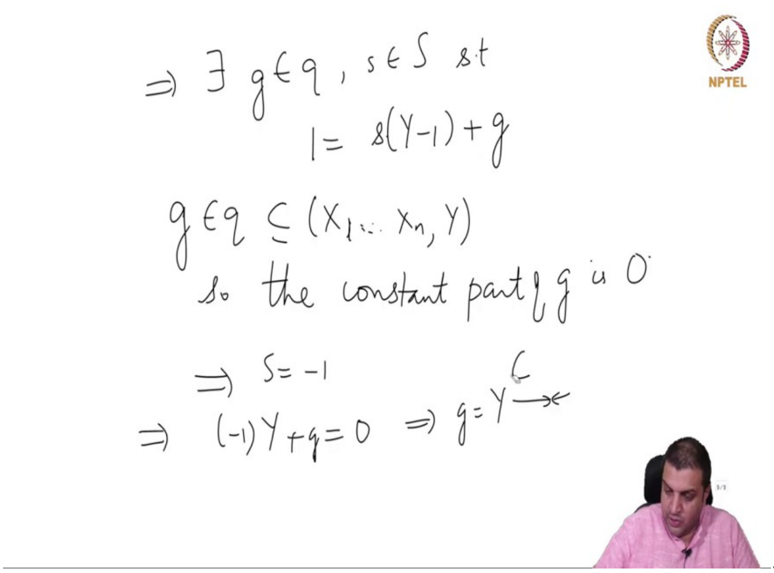
(Refer Slide Time: 12:18)



This means that, we can write 1 as ϕ of something. So, this implies that $1\in\phi^{-1}(\phi(q))$, but ϕ is a surjective ring homomorphism, so $\phi^{-1}(\phi(q))=q+(Y-1)$.

Because, the kernel of the ring homomorphism is $(Y-1)$. So, if $1\in\phi(q)$ then $1\in\phi^{-1}(\phi(q))$ also. And therefore, this is true.

(Refer Slide Time: 13:26)



So, this now means that there exists some $g\in q$ not necessarily homogeneous, g need not be homogeneous in this thing. And $s\in S$ such that $1=s(Y-1)+g$.

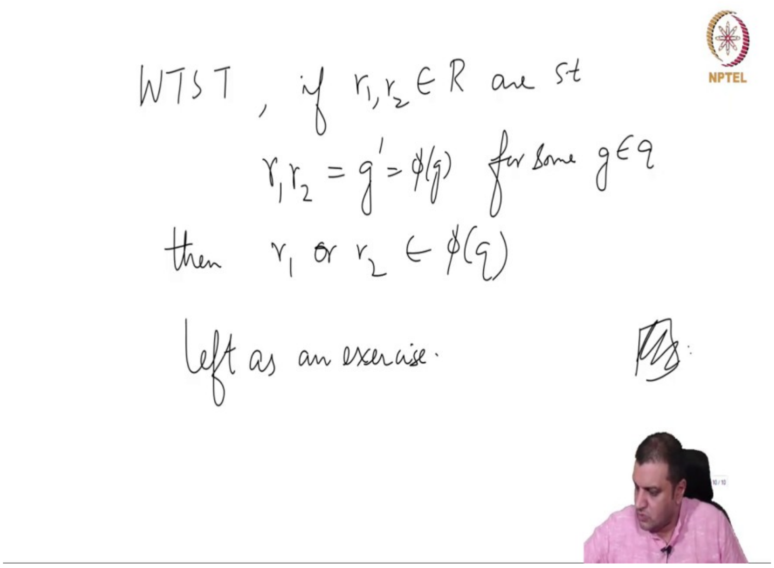
So, this is the relation inside the polynomial ring S. Let us break it degree by degree. So, it says that the constant part on the right side is 1, there is a constant part is on the left side is 1. And for every $i>0$, the degree i part on the right side is 0 that is what this one says.

So, what is the constant part on the right side? Of course, there is an s from here and then there is a constant part of g, but $g \in q \subseteq (X_1, \ldots, X_n, Y)$. So, there would not be any constant part because, it will be some S linear combination of these elements. So, the constant part of g is 0; which now implies that $s=-1$.

But, then in higher degree so, this is analyzing the constant part. In the non constant part, we also see that $(-1)Y+g=0$, which means that $g=Y$, but that is a contradiction as $g=Y \in q$ but we assume that q does not contain Y.

So, this is if you have a prime ideal that does not contain Y a homogeneous prime ideal, then when you dehomogenize it we get a prime. So, that is this is a we get a proper ideal, we have not yet proved that it is a prime we get a proper ideal. So, this proving that it is a prime is a little so, I will just sketch it and I will outline the details the steps in the exercises. So, I will just only give you the idea.

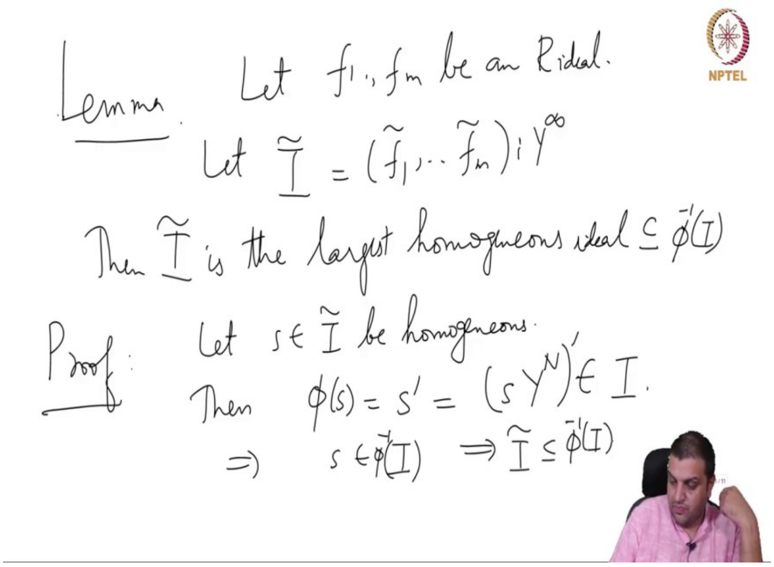
(Refer Slide Time: 16:15)



So, now, we want to show that, if $r_1, r_2 \in R$ are such that $r_1 r_2 = g' = \phi(g)$, for some $g \in q$. Then r_1 or $r_2 \in \phi(q)$. this is what we want to prove. So, this requires a little bit of calculation which it is not very illuminating it is not surprising either.

If we just write out what details has to be checked and they can be checked and this will follow. So, this will be left as an exercise. So, this is so, we will assume that we have proved the lemma so, recall that we had wanted to homogenize the elements, and then saturate out Y homogenize the generating set and then saturate out Y.

(Refer Slide Time: 17:51)



So, let us try to understand that ideal lemma. Let $(f_1,\ldots,f_m)$ be an R ideal. Let $\tilde{I}$ be that ideal that we were considering $(\widetilde{f_1},\ldots,\widetilde{f_m}):Y^{\infty}$. Then, $\tilde{I}$ is the largest homogeneous ideal, inside $\phi^{-1}(I)$, this itself is not homogeneous. Because, we call that $Y-1$ is there in $\phi^{-1}(I)$. So $Y-1$ is in the kernel. So, it is in $\phi^{-1}(I)$. So, it is not homogeneous, but this is the largest homogeneous ideal. So, we will prove two containments.

So, if you take let $s \in \tilde{I}$ be homogeneous. So, we want to prove that $\tilde{I} \subset \phi^{-1}(I)$, $\tilde{I}$ is homogeneous ideal. So, if we just prove homogeneous elements inside here are inside this, then $\tilde{I}$ will be homogeneous. Then, if you apply $\phi(s)$ which is just dehomogenize s notice that, if we yeah this is just dehomogenizing of this thing, but if you take s and then multiply by any power of Y.

And then dehomogenize it we would not get anything different so, it is the same as this. But for large enough N this is inside this ideal. So, it can be written as an S linear combination of these things. And then when you dehomogenize that we would see. I mean so when we dehomogenize that let us go back here, if you take an element of R homogenize and then dehomogenize, we would just get the element back.

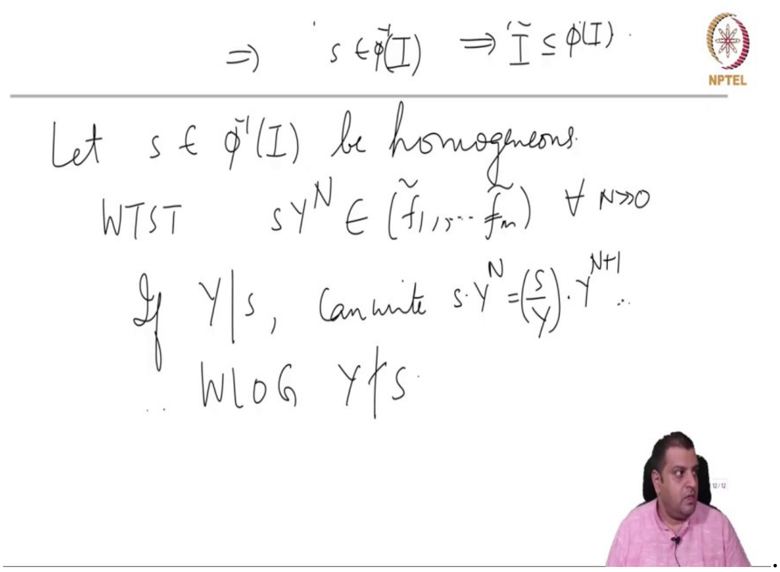
So, therefore, this sY^N for large enough N is in this ideal generated by the homogenizations. Therefore, it can be written as some S linear combination of these elements dehomogenize, which means it is a ring homomorphism. So, it is an R linear combination of their respective dehomogenizations.

But $\widetilde{f}_1$ dehomogenization is f_1 so, this is inside I. So, this proves that $s \in \phi^{-1}(I)$, but I tilde as I said is homogeneous. So, now, this implies that $\tilde{I} \subset \phi^{-1}(I)$. So, let me just quickly say this once more here $\phi(s) = \tilde{s}$ and $\widetilde{sY^N}$, they are the same so, we can multiply.

So, then after we multiply it would come inside this ideal generated by these tildes S linear combination of these $\widetilde{f}_i$, when you apply the prime here which is to apply the ring homomorphism ϕ. It would be ϕ applied to the coefficients ϕ applied to them ring homomorphism.

So, now, when S linear combination becomes R linear combination and that is I. This proves this so, this proves one direction this direction. Now, let us take a homogeneous element inside here and we want to prove it inside here.

(Refer Slide Time: 22:06)

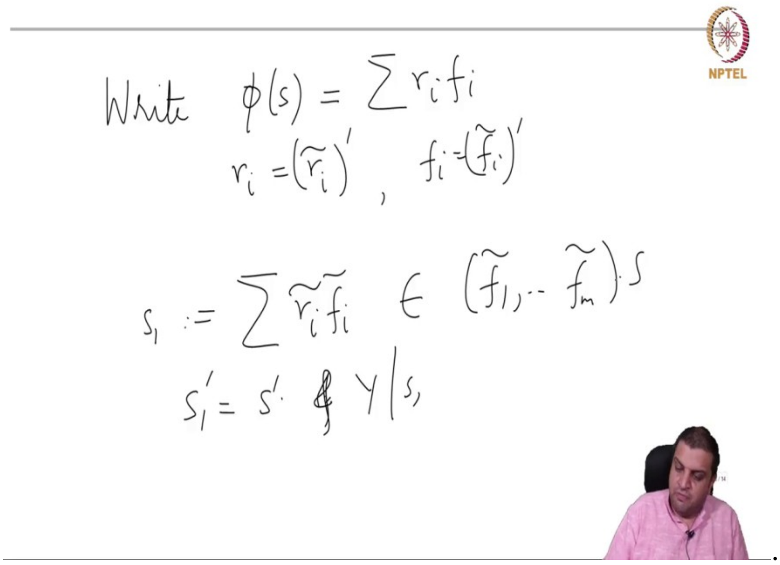


So, let $s \in \phi^{-1}(I)$ be homogeneous. So, we wanted to show that $sY^n \in (\widetilde{f_1}, \ldots, \widetilde{f_m})$ for all large N, I mean for some N is enough, then it will be true for larger N. So, this is inside here. So, we want to show this is the homogenizations of the generating set not ok.

If Y divides s since we only have to prove this, we can rewrite this. We can write sY^N as $\left(\frac{s}{Y}\right)Y^{N+1}$ remember this is the UFD. So, it makes sense to write such things $\left(\frac{s}{Y}\right)Y^{N+1}$. So, we can adjust this the number of Y times Y appears and divides s on this factor and we can just remove this thing here and then and so on.

So, therefore, without loss of generality Y does not divide s. So, we have taken a homogeneous polynomial and then we have assumed we can assume therefore, that its that is not divisible by Y.

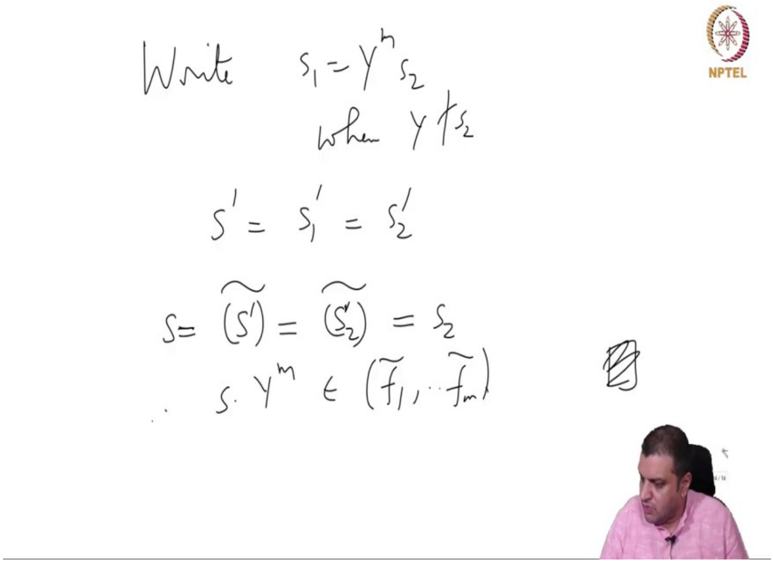
(Refer Slide Time: 23:53)



So, now write $\phi(s) = \sum r_i f_i$ which is the dehomogenization. Then $r_i = (\widetilde{r_i})'$ and $f_i = (\widetilde{f_i})'$.

So, let us take this sum and then take their prime. So, taking this prime is a ring homomorphism sorry not this sum take this sum, let us take that. Call this thing s_1 applying prime, which is applying the ring homomorphism ϕ, is a ring homomorphism I mean (Refer Time: 25:20) therefore, s_1' prime is same thing as s'. So, we have two homogeneous elements with this property that, their primes are the same. And Y does not divide s and Y does not divide s this is what we have two homogeneous elements Y might divide s_1.

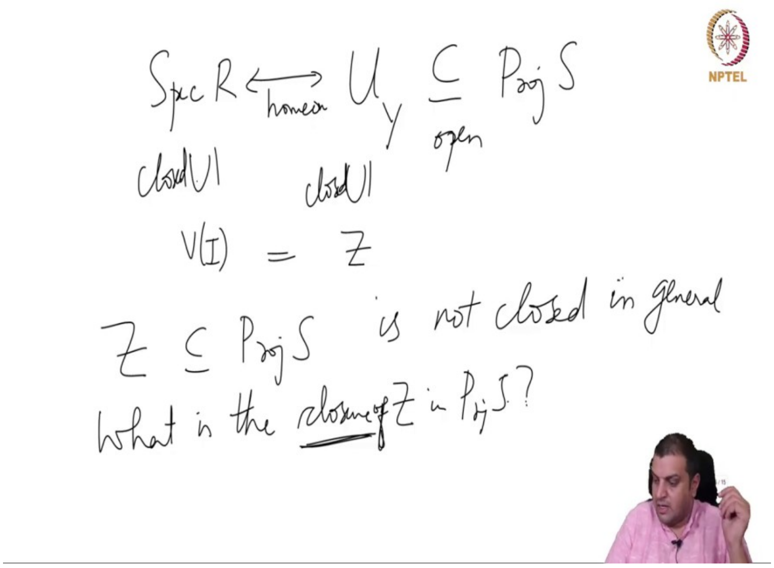
(Refer Slide Time: 25:53)



So, write $s_1 = Y^N s_2$, where Y does not divide s_2. So, then s' is same thing as s_1', the same thing as s_2', if you homogenize these things, because Y does not divide s. s is this ok, that is same thing as s_2 that is just prime. All that we have done is to just replace what is inside the tilde, but this is the same as s_2 because, Y does not divide s_2 also so s is equal to s_2.

In other words yeah $s = s_2$and the point here is that this is in that ideal. So, therefore, some $s = s_2$ therefore, $s Y^m$ is inside the ideal generated by the tildes of the f's. So, this proves the lemma. So, now, we will conclude that why is it that we are interested in this calculation altogether yeah. So, what is it that we are trying to do what we are trying to do is the following.

(Refer Slide Time: 27:48)

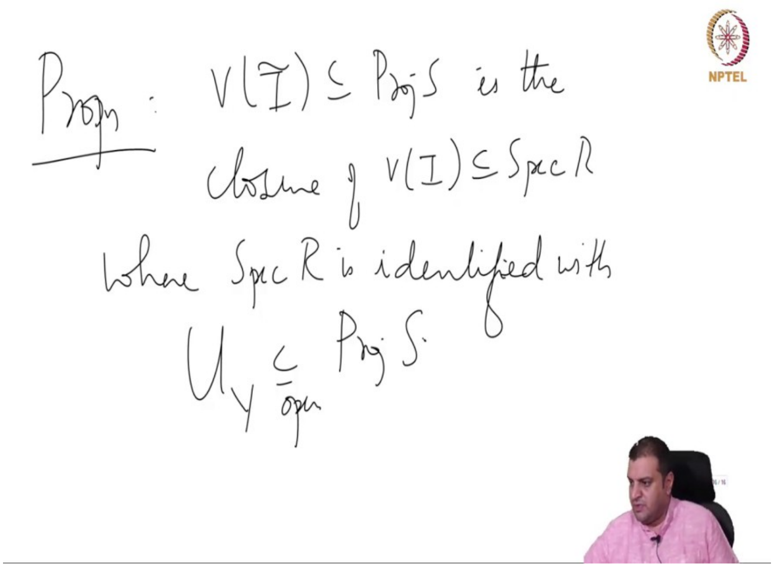


So, we had Spec R which we said is homeomorphic to U_Y, this is an open set inside Proj S. So, this is homeomorphic we did not prove it, but you know it is something. So, now, here we have $V(I)$ so, here abuse of notation there may be abuse of notation, but in this picture it will be clear. So, this is $V(I)$ here therefore, this defines a closed set inside U_Y, it is a closed set here.

So, this defines a closed set here this is open. So, let us call this Z for now. So, this is just Z here it is closed because, this is homeomorphism Z is closed in U_Y, but this is open. So, Z is not closed in Proj S. I mean, there are situations where it is closed, but in very specific cases. Z inside Proj S is not closed in general. There are only some very special cases where it will be closed. What is the closure of Z in Proj S?

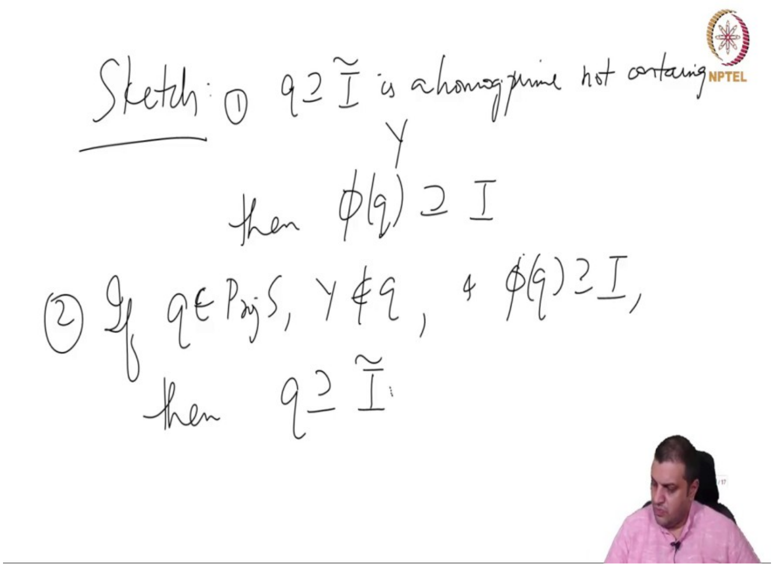
So, this is often referred to as the projective closure of $V(I)$ in that projectives in that topological space. So, this is the question what is the closure of Z? And the answer is it is given by that $\tilde{I}$.

(Refer Slide Time: 30:03)



So, here is a proposition $V(\tilde{I}) \subseteq Proj\, S$ is the closure of $V(I) \subseteq Spec\, R$ V I inside Spec R, when Spec R is identified with an open set U_Y inside Proj S. So, this is the proposition, we the proof is just applications of the previous lemmas. So, what we need is that.

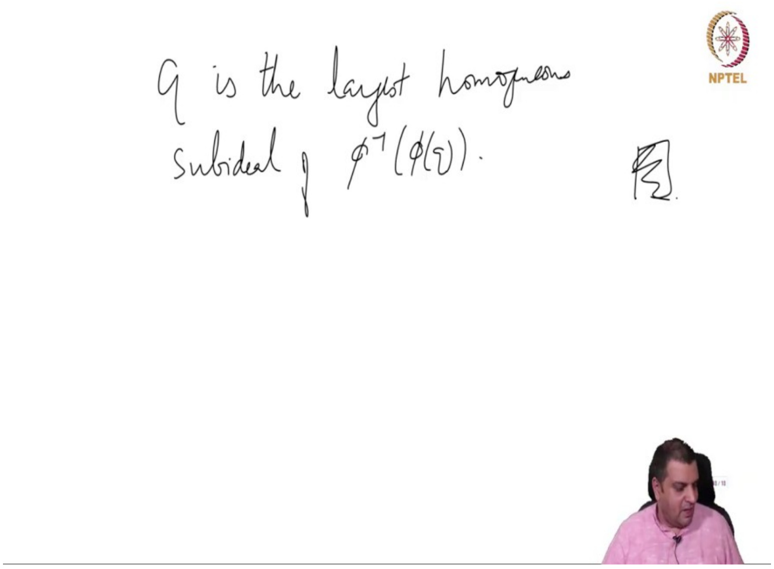
(Refer Slide Time: 30:56)



So, I will only just sketch it because, I would like to show an example of this the issue that you have to worry about in this case. So, if $q \supseteq \tilde{I}$ is a homogeneous prime not containing Y, then $\phi(q)$ which is the dehomogenization of q contains I that is 1. check. And the other statement is if q is a prime with the same property, such that this is true then q. So, this is 1 we have to check.

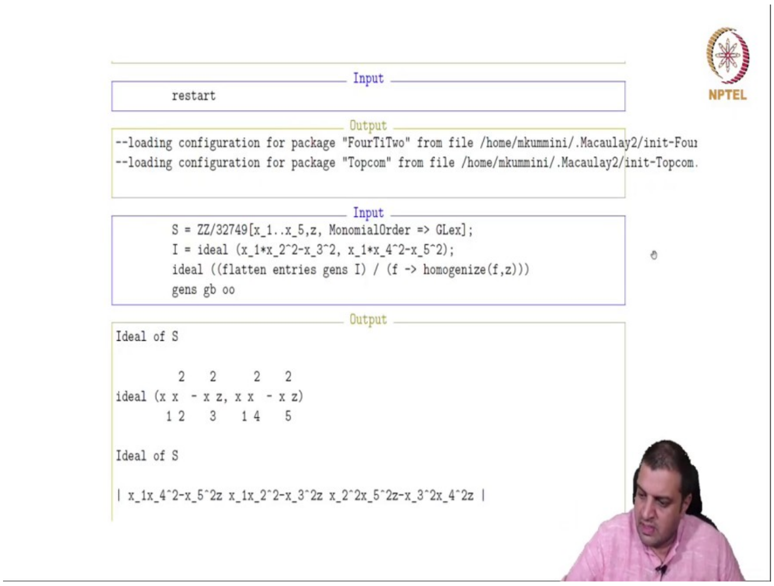
Second statement if q is in Proj S, $Y \notin q$ this is the same as this. And $\phi(q) \supseteq I$, then q contains $\tilde{I}$ this is the second thing that we have to check. And for this we have to check the basic thing that we have to check is that.

(Refer Slide Time: 32:31)



So, to do this we need to check that q is the largest homogeneous sub ideal $\phi^{-1}(\phi(q))$. So, if you check these things then this proposition will be proved. So, I want to show an example here, using Macaulay and to see to show why the saturation is necessary, and in fact, how it can easily be done without proof.

(Refer Slide Time: 32:10)



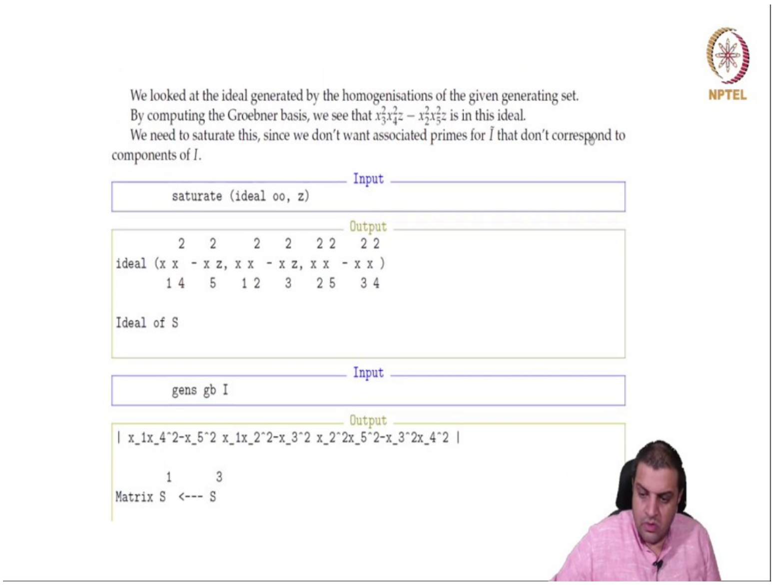
So, this is the example. You can ignore this. If you start your Macaulay session a fresh at this point, you can ignore this is to restart the Macaulay session, I wanted to forget what the x y z. So, now redefine everything that is just a restart you can ignore this input and this output, if you are going to start a fresh a new session.

So, S is the polynomial ring in 6 variables $x_1,\ldots,x_5$ and z and we use monomial order GLex. Because, we are going to see an issue. So, now let us take an ideal $(x_1x_2^2-x_3^2, x_1x_4^2-x_5^2)$. So, it is a cubic minus degree 2. Similarly degree cubic minus quartic cubic minus quadratic.

And then we ask the homogenize is the Macaulay command homogenize f with respect to z. And so, apply the function f goes to homogenize f goes to z to the list flatten entries gens I, gens I is a matrix entries will give a lists of that matrix and flatten will give just them in 1 list. So, take this list and then apply this function and then construct the ideal. So, we got this ideal, then we just ask for it to generate a Groebner basis and show its.

So, Groebner basis involves these two polynomials and the S polynomial between these two the multiply this by x_4^2, multiply this by x_2^2 and take the difference, and that is inside here. So, this is so, that is also and its leading term is not divisible by these. So, we need that also.

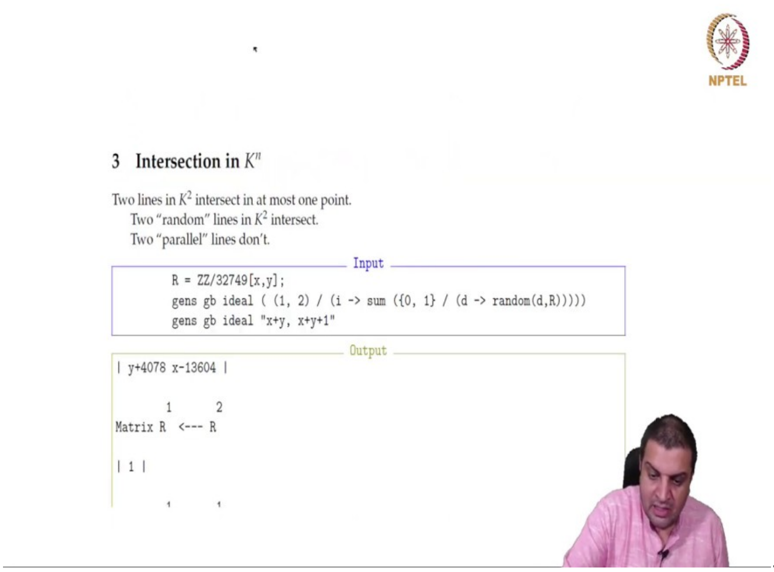
(Refer Slide Time: 35:32)



So, now let us homogenize, let us homogenize these things and at that let us saturate z. So, we got this also, but if you look at we already sorry this is just I this is repeated this command without ok.

So, the reason is that if you just take the homogenizations of the polynomials. Then this polynomial $x_3^2 x_4^2 z - x_2 x_5^2 z$ is there in the ideal, which is just nothing other than just doing the S polynomial calculation for this pair.

(Refer Slide Time: 36:17)



For this it is nothing more than just doing the S polynomial calculation, even if you had a z here and a z here which was for the homogenization. The S pair cancellation would be the same and we would just get instead of this thing here, we would just get z times this that is what this thing here is.

So, we need to saturate this since we do not want associated primes that correspond to components of I. And so, we saturated that and we got this one and so, what really has done is we took a Grobner basis in GLelx for this one then we saturated that. So, this one when we saturated we would get a z here, that is what this is this one when we saturate we would get a z in for the second term $z x_3^2$, that is this term. And this is homogeneous. So, when you saturate it just comes when you homogenize it just comes as it is.

So, the idea is that if you take a GLex Groebner basis with this with the variable the homogenizing variable at the end, then you take a Groebner basis for that thing and just homogenize that we would get a Groebner basis for the homogenization. And we do have to worry about saturating I mean, this saturating the $\widetilde{f}_1, \ldots, \widetilde{f}_m$ if it is not a Groebner basi. So, this is the algorithm and so, this is how we translate problems about non graded things to graded things.

In the next lecture we will look at a couple more things about graded algebras, graded quotients of polynomial rings.