Computational Commutative Algebra
Prof. Manoj Kummini
Department of Mathematics
Chennai Mathematical Institute

Lecture – 51
Proj of a graded Ring

Welcome. In this we look at the relevance of graded ideals in geometry it is through a construction called Proj of a graded ring and so, we will only discuss the very basics just enough to motivate why these computations with graded ideals are useful. If you want to look at more understand more about these constructions, then you should read a book on scheme theory or for example, Hartshorne chapter 2 or any book of that sort which covers that sort of material.

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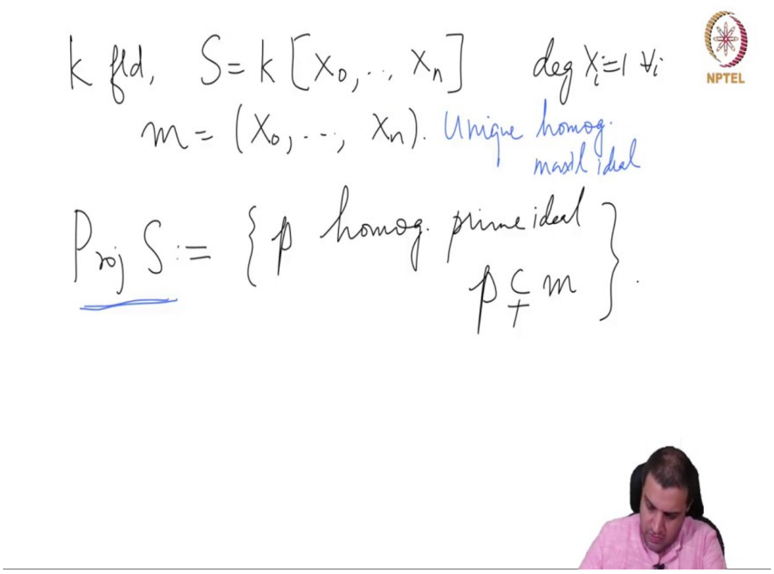

So, for us k would be a field. So, this definition who is restricted to the case where k is a field, the general definition is a little bit different, but we will just keep it simple for us. So, S is a polynomial ring $S=k[X_0,\ldots,X_n]$ and degree of X_i is 1 for all i and $m=(X_0,\ldots,X_n)$ is a homogeneous maximal ideal. So, this is the unique homogeneous maximal ideal and we define Proj of S to be this set $Proj\,S=\{p\, homogeneous\; prime\; ideal\; p\subsetneq m\}$ and we will give it a topology in a minute.

So, these are homogeneous prime ideals strictly contained inside the homogeneous maximal ideal, in the general case this would just say m is not inside p, but this is the for us this is its equivalent to this. So, we will just keep this as our definition.

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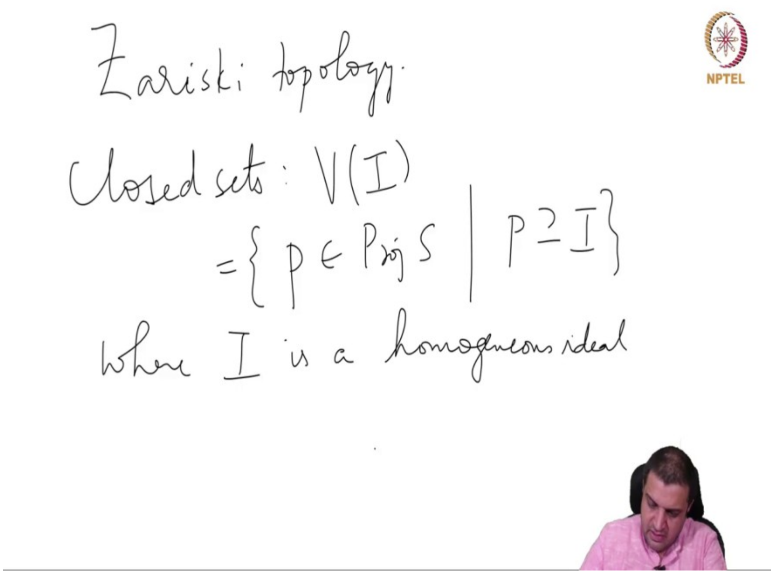

So how we give it the Zariski topology? So, close sets. So, here some abuse of notation I do not want to introduce some new notation just for this one. So, here when we mean $V(I)$, where I is a homogeneous ideal of S then we look at the elements $\{P \in Proj\, S | P \supseteq I\}$.

So, that contain I. So, all the ideals in this page here are homogeneous and p is a prime ideal not equal to the homogeneous maximal ideal. So, this is Zariski topology.

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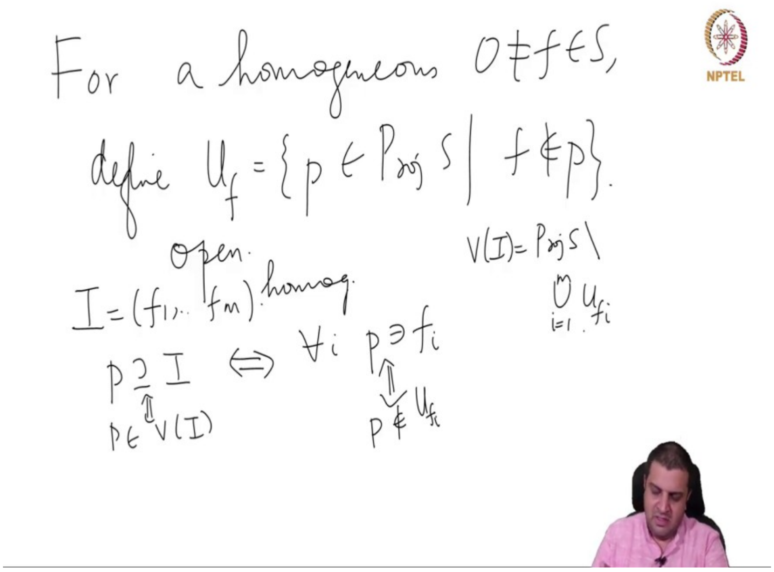

So, for a homogeneous non-zero element f we can define $U_f = \{P \in Proj\, S \mid f \notin p\}$. So, this is the complement of $V((f))$. So, these are open and these will be the basic open subsets in this topology. So, suppose that I is generated by some $f_1, .., f_m$ a homogeneous. Then, if $P \supseteq I$ if and only if $\forall i, f_i \in P$ this is the condition that P is inside $V(I)$ homogeneous prime.

So, the P is also homogeneous in this equation and this is the condition that $P \notin U_{f_i}$. So, it says that $P \in V(I)$ if and only if $P \notin U_{f_i}$ for all i. So, then this would say that $V(I) = Proj\, S \setminus \bigcup_{i=1}^{m} U_{f_i}$. So, every closed set or in other words the open set $Proj\, S \setminus V(I)$ is a union of these things.

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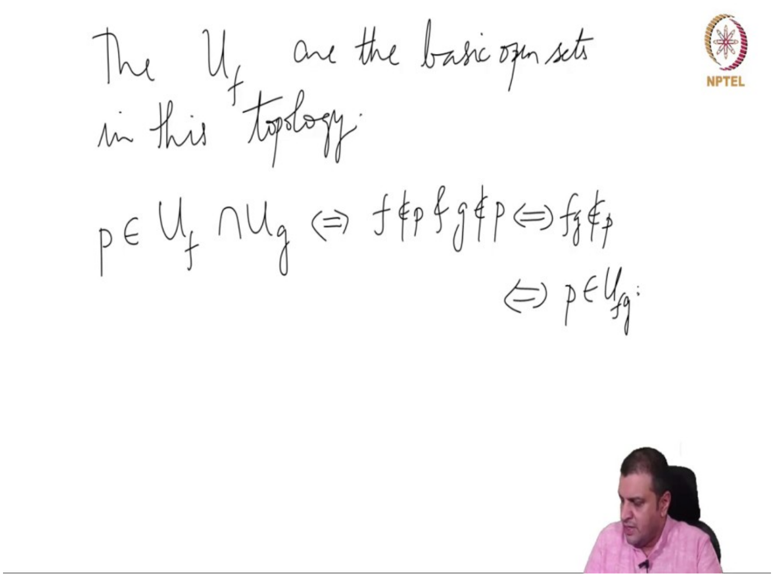

So, this is we also need to check that the U_{f_i} are meant to be basic open sets in the topology we need to check one more thing which is that $U_f \cap U_g$ is contains some open set of the same kind the basic open set.

So, let us ask this. So, $P \in U_f \cap U_g$ if and only if $f \notin P$ and $g \notin P$ if and only if the product $fg \notin P$ this is where its prime is used I mean, it is used in these definitions also but you know because it is prime, but this is the same thing as U_{fg}.

So, therefore, this base intersection of two open basic open sets is itself is a basic open set. So, then it satisfies the requirements to be a basic collection of basic open sets. So, this is Zariski topology on Proj S and. So, there is a relation between spec of a polynomial ring in n variables and Proj of polynomial ring in n+1 variables and this is what it is and we will explore this in the next few lecture.

We will try to understand this carefully in the next few lectures in not in the immediate one, but some lecture later.

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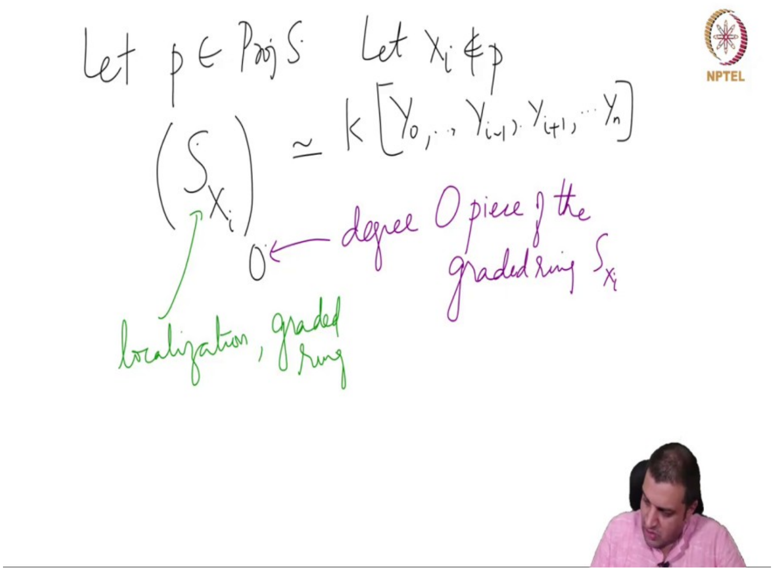

So, let $P \in Proj\, S$. So, it is a homogeneous prime not equal to the maximal ideal then let $X_i \notin P$. If every X is in P then P has to be m the homogeneous maximal ideal. So, there is some i with this property. So, now, take S localize at X_i this is just localize it. Now, this is a graded ring and take its degree 0 piece.

So let us just write down this is the localization just inverting X_i it is a graded ring and this whole thing is is just taking the degree 0 piece of that graded ring. So, this is the degree 0 piece of the graded ring S_{X_i}.

So, now take this one. So, that is the the relation this is isomorphic to k adjoint some new n variables $k[Y_0, \ldots, Y_{i-1}, Y_{i+1}, \ldots, Y_n]$ where $Y_j = \frac{X_j}{X_i}$.

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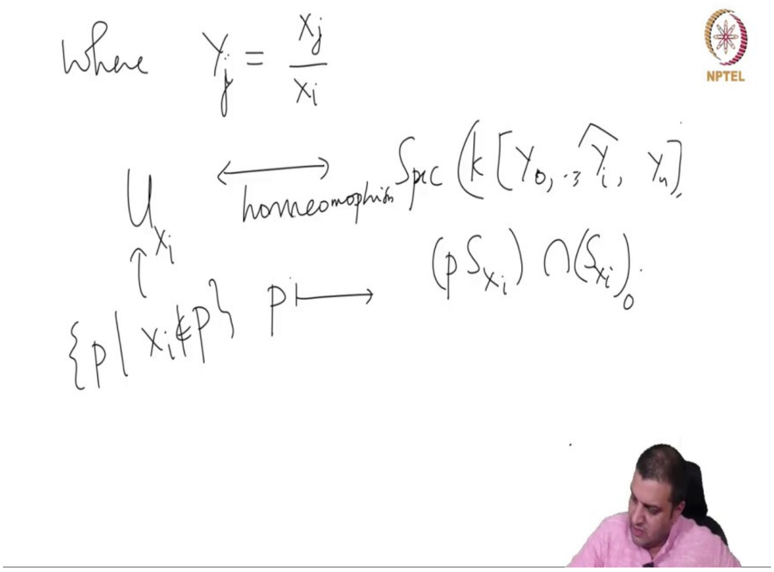

So, this one can check. So, this is what this is and. So, we have now two topological spaces we have U_{X_i} which was all the homogeneous prime ideals not containing X_i. So, this is P such that P in Proj S not containing X_i and on this side we have some polynomial ring.

So, I will just write for simplicity $\widehat{Y}_i$ with the hat to denote that has been omitted. So, there is only although the list has only n+1 and i th has been omitted. So, it is just this. So, there is a spec of this ring. So, there is a homeomorphism between these two. So, these things I have not proved because proving them will take sort of outside I mean it will take a little while.

So, you can look at you know book in scheme theory to understand this is mostly to motivate why you know I mean this part is mostly to motivate how to switch from the non graded case to the graded case we will see later and then also that there are certain advantages working in the graded case.

So, this is what it is and what exact what is the map one could look. So, this is the map one takes P here. So, in that one direction one can give the map one takes P here and then P goes to well first we inverted X_i. So, this is a prime ideal in S_{X_i} and then contracted it to the smaller sub ring S_{X_i} in degree 0 and that is that would give this map.

So, prime ideal corresponds to invert X_i and then just look at its degree 0 piece. So, this gives a this is a homeomorphism as we know. So, we will use that fact perhaps without I mean without any further explanation.

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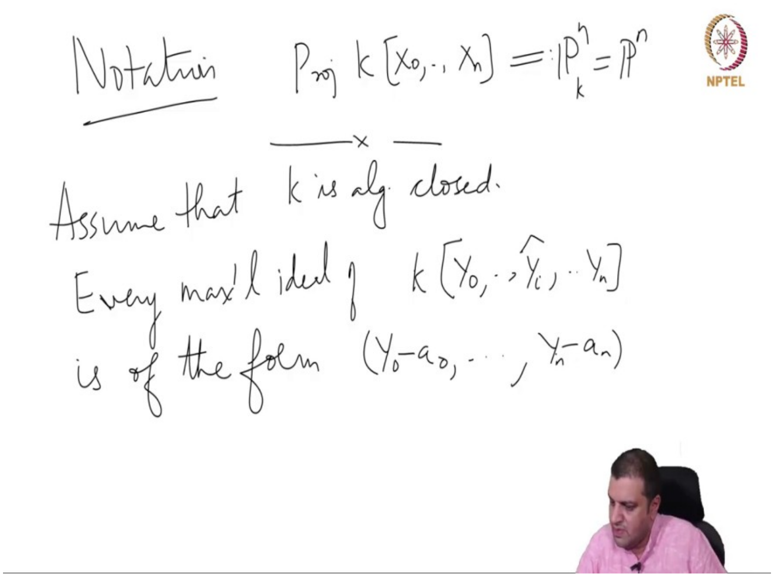

So, notation; $Proj\, K[X_0,\ldots,X_n]$ will be denoted as P_k^n and this we will just call it P^n we are not going to work with multiple fields we will just call it P^n. So, one can just to get a some geometric intuition behind this thing now assume the k is algebraically closed then every maximal ideal sorry.

So, remember the identification between these two rings is that Y_i here is the ratio $\frac{X_i}{X_j}$. So, every maximal ideal of this ring $k[Y_0,..,\widehat{Y}_i,..,Y_n]$ is of the form $(Y_0-a_0,\ldots,Y_n-a_n)$ there is no i-th term there is just n terms inside this here. Now we can go I mean work backwards through this and see what P is.

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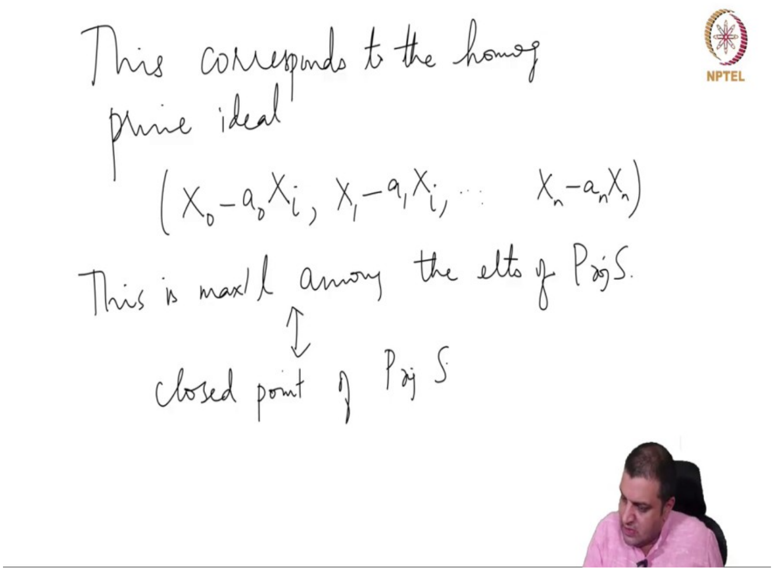

So, this corresponds to the homogeneous prime ideal $(X_0 - a_0 X_i, X_1 - a_1 X_i, \ldots, X_n - a_n X_i)$ and there will be no relation indexed by i inside here. So, this still has only n terms in it not $n+1$ and so, this is a prime ideal that one can check.

So, this is a ring. So, this is maximal among the elements of Proj S. I mean this is not a maximal ideal in S, but if you consider homogeneous prime ideals which are not equal to homogeneous maximal ideal this is in that collection this is maximal. So, in other words this gives a closed point of Proj S. I mean there is in the closure of the singleton set there is nothing else; so, just a closed point. So, this is what we mean.

So, let us go back to. So, this is actually what would be a point inside Proj S and we know in this ring this actually did correspond to a point.

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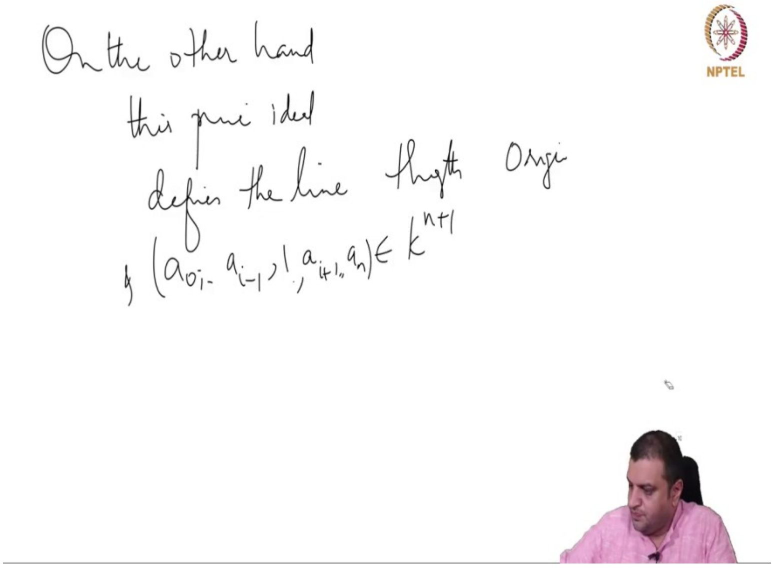

But on the other hand this prime ideal defines a line with these slopes. So, its a line in k^{n+1} with these slopes $\frac{X_0}{X_i}$ slope was a_0 and $\frac{X_1}{X_0}$ slope was a_1 and so on. So, we can write it defines a line or defines the line through the origin and $(a_0,\ldots,a_{i-1},1,a_{i+1},\ldots,a_n)\in k^{n+1}$.

It is the line through this point and this says that the those ratios the slopes are, X_0 is $a_0 X_i$ and so on. X_j is $a_j X_i$ for j different from i. So, it corresponds to the line through the origin and this point inside k^{n+1} that is. So, let us just quickly check without actually I will not write it.

So, if you have a point that satisfies these equations then well to actually identify the point all that we have to give is just to specify what the value of X_i is. If the value of X_i is non zero then it will be a multiple of this point if the value of X_i is 0 then this point will be the origin that is the only solution. So, this defines a line through this origin and. So, what we have done is. So, any multiple any non zero scalar multiple of this is also the same point in the projective space it corresponds to the same ideal.

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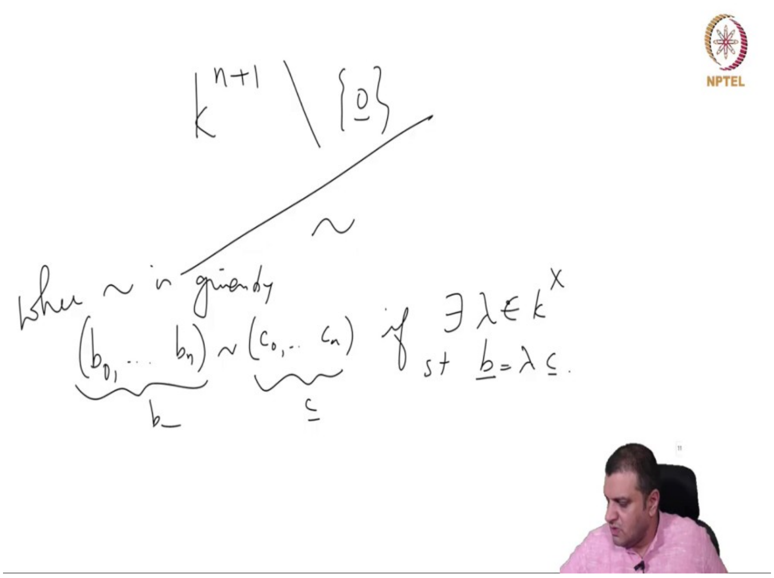

So, what we have done really is constructed topological space with the following property we took k^{n+1}, when we took away the origin and on that we acted by the. So, let us say equivalence relation and what is the equivalence relation? $(b_0,\ldots,b_n)$ $(c_0,\ldots,c_n)$ if and only if $\exists\lambda\in k^{¿}$ such that $\underline{b}=\lambda\underline{c}$.

So, it is identify at the level of points its identifying things like this and, but you know the space already has some prime ideals.

So, its a little bit more difficult to describe it at that level, but for a picture one can keep this in our mind. So, the first observation that we would like to make is this is built from the previous lectures proposition dimension of $V(I)$.

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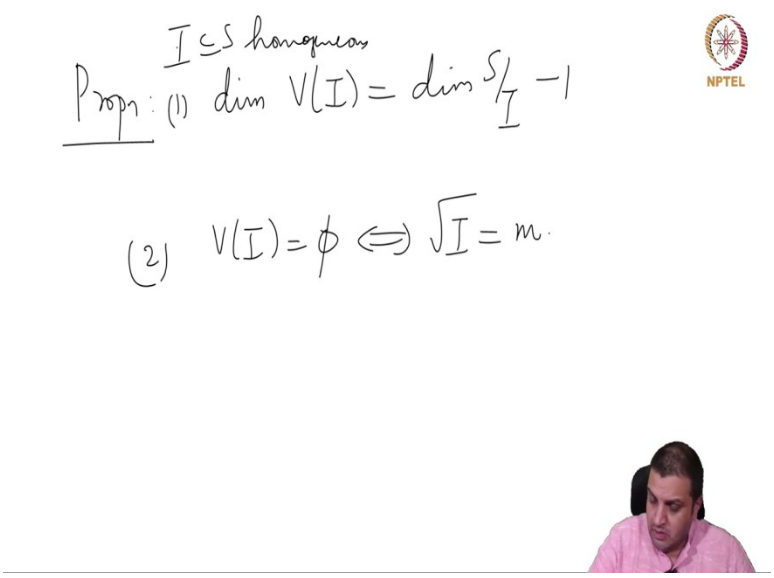

So, remember this is inside a Proj S not inside Spec S. So, $I \subseteq S$ homogeneous, $\dim V(I) = \dim \frac{S}{I} - 1$ and this is the first statement and 2, $V(I) = \phi$ if and only if $\sqrt{I} = m$ that is I is primary to the homogeneous maximal ideal in other words for every variable some power of the variable appears inside I.

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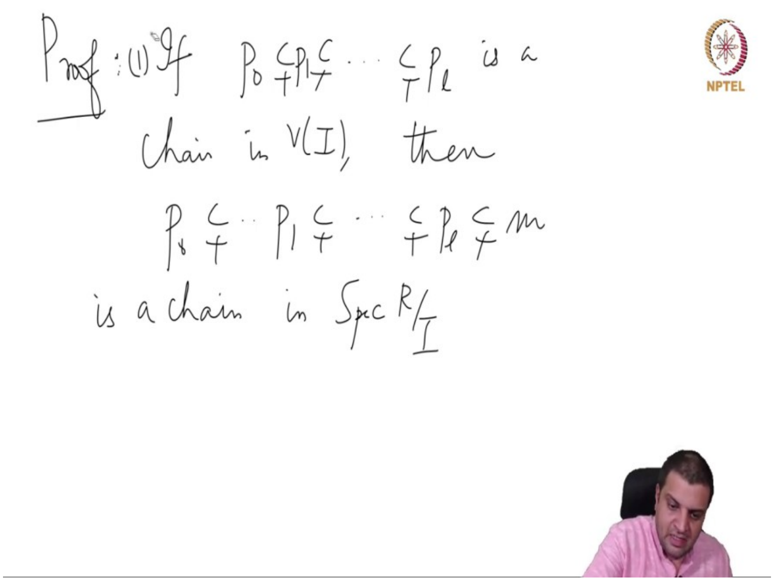

So, this is what we have. So, let us prove these things this is not very difficult. So, when we say dimension of $V(I)$ we mean a chain of primes inside $V(I)$. We talked about why the

Krull dimension for ring is same thing as the topological notion of dimension by filtration by irreducibles for topological spaces and it is the same thing here. So, if so, I will just sketch this part I mean I would if $P_0 \subsetneq P_1 \subsetneq \ldots \subsetneq P_l \subsetneq m$ is a chain in $V(I)$, but so, this is a homogeneous prime ideal which is not equal to the maximal ideal then $P_0 \subsetneq P_1 \subsetneq \ldots \subsetneq P_l \subsetneq m$ is a chain in Spec $\frac{R}{I}$.

Now, the dimension of the ring is determined by this value and dimension of $V(I)$ is determined by this. So, therefore, it is always at least one more and then if you have any chain like this then it is if this is a chain in $V(I)$ then this is a chain in I mean if this is a chain in Spec $\frac{R}{I}$, then this is a chain in $V(I)$. So, in fact, we can remove this if here and then this thing here then say if and only if.

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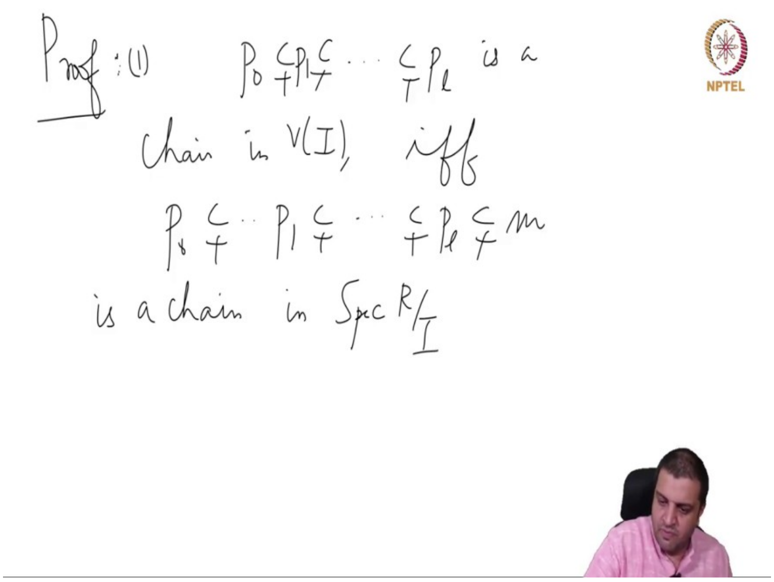

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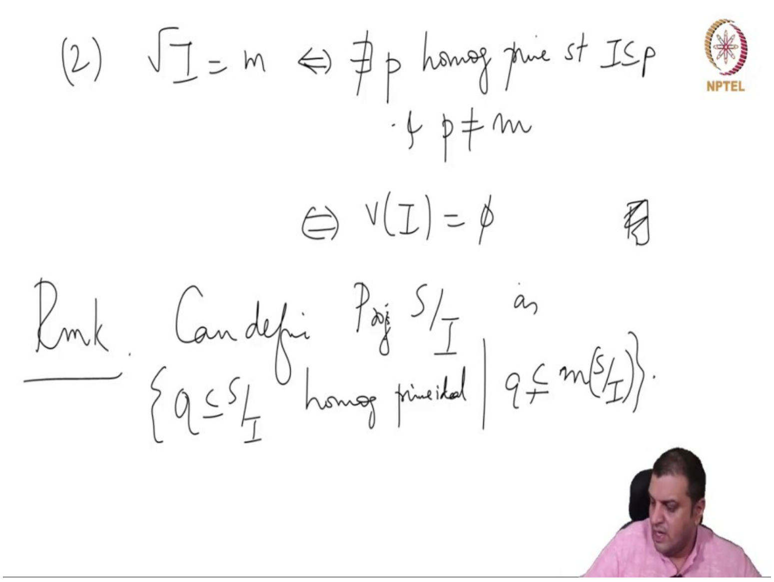

So, this is the; this is the proof of 1. And proof of 2 is that remember if. So, 2 said that if and only if $\sqrt{I}=m$. If $\sqrt{I}=m$ then this implies that there does not exist any P homogeneous prime such that $I \subseteq P$ and P is not m that is right that is because minimal primes is just m.

So, this will not be and this is this one implies that $V(I)$ is empty because any such P is what would make up $V(I)$ and if $V(I)=\phi$ then this is true and if there is not any prime ideal containing I and P is not equal to this, but remember I is homogeneous. So, the minimal primes over I are themselves homogeneous. So, that goes the other direction and so on. So, this is the proposition.

So, just to remark we can define Proj $\frac{S}{I}$ also same way in a similar way. Homogeneous ideal as the set $\{q \subseteq \frac{S}{I}$ homogeneous prime ideal ¿ $q \subsetneq m \frac{S}{I}\}$ which is the unique homogeneous maximal ideal of $\frac{S}{I}$.

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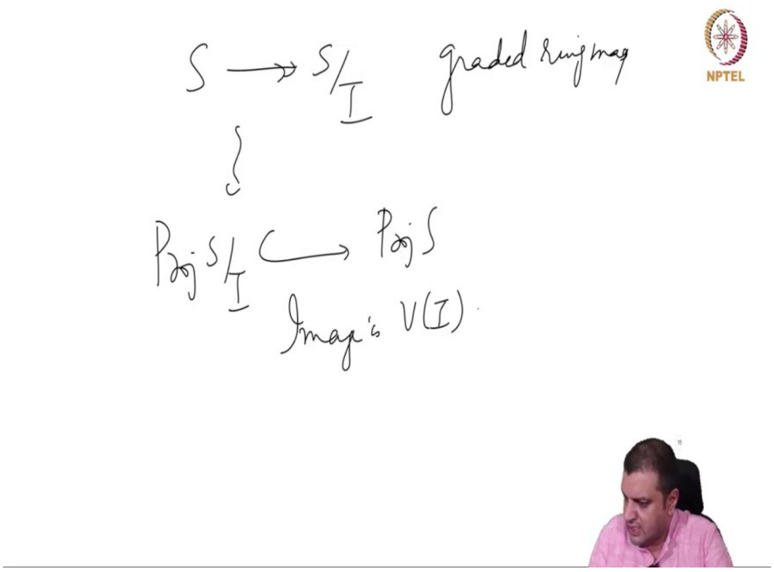

So, you can take this and then just as we did it for Spec we notice that, S surjects on to $\frac{S}{I}$, this is a graded ring map preserving degree etc then this one gives a Proj $\frac{S}{I}$ inside Proj S and image is exactly what we defined is $V(I)$. So, just warning it is not true that every graded map will give a map of Proj.

If you have just a map of rings it will give a map of Spec. Proj is not that straight forward there has to be some conditions, but there will be I mean in this particular case one can sort of explicitly verify that this surjective map induces this injection and then and this is precisely this is.

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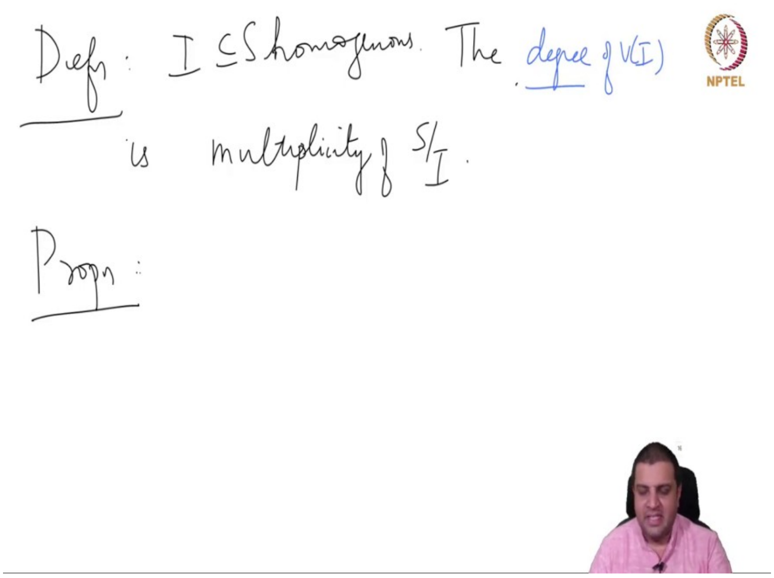

So, now let us look at a definition. So, let us try to use some of the things that we learned so far. Definition, $I \subseteq S$ homogeneous. So, sorry the assumption that k was algebraically closed was only till where was it only at this point just to get a geometric picture. So, from this statement onwards it is not relevant, I mean it is for any field. Just to clarify $I \subseteq S$ homogeneous, again no assumption on k here then the degree of $V(I)$ is multiplicity of $\frac{S}{I}$.

So, this is a thought of as a closed the correct word is sub scheme, but we will not I mean since we have not defined those things I will not use it at least it is a closed subset of Proj of S is the degree of this that is a definition. So, this is what we will call the degree of this thing and here is a proposition, the sort of geometric explanation behind degree with notation as above assume k is infinite; let $d = dim\, V(I)$ then for general linear forms $l_1, \ldots, l_d$

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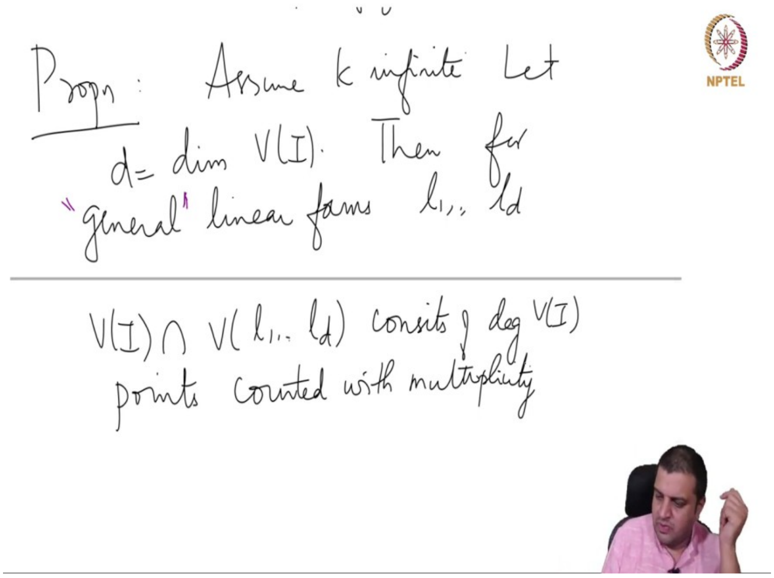

So, what does that mean? For general linear forms. So, what does this mean here? It means that just randomly pick something. As long as we allow the coefficients to be something arbitrary; so, general linear forms $l_1,\ldots,l_d$, $V(I)\cap V$¿) consists of degree $V(I)$ points counted with multiplicity, I will explain what would be counting with multiplicity mean once we see the proof we will ok proof.

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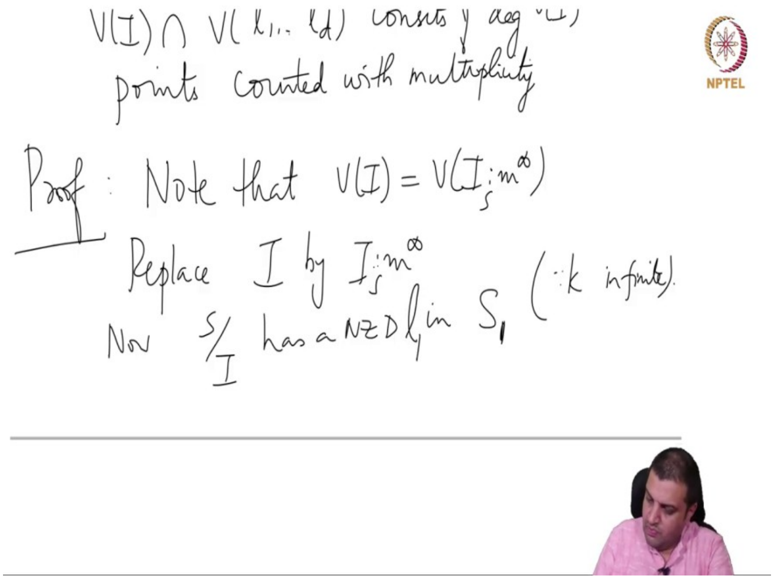

If we go back to the proof of how we said how we sort of calculated this expression for the Hilbert series which is some numerator (Refer Time: 27:51) series divided by $1-t$ to the some exponent, if we went back to the proof the proof relied on going modulo non zero divisor and when there is no non zero divisor one of two things could be done.

So, we will do the second one that we did which was to saturate out the finite length sub module. Remove everything that has been that is there which is killed by a power of the maximal ideal. So, then we will pick up a non zero divisor and then we go modulo that. So, now, we have brought down dimension and keeping the same multiplicity. So, this is what we want. So, note. So, we will do the same thing.

Note that $V(I)$ is same thing as $V(I:m^{\infty})$ meaning we saturated out the component of m. So, this is the observation that any prime homogeneous prime ideal which contains which contains this which contains remember I is a smaller ideal than this or not I is subset of this thing.

If a prime ideal contains this and maximal ideal multiplies something to 0 in that thing, then the maximal ideal will multiply it into the prime ideal, but maximal ideal is not a subset of the prime ideal. So, that something should be in the prime ideal; so, therefore, this. So, the saturation is also $V(I)$ is equal to V of this. So, doing this. So, replace I by $I:m^{\infty}$. So, this colon is in S.

So, now $\frac{S}{I}$ has a non zero divisor. So, this was the argument that we did earlier. So, now, go modulo that thing we have reduced to dimension and now use induction on dimension. So, $\frac{S}{I}$ has a non zero divisor in degree 1 since k is infinite. k has to be large k have so many elements that it is not the degree one; the degree 1 part is not contained inside the union of the associated primes.

So, this is k is infinite and. So, with this we are able to we. So, let us call this non zero divisor l_1.

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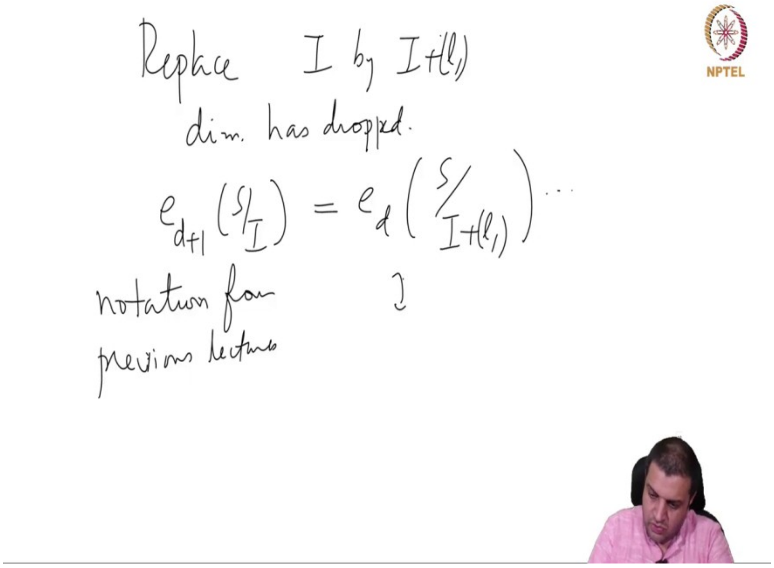

So, now, replace I by $I+(l_1)$. So, now dimension has dropped and we noted that. So, remember that the multiplicity $\frac{S}{I}$ is $e_{d+1}\left(\frac{S}{I}\right)$. So, this is from the notation from previous lectures notation from previous lectures this is same thing as $e_d\left(\frac{S}{I+(l_1)}\right)$ and then we can just repeatedly do this and then get to that situation where.

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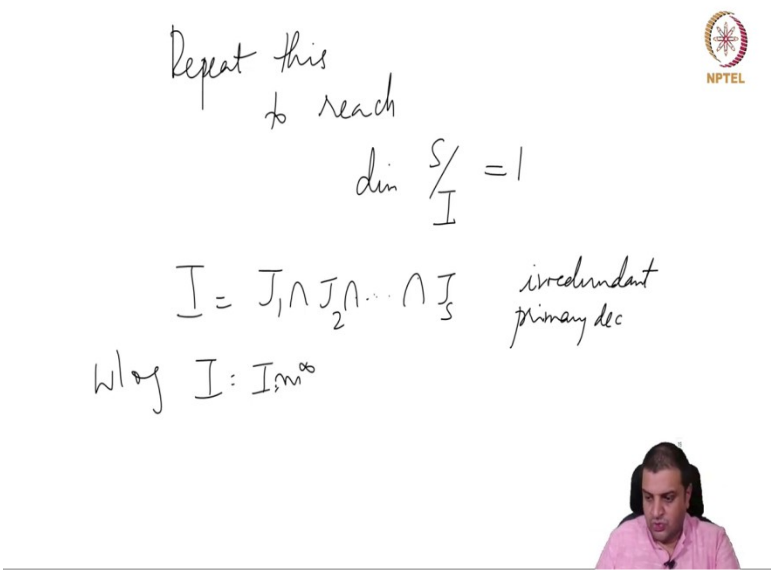

So, now, repeat this to reach the situation $dim\frac{S}{I}=1$ again we can factor out. So, now if we write I as some $J_1 \cap J_2 \cap \ldots \cap J_s$ and again without loss of generality I could be maximal ideal need not be associated to this and. So, this is a irredundant primary decomposition and again without loss of generality $I:m^{\infty}$. So, none of this is associate the maximal ideal is not associated with that list.

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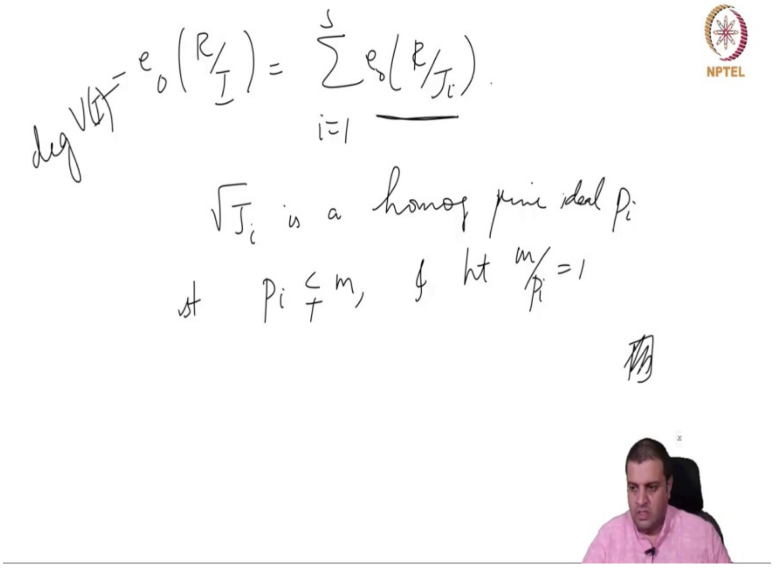

So, then $e_0\left(\frac{R}{I}\right)=\sum_{i=1}^{d} e_0\left(\frac{R}{J_i}\right)$ now each of these things. So, $\sqrt{J_i}$ is a homogeneous prime ideal. So, let us say P_i such that P_i is not equal to m and there is nothing in between and height of $\frac{m}{P_i}$ is 1.

So, it is this maximal in that thing. So, these are the close points of the spectrum of the Proj and, but this particular thing will have a multiplicity not equal to 1 and that is what we meant by saying counted with multiplicity.

So, these are s distinct points, but the point corresponds to the maximal corresponds to the prime ideal P_i, but it has a multiplicity which is the multiplicity which is this number ok. So, that is what we meant by saying the degree. So, this is the degree by the argument this was the degree of $v(I)$ is after we reduce it is the number of points counted with multiplicity.

So, this is how it is calculated. So, we will stop at this lecture now, in the next lecture we will look at degree in a slightly different way and then we will do some Macaulay example which illustrates these are these points.