Computational Commutative Algebra

Prof. Manoj Kummini
Department of Mathematics
Chennai Mathematical Institute

Lecture – 52
Homogenization – Part 1

So, now this is lecture 52 and we are continuing with our discussion about degree.

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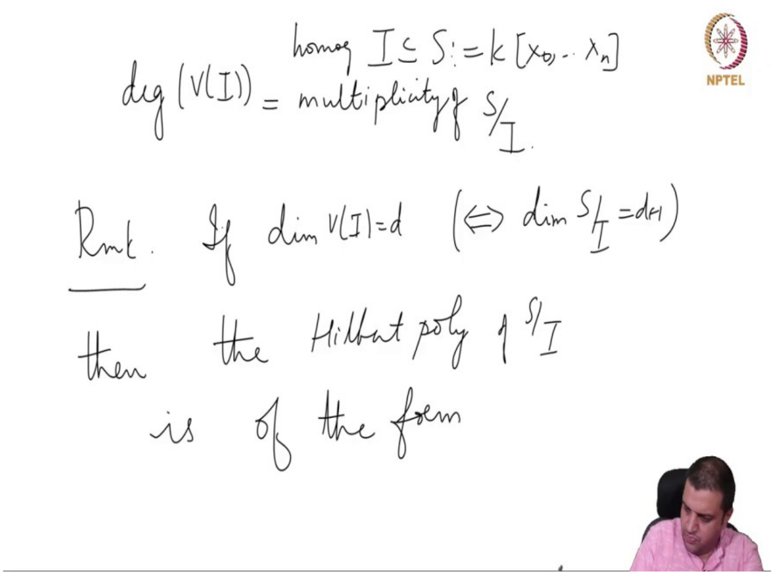

So, recall that degree of $V(I)$, I is a homogeneous ideal. In a polynomial ring $n+1$ variables so, this is homogeneous ideal. So, degree of $V(I)$ we defined this to be the multiplicity of $\frac{S}{I}$. And just a remark this is not really new result, if dimension of $V(I)=d$ or equivalently the dimension of $\frac{S}{I}$ is $d+1$.

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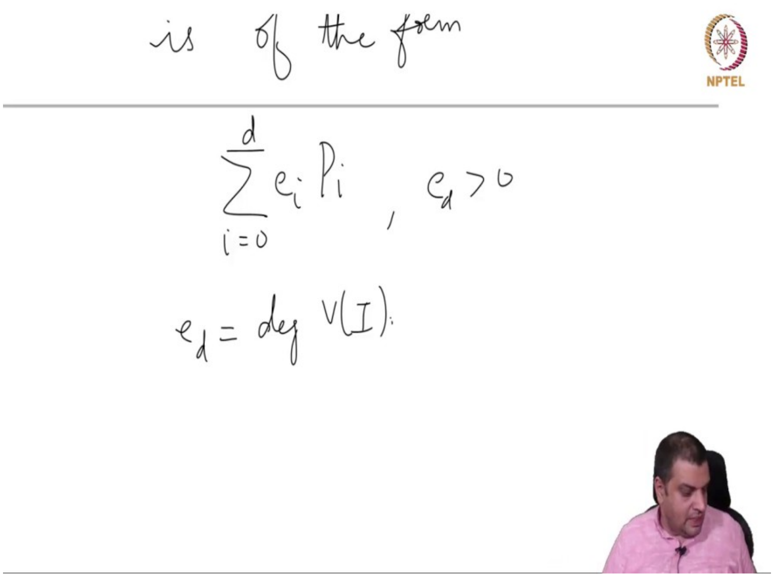

Then the Hilbert polynomial of $\frac{S}{I}$ is of the form $\sum_{i=0}^{d} e_i P_i$, d is 1 less than the dimension of the ring. And with e_d is not 0, e_d is actually positive these are integers. And so, then e_d is the degree of $V(I)$, it is just a remark that you should be familiar with.

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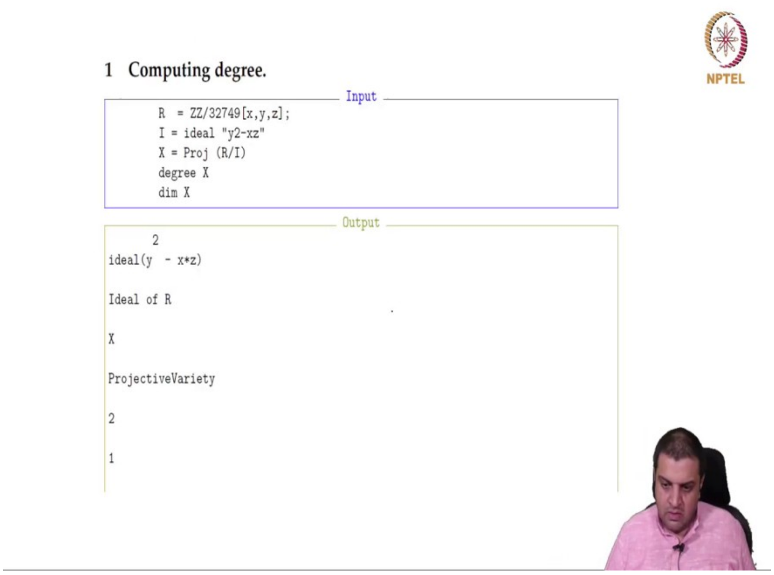

Let us look at couple of examples. So, here is an example of computing degree. So, we take a polynomial ring in some large characteristic, for these polynomials you know even if you take such a even if this is a finite field. But, one would still have enough linear forms at least the way we have set this up. So, we take the ideal $(y^2 - xz)$ and in Macaulay there is a command called Proj for a graded quotient ring.

So, we can just say X is Proj of $\frac{R}{I}$. So, let us look at this so, this is a three dimensional polynomial ring and in which we are going modulo one equation. So, this is just happens to be prime. But we are going modulo one equation, and then we know that the dimension drops exactly by 1.

This was a consequence of the dimension theorem Krull principal ideal theorem etc. So, dimension drops exactly by one therefore, $\frac{R}{I}$ has dimension 2. So, we ask questions to Macaulay.

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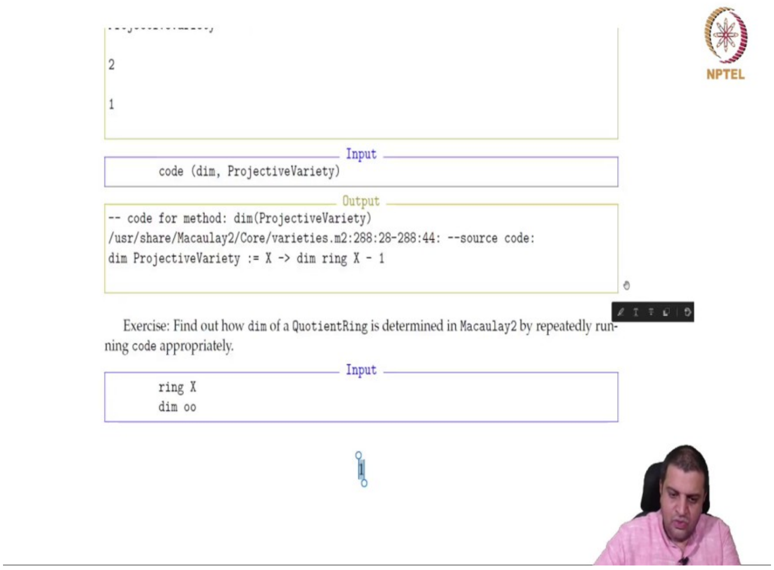

So, we ask for its degree. So, if we ask X is equal to Proj $\frac{R}{I}$ it would say X ProjectiveVariety. So, it is of type projective variety and you can ask for you know help or methods to figure out what are all things can you do with the projective variety. So, one command is degree of X. So, we ask for degree of X, it gives out 2 and dimension of X it gives out 1.

So, let us we will worry about the degree I mean afterwards, but let us just look at dimension.

$\frac{R}{I}$ is as I mentioned 2 dimensional, ambient polynomial ring is 3 dimensional, going modulo one equation so, we have 2 dimensional and then we have this. So, let us ask so, we can ask this question to Macaulay. How is it computing dimension of a ProjectiveVariety?

So, this is command called code. So, code and then we give the you know as a sequence we just give all of them. So, dim is a command and this is the type of the input to the command so, code. So, this is; so this is asking how what is the code to compute dim of a ProjectiveVariety?

So, it says dim of a ProjectiveVariety. So, this is a function. So, it is a function which takes X to the dimension of the ring of X to -1. So, this is how I mean we have been we have been discussing dimension of ProjectiveVariety.

So, this is not surprising at all and it is a good idea to sort of try to understand how something is computed at least so, that maybe we do not understand, but at least as much as we can understand we can see that the computations are correct. You know Macaulay has been around for some time and all of this code is public.

So, it is if there are mistakes in these people would have noticed it, but you know. So, let us ask, so, now, you should do this yourself, how is the dim of a quotient ring computed. So, you run you give code you know in brackets dim comma quotient ring, it will tell you how that is computed and then sort of you will do it multiple times repeatedly to figure out how it is going. So, as an exercise do this.

But, let us ask I mean what these quantities are let us ask for ring of X, ring of X is this then ask for its dimension ring of X and then dim oo is just the dimension of the previous output.

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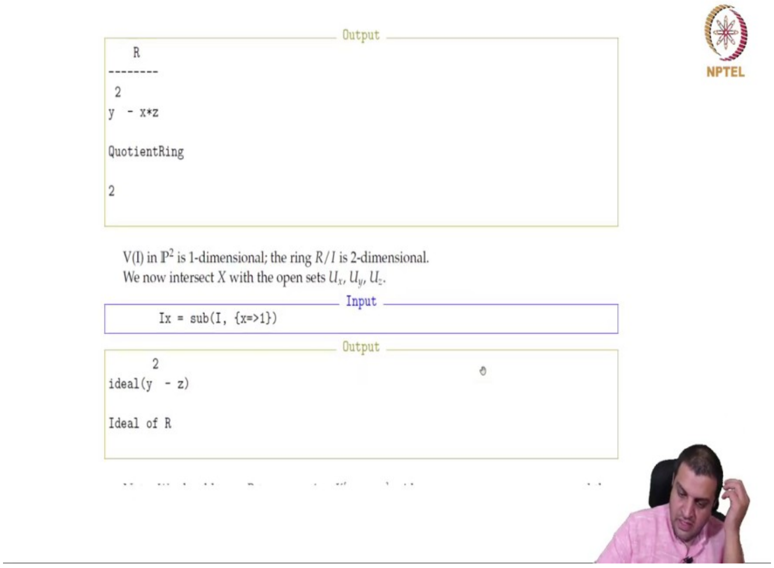

So, we ask for dim of X. $\frac{R}{y^2 - xz}$, I mean we knew this, but its good idea to run these commands and check and it is of type quotient ring. And we ask for its dimension it is 2. So, I mean this dimension of the ProjectiveVariety is computed exactly as we were discussing in the lecture.

So, the ring has 2 dimensional and $V(I) \subseteq P^2$ is 1 dimensional. Now, let us see what this intersection with open sets looks like.

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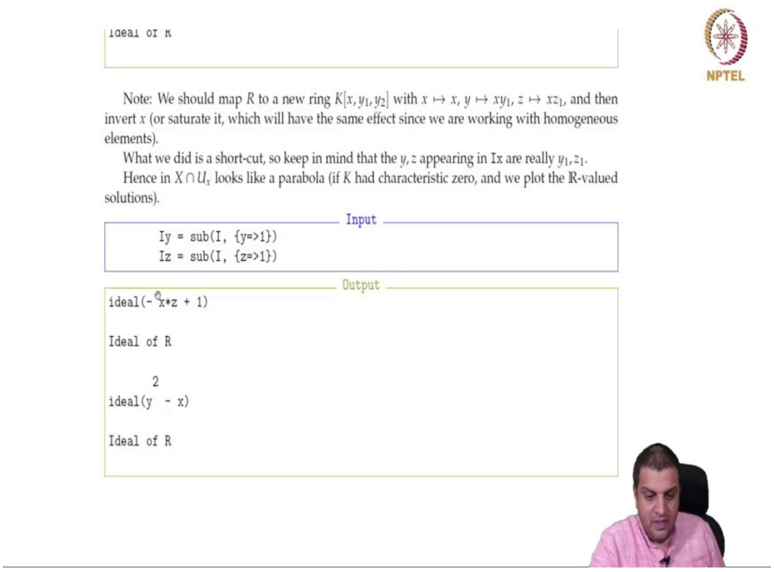

So this is a little trick what we really need is we should map the polynomial ring to a new ring. So, it should be mapped to a polynomial ring in three variables x, y_1, z_1. y_1 is $\frac{y}{x}$, z_1 is $\frac{z}{x}$.

So, with this ring map if you extend I and then invert x, then we would just get a polynomial which involves constants, y_1 and z_1 because it is homogeneous. So, just think about this if you have a homogeneous polynomial and we make this substitution, homogeneous polynomial x,y,z. And y we substitute as $x y_1$ and z we substitute as $x z_1$.

Then that whole polynomial will now be x to the common degree the homogeneous degree times, some polynomial which does not involve x at all which involves just constants y_1 and z_1. So, that is what you have to do, but the cheap trick is just substitute x to 1. So, this is just I have just a trick it just works. So, Ix we just substitute as x equals 1. So, remember this is the equation.

So, if you do this correct operation we would just get $y_1^2 - z_1$ but just call y_1 and z_1 as y and z we will just got this. So, on that open set which is like Spec k[y,z], this x is defined by $y^2 - z$. So, in real picture it is a parabola. So, what we have done here is just a shortcut the y and z that we see here are really this y_1 and z_1 so, keep that in mind. So, $X \cap U_X$ looks like a parabola, I mean assuming that we are plotting the real values.

But, it is only to keep a picture in mind. And we can similarly do Iy in which we divide the polynomial by y and then remove that part. And so, then it would be $1 - xz$. So, it is just like setting y to 1.

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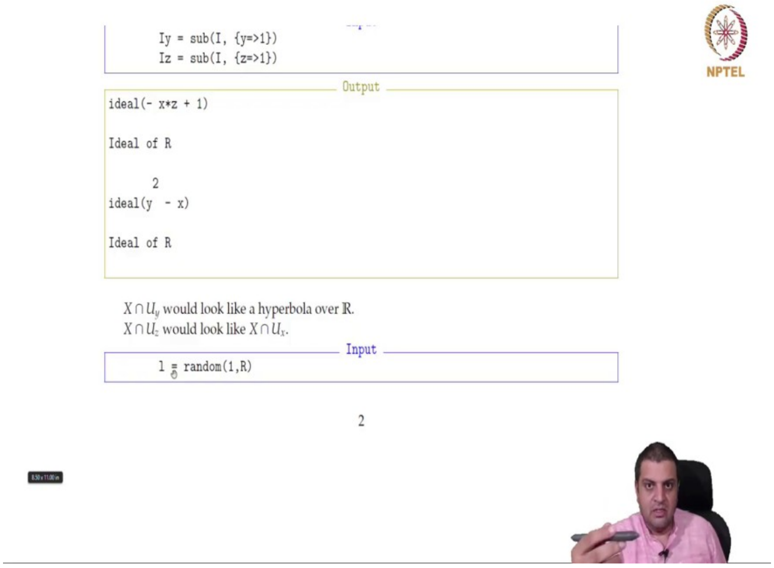

Again remember that this x and z are not the x and z of R this is really the ratios of x and z to y i.e. $\frac{x}{y}$ and $\frac{z}{y}$. So, that is the so, it is like a overall its looks like an hyperbola. And in this is the case of z should be same as I same of same as x because, this it is not I mean x and z are interchangeable in the way, it is described. So, X this looks like U_X.

So, now we what we would like is to go modulo, this is a 1 dimensional set we saw that topological. We saw that if we can go modulo one equation and then check what happens.

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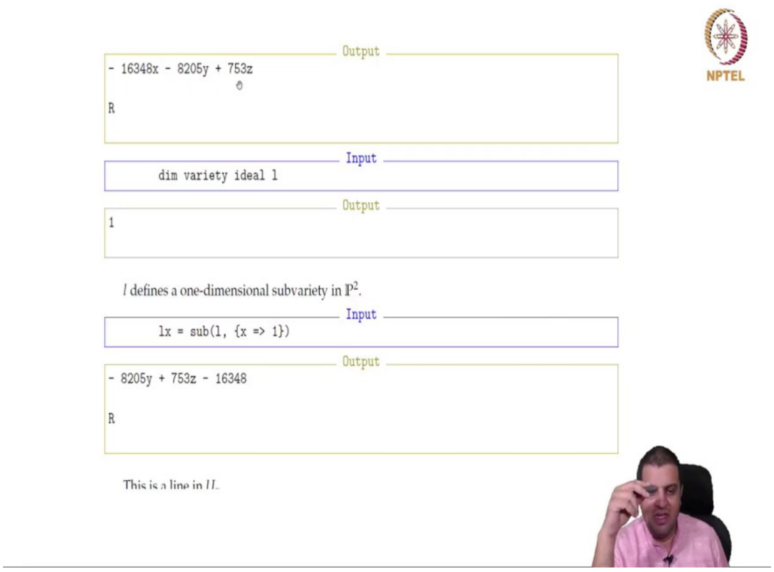

So, I took a random polynomial and as we talked about it earlier, if the field is large enough then any I mean theorems for which we are look for general elements will probably work, if you run the random command. Again this is just a belief there is no proof, but if you run the random command enough times the result that you that, we would agree with what the general linear form argument would give.

So, now, we can ask dimension of the variety of ideal of l, we could have just ask ProjectiveVariety also. If a homogeneous ideal variety command will produce a ProjectiveVariety and it says it is 1, again in P^2 we go modulo one equation. So, the 3 variables became now, 2 and its Proj is one dimensional. So, that is what we have here.

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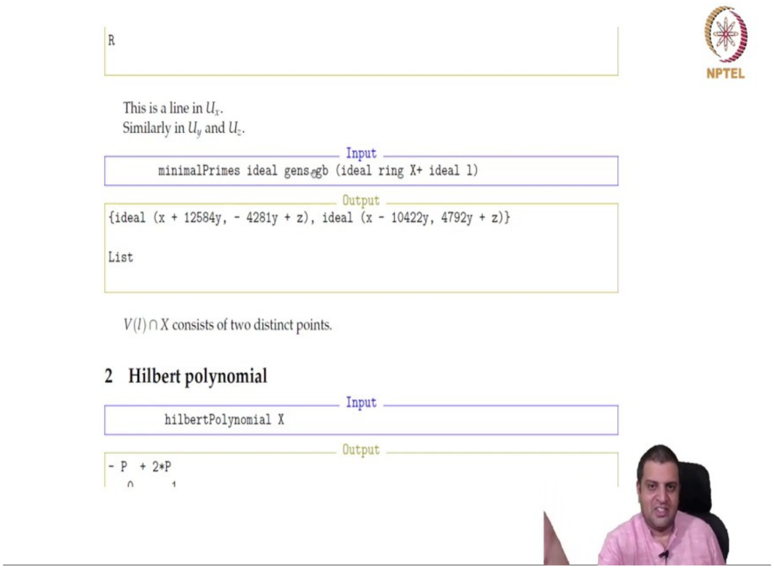

And let us look at why is it called a linear form, well look its if you do this here inside P^2 it looks like a line. This is some linear equation whose solution would be y. So, this is a line in U_X and similarly by looking at this if you invert y if you invert z any of them, it will all look like lines.

Now, let us ask if you take minimalPrimes. So, this is slightly longest command ideal of the ring of x, which is (y^2-xz) plus ideal of l which is the general form that we just constructed here on top here this one. So, if that so then let us take a Grobner basis, because if we just write the sum of these two ideals it will just put the generators together, it would not tell us it would not sort of simplify it for us.

So, we ask for a Grobner basis then we ask for its generators and so, we want to convert a I mean we do not want Grobner basis the output of gb command is of type Grobner basis. We want an actual ideal, we do not want the extra structures associated for that we just do ideal gens gb. So, much of it tells us what this ideal is $y^2 - zx$, general linear form, that ideal is.

And then, we ask for its minimalPrimes, which as I said we have not discussed how that is computed, but at least we know that there is a command which will compute it. And it shows us two ideals, if we notice both of these are homogeneous linear forms. So, let us look at this ideal inside P^2 this will cut down dimension by 1, this is linearly independent from that.

So, it will cut the dimension once more by 1 and so, this is actually a point with multiplicity 1, there is no other there is no extra multiplicity here. Similarly the same argument these two forms are linearly independent. So, they will cut 2 they will cut dimension by 2. So, we started with P^2 I mean every chain of homogeneous primes at length 2 in it, excluding the maximal idea had length 2 in it and we cut two of them.

Now, it has length 0. So, this is also a point and its multiplicity is 1 because, if you write out this ideal and compute its Hilbert polynomial, we would just get P_0. So, this is the thing and so, $V(l) \cap X$, this is what we I had mentioned in the theorem consists of two distinct points. So, this is the theorem in the previous lecture.

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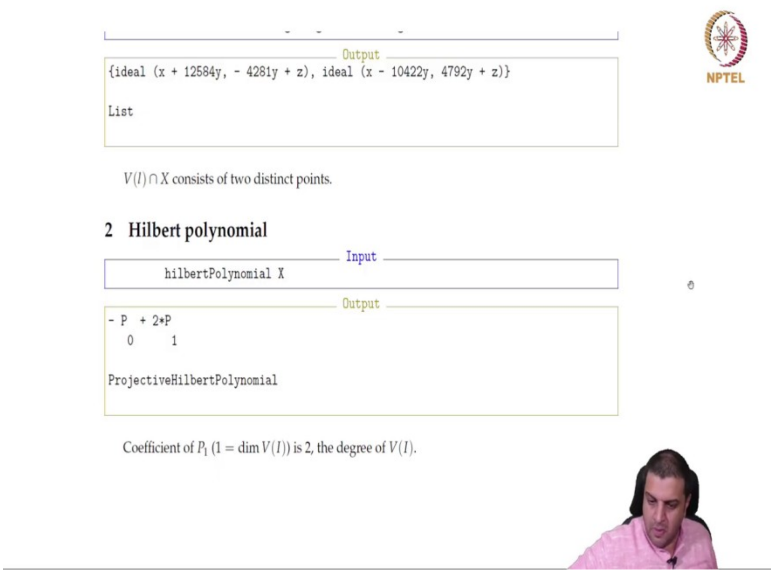

So, now, we look at Hilbert polynomial of X. So, it is $-P_0+2P_1$ and as we notice that its dimension is 1, this P_1 is 1 says its dimension is that and coefficient of P_1 is 2 which was a degree. So, if you just compute the Hilbert polynomial of a ProjectiveVariety in Macaulay, it would tell us what the degree and dimension is just this exponent here, is the degree is the dimension. And this sorry this coefficient 2 here is the degree. So, this is just preliminary things about computing degree.

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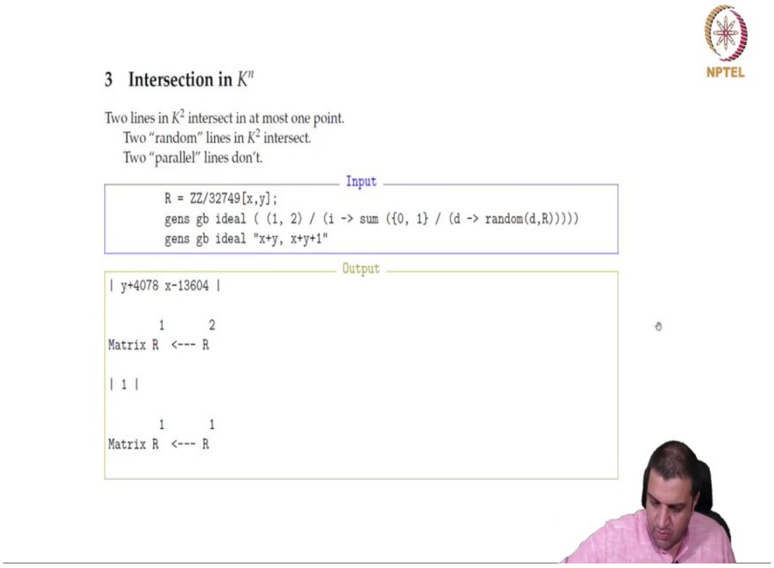

So, the next example we will discuss little later. So, now, we want to worry about why care about this homogeneous Proj and homogeneous ideals etc why not just work with Spec and k^n.

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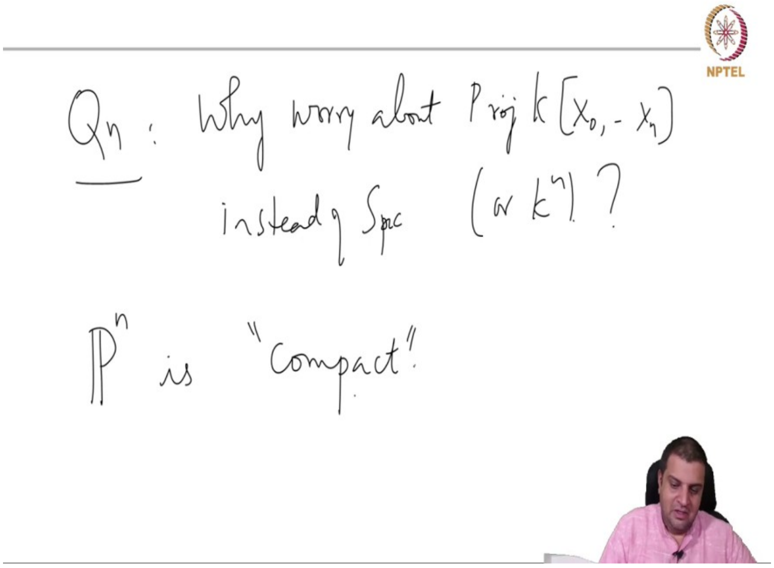

So, this is question why worry about Proj already its very complicated construction P^n, you know ring in n+1 variables. And you know instead of Spec or k^n, I mean equivalently if you are working over an algebraically closed field just k^n. Why not I mean why we do this, what is the advantage of doing graded things over non graded things.

So, this is a little I mean there is no direct justification for this. So, we will see a few things here and there in the next two to three lectures. So, the motivation is that P^n well this is not this is a very new statement is compact of course we have not that is not in a fine space. If you look at the risky topology that is also compact, it would satisfy the definition of compact that is not what we mean this thing.

So, I put that in quotes and so, this projective space or the Proj of a ring has of a polynomial ring over a field, has certain more desirable topological properties than A^n. We have as we said we have not discussed lot of the scheme theoretic issue, I mean any of the scheme theoretic issues behind this I we have been even defined what a scheme is.

So, this I will not attempt to justify this, but let us we have we it is possible, I mean it is likely that you have seen an example of this in in undergraduate complex analysis. So, let us if you have not seen it is just ignore this.

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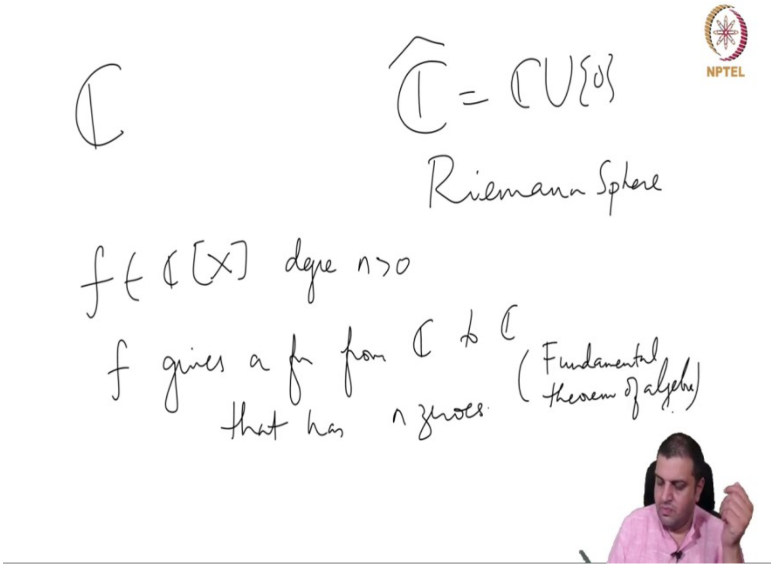

So, consider C and consider the Riemann sphere. So, which I will just denote by this is $C \cup \{0\}$ it is a one point compactification Riemann sphere. Suppose we take a polynomial in one variable. Then if we look at C well, it has no poles its you know its an entire function it is a holomorphic all over on C. And let us say degree is n and for simplicity let us assume it is not constant.

So, positive degree I mean it is true for constants also, but if we think of C as a function from f as a function from $C \to C$ it has n zeroes that is the fundamental theorem of algebra. f gives a function from $C \to C$ that has n zeros fundamental theorem of algebra. I mean some version of it saying polynomial splits into linear forms.

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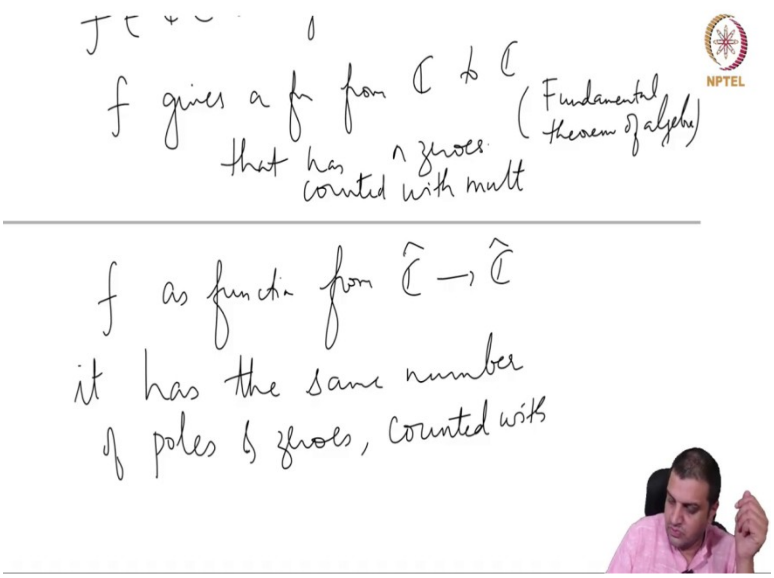

So, this is I mean, but, if you think of f as a function from the Riemann sphere to itself. Then it has the same number of zeros and the same number of poles. Because, poles I mean well infinity is a pole of order degree of n.

So, it is same number of poles counted with multiplicity poles and zeros, counted with multiplicity. Again when we say it has n zeros this is counted with multiplicity, I mean it is possible that f is just some $(x-1)^2$ counted with multiplicity.

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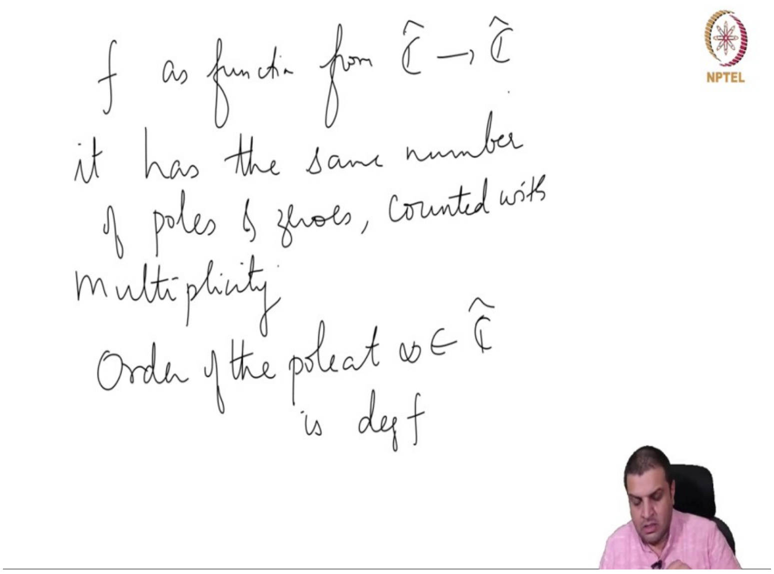

I mean if you have a polynomial of degree n around the point infinity inside here, it would look like I mean if you change coordinates, it would look like z^{-n}. So, therefore, its number of poles is equal to the degree that is the thing. So, that the point is order of the pole at infinity inside this point is degree of f.

So, there is some so, bringing in this point to make C the Riemann sphere, has this advantage that in functions for polynomial functions, it brings in a symmetry between poles and zeros.

Now, if you look at rational functions, then this is I mean therefore, this is true for all rational functions also. If you have a rational function $\frac{f}{g}$, well the poles and for both f and for g because, they are polynomials the statement is true. So, therefore, its true for $\frac{f}{g}$ also and we know.

So what is a pole for f would be a 0 for $\frac{1}{f}$ and so on, if you look at if you work in this context. So, it brings in some extra structure or some symmetry that can possibly be used. So, this is one say let us say one motivation, it is not without actually proving what without actually making sense of what this quote unquote compact is and proving the statement. I would not be able to make sense of it any further, but and I will not try to attempt that in this course.

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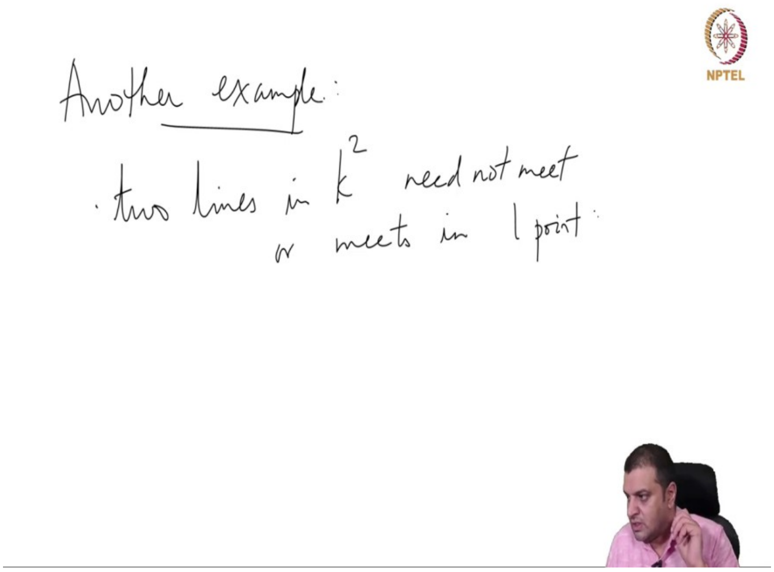

So, another point, so, this is one thing another example I mean why going for this bigger space is desirable, in fact, I again we will improve we would not prove, it we can think of C as Spec $C[X]$ and then $\hat{C}$ as Proj of $C[X,Y]$ this is ok. So, therefore, the passage from $C \to \hat{C}$ is exactly like the passage from Spec of a polynomial ring in 1 variable to Proj of a polynomial ring in 2 variables.

Remember, if there are $n+1$ variables in the corresponding affine piece there are only n variables. Corresponding piece where you look at the Spec there are only n variables. So, another example is if you take two lines in k^2 need not meet or meets in 1 point.

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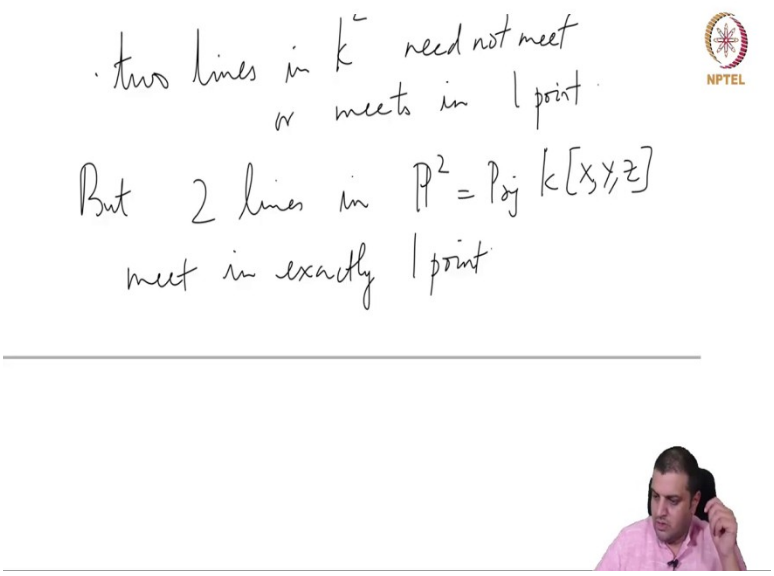

So, this is but if you take two lines in P^2, but 2 lines in P^2, which we said is Proj of $k[x,y,z]$ always meet in exactly 1 point.

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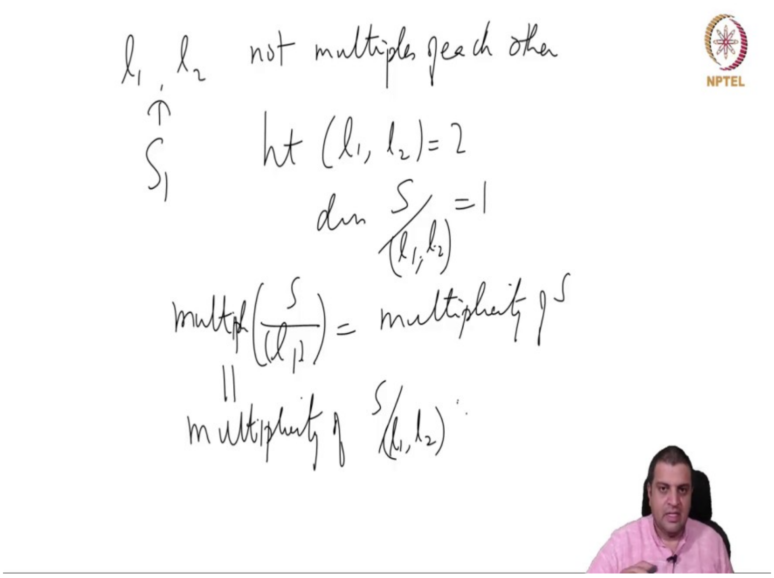

And why is that? If you take so, let us $l_1, l_2 \in S_1$ not multiples of each other. Then height of the ideal (l_1, l_2) is 2. So, therefore, $dim \frac{S}{(l_1, l_2)} = 1$. So, it gives a point and more importantly multiplicity of $\frac{S}{(l_1, l_2)}$. So, if you go modulo 1 element, this is the same as multiplicity of S that is because going modulo 1 non zero divisor does not change multiplicity. I mean we used it in the other direction to prove something about the numerator of the Hilbert series, but this I mean the quality is what we want. And this is the multiplicity of $\frac{S}{(l_1, l_2)}$.

So they do intersect with one point without with no multiplicity one simple point. So, two points in projective space two lines in projective space always intersect while two lines in k^2 need not intersect. So, there are these nice properties that one has when one works with examples like, I mean within projective in the graded situation.

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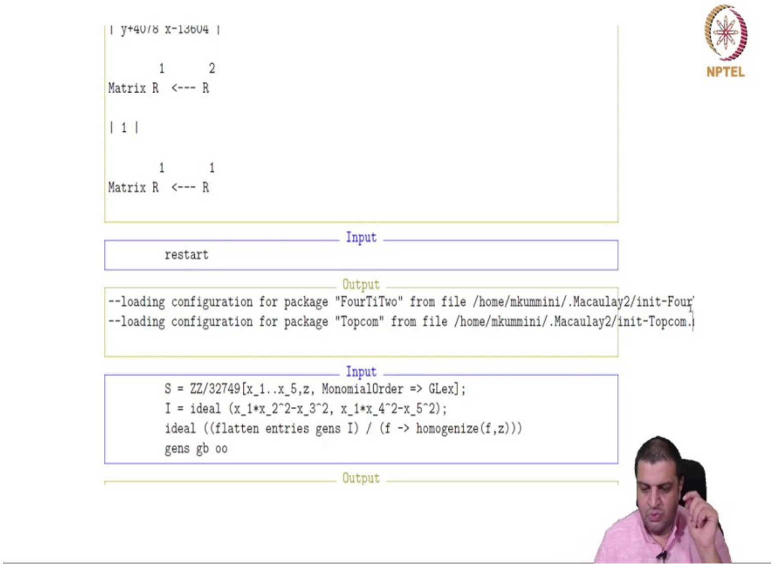

So, let us look at this Macaulay code. So, this is just restating whatever we did just to get. So, two lines in k^2, two random lines in k^2 will intersect. So, what have we done? So, here so it is a long way of just generating two random elements. So, what have we done? So, forget about gens gb ideal what we are concerned about is what is this; what is this thing between this left most bracket and right most bracket do.

So, here is a sequence 1,2, now apply to every element in this so, this sequence slash a function is just shorthand for the apply function. So, apply to every element of this the function, which takes i to the sum of some things. Now, what are those things take the list 0,1 and to this list apply the function d. So, d is either 0 or 1. I mean d is first 0 and then 1 a random element of degree d in R.

So, what it is saying is for each i pick first random element of degree 0 and then a random element of degree 1 take this sum. So, this is just an inhomogeneous polynomial of degree at most 1. So, that is what this sum open bracket corresponding close bracket gives and do this for $i=1$ and $i=2$ meaning to generate 2 inhomogeneous linear polynomials, I mean linear polynomials with constant term.

So, and then we asked to simplify the expression so, gens gb ideal. So, then it says 2 so, this thing I mean that ideal has a solution $y+4078=0$ and x minus this number is 0. So, in other words 13604,4078 is a solution to this equation. So, these two line, I mean the lines that we are randomly generated they do meet.

So, just understand it and parallel lines it is very easy just like $x+y$ and $x+y+1$ and clearly, if you ask for is I think 1 is a I mean 1 is in the ideal. So, this is not going to meet anywhere. So, that is what we mean? So, this has to do with the next topic so, which we will do in the next lecture.