Computational Commutative Algebra
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Lecture – 50
Hilbert series - Part 2

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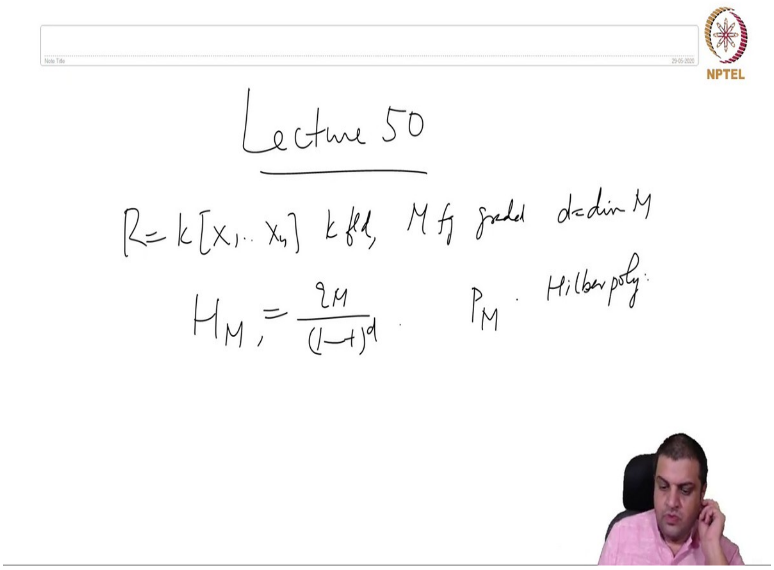

Welcome. This is lecture 50. In this, we want to prove a relation between the Laurent polynomial q_M which is the numirator in the way we wrote the Hilbert series and the Hilbert polynomial P_M. So, it is not very surprising because the difference between the Hilbert polynomial and the Hilbert function is only in some finitely many degrees. After that the behavior of the Hilbert function is the same as the behavior of the Hilbert polynomial .

So, if you write the generating function for both of these, then asymptotically, we should get the same behavior and the denominator should be the same. So, it is not very surprising, but so let us explore this in the let us try to understand this. So, we continue the notation $R=k\left[X_{1,}\ldots,X_1\right]$ and M is a finitely generated graded module.

And we have various things that we have defined so far which is we know the Hilbert series. Then, this is of the form

. So, let us put d to be the dimension of M and then, we also have P_M which is the Hilbert polynomial. So, this is; so here is a proposition.

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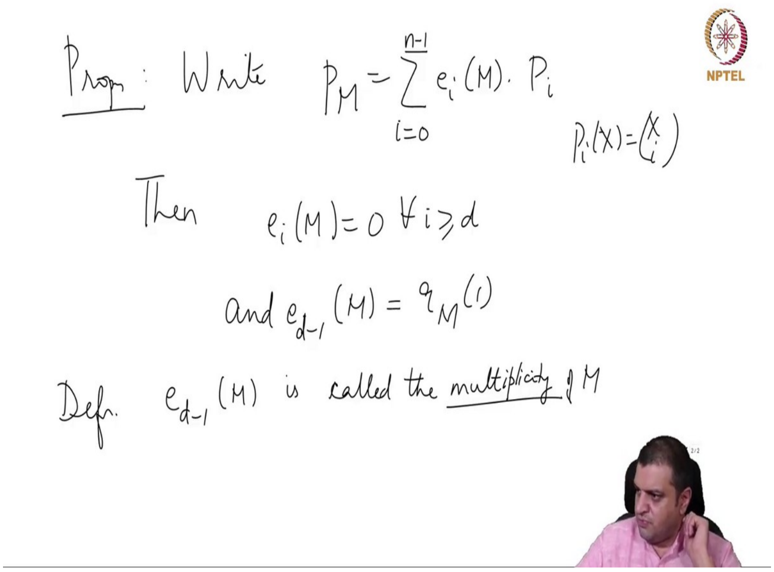

So, write we know that the Hilbert polynomial $P_M = \sum_{i=0}^{n-1} e_i(M) P_i$

. This was the binomial looking polynomial $P_i(X)$was X choose i. So, it is like a binomial function ok.

So, assume. So, write P as this, then so what is this expressions relation to the Hilbert series expression in terms of the Hilbert series? Then, e_i(M) = 0 for all $i \geq d$ and $e_{d-1}(M) = q_M(1)$. This is a number that we had studied and established was that this is a positive integer in the last proposition.

So, this is the number of copies in which the top degree binomial polynomial P_i shows up. So, this is non-zero and this is these are the ones above that are 0. . This number here $e_{d-1}(M) = 0$ is called the multiplicity of M. We will see this when we look at projective geometry multiplicity of M. So,proof is not, it just a sort of not very difficult at all, I mean we are just comparing.

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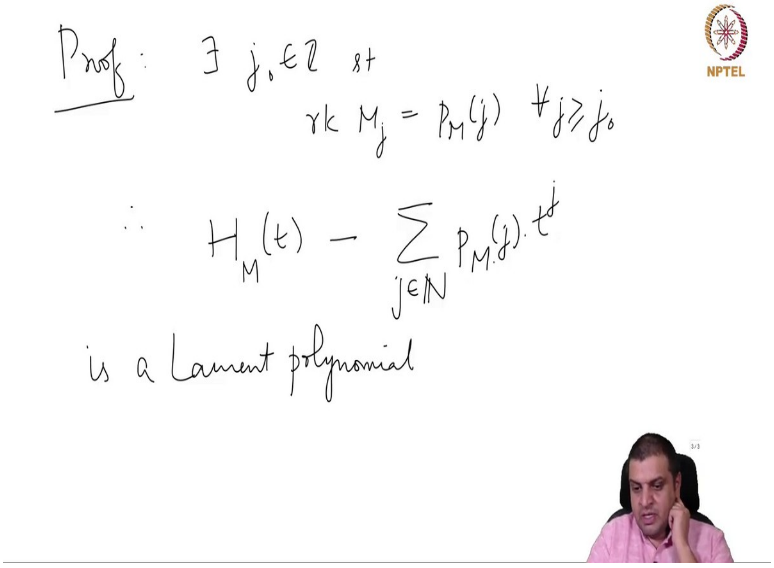

So, notice that there exist some j_0 integer such that the Hilbert function $rk(M_j)=P_M(j)$, for all $j \geq j_0$. That this is true. There is a polynomial with this property is what how we defined P_M; I mean what gave P_M . So, this is true.

So, in therefore, if we look at the following two sums $H_M(t) - .\sum_{j \in N} P_M(j) t^j$ is a laurnt polynomial. We want to restrict this to the non negative part because if we work out what the value of $P_M(j)$ is, well that will depend on so what the value of this binomial polynomial P_i is and it is 0 for some short interval, short interval 0, - 1 and so on. After that it becomes non-zero and from 1 or it is 0 for a short interval which includes up to -1.

So, -1, -2 and a few numbers this function will take value 0 and thereafter, for smaller and smaller integers, I mean more negative integers, this polynomial takes a non-zero value and one would expect the same behavior for $P_M(j)$also.

So, for j sufficiently small this will be a non-zero thing. So, we want to sort of restrict the right hand side to the non-negative and left hand side, we do not care.

The point is that the difference, so there will be some t to the minus exponents coming from might be there from this part. But eventually, for $j \geq j_0$; this will cancel each other . So, this is a Laurent polynomial.

So, now Hilbert polynomial itself is a Laurent polynomial divided by $(1-t)^d$, this we established . So, now, if you take H_M and subtract a Laurent polynomial, we will get that this is also of the same form.

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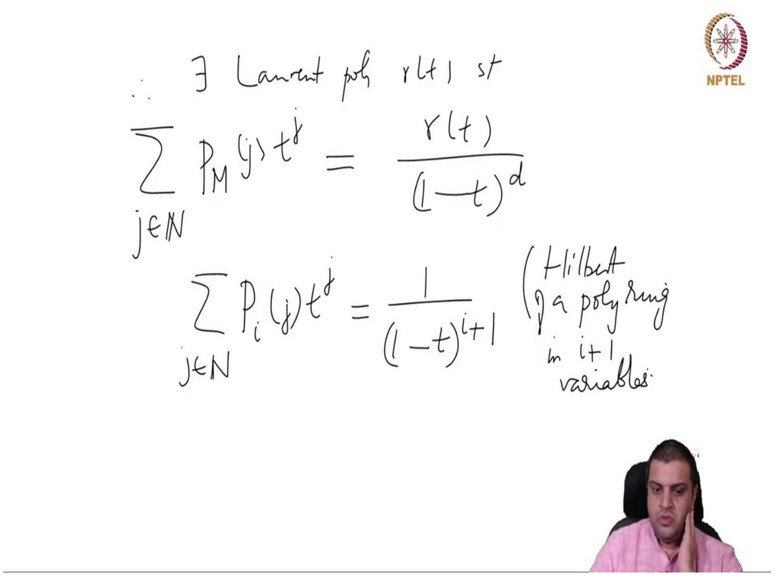

So, therefore, there exist a Laurent polynomial r (t) such that $\sum_{j \in N} P_M(j) t^j = \frac{r(t)}{(1-t)^d}$. On the other hand, if you take each one of these binomial polynomials if you take $\sum_{j \in N} P_i(j) t^j = \frac{1}{(1-t)^{i+1}}$. In fact, we earlier observed that P_i is the Hilbert series of a polynomial ring in i + 1 variables and this is exactly the Hilbert series of that polynomial ring. So, this is one can check this directly. So, this is a Hilbert series of a polynomial ring in ; i + 1 variables. So, this is the basic thing.

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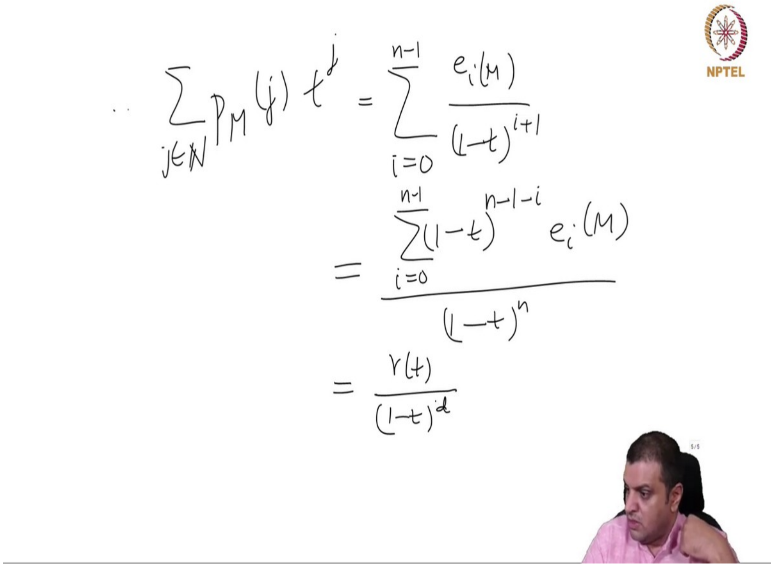

So, therefore, if we take $\sum_{j\in N} P_M(j)t^j = \sum_{i=0}^{n-1} \frac{e_i(M)}{(1-t)^{i+1}} = \sum_{i=0}^{n-1} \frac{(1-t)^{n-i-1} e_i(M)}{(1-t)^{i+1}} = \frac{r(t)}{(1-t)^d}$

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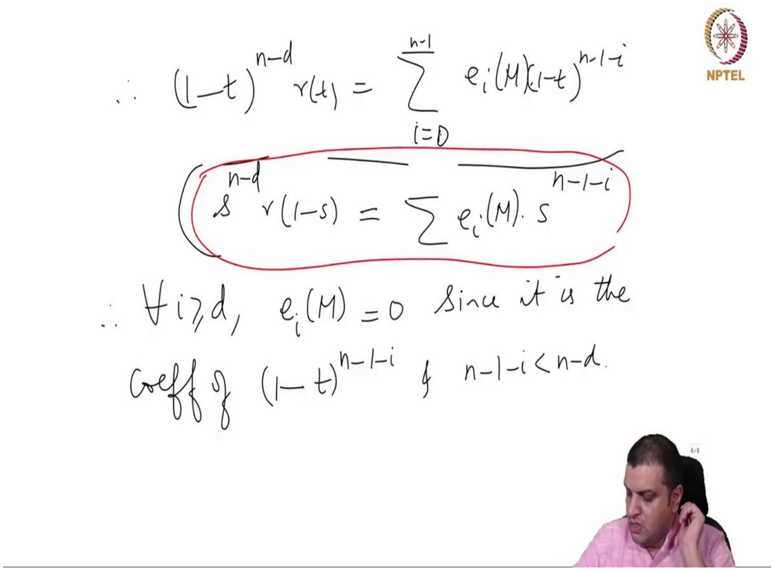

So, if you multiply both sides we would get $r(t)(1-t)^{n-d} = \sum_{i=0}^{n-1} (1-t)^{n-i-1} e_i(M)$

So, this is some polynomial; $s^{n-d} r(1-s) = \sum (s)^{n-i-1} e_i(M)$. So, therefore, now let us rewrite that thing. Therefore, for all $i \geq d$ $e_i(M) = 0$. Why? Because, since it is the coefficient of 1- t.

So, this I am just writing to explain; i just go back to the. So, just it just an aside; all that we want to say is that the if you write this if you think of $1-t$ as the variable. Then, it cannot have coefficients in small degree coefficient of $(1-t)^{n-1-i}$ and $n-1-i<n-d$ that is coefficients for every exponent less than n - d is 0, that is all that we have used and this gives us conclusion . So, that proves that statement.

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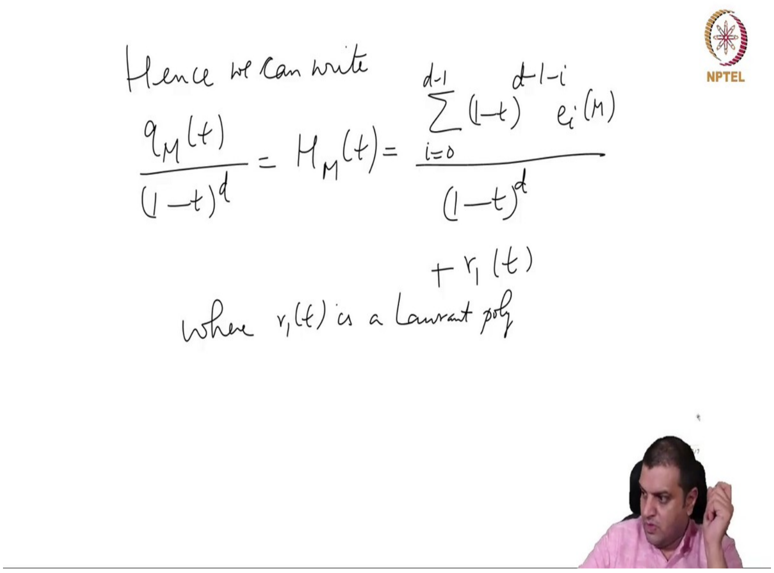

So, hence, we can write $\frac{q_M(t)}{(1-t)^d} = H_M(t) = \frac{\sum_{i=0}^{n-1} (1-t)^{n-i-1} e_i(M)}{(1-t)^d} + r_1(t)$.

Remember this expression was for the generating function for P_M for the non-negative degrees that differs from the Hilbert series by some Laurent polynomial; where, $r_1(t)$ is a Laurent polynomial.

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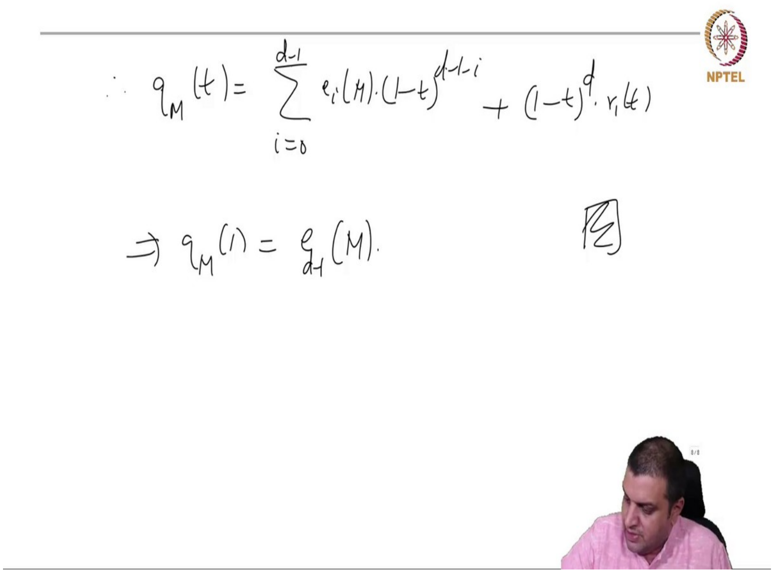

So, now what we need to clear; make the denominator same, then compare. So, therefore, $q_M(t) = \sum_{i=0}^{n-1} (1-t)^{n-i-1} e_i(M) + r_1(t)(1-t)^d$ That is how the denominator the numerators would look like.

And this now implies that $q_M(1)$, all of these terms involve a 1 - t this is the proof $q_m(1) = e_{d-1}(M)$. So, this is a description of the multiplicity. This quantity is called the multiplicity; this $e_{d-1}(M)$.

So, it is the coefficient, it is the; it is the coefficient of the largest j such that the binomial looking polynomial p_j appears with a nonzero coefficient. So, this is what the multiplicity is and now, we want to do to understand this for ideals for future in the next one or two lectures applications towards projective geometry.

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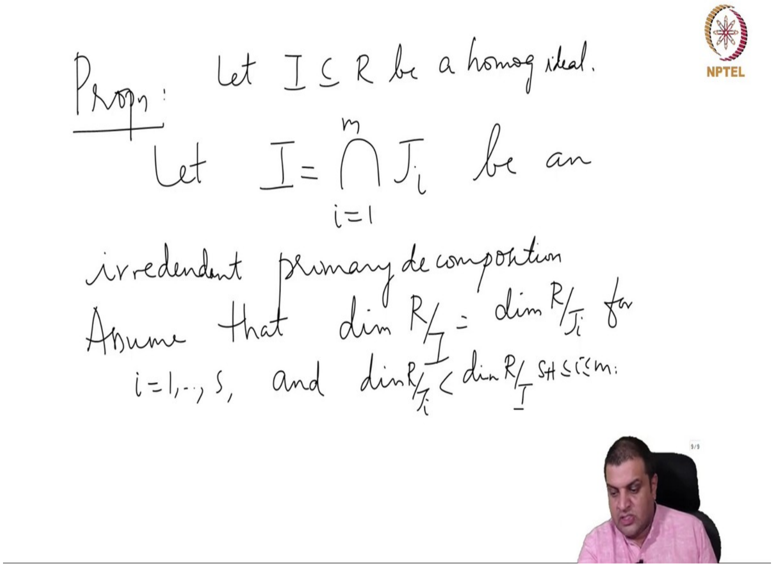

So, proposition how would one do? So, one can of course, do this for modules also; but I do not plan to do it to apply it in that generality and perhaps, the better intuition is there one when one works just with ideals.

So, now let I be a homogeneous ideal, , let $I=.\bigcap_{i=1}^{m} J_i$ be an irredundant; meaning, the radicals of the J_i's are pair wise distinct primes.

There is no there are not; there are not two different J_i's with the same associated prime irredundant primary decomposition. we have ordered the J_i assume that $dim\left(\frac{R}{I}\right)=dim\left(\frac{R}{J_i}\right)$ for i=1,..,s and $dim\left(\frac{R}{J_i}\right)<dim\left(\frac{R}{I}\right)$ for $s+1\leq i\leq m$ ok. So, we have arranged the J_i. So, that the first s of them are the ones which have maximum dimension and the rest of them have smaller dimension.

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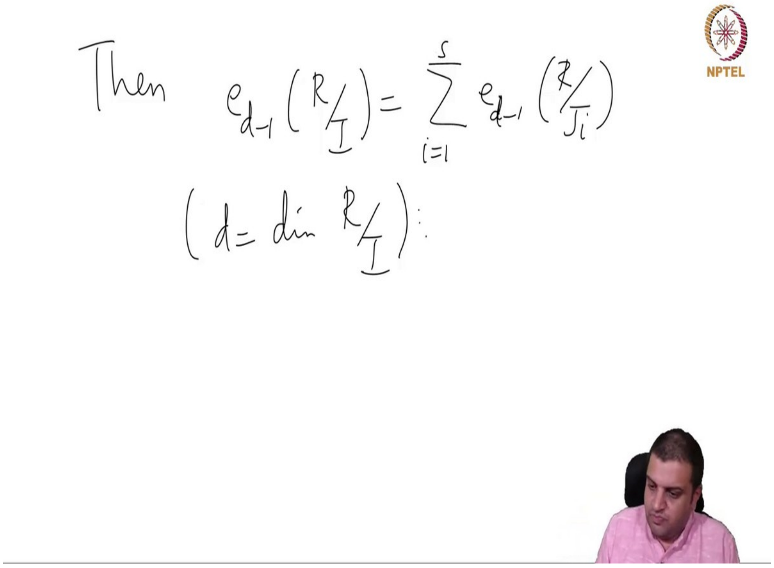

Then, their $e_{d-1}\left(\frac{R}{I}\right)=\sum_{i=1}^{s} e_{d-1}\left(\frac{R}{J_i}\right)$. So, sorry d here is the dimension of $\frac{R}{I}$.

The other ones do not count in this calculation at all and we will see a geometric reason behind this, once we go away. So, far, we have seen rings and ideals and correspondingly somek^n and algebraic varieties inside k^n that is I mean we will change that picture a little bit.

Now, we will worry about what is called homogeneous prime spectrum and that will be there we can see this I mean we will see what this means. And then, we will try we will see how this can be computed quickly etcetera or at least some ways of computing them quickly; there are other ways.

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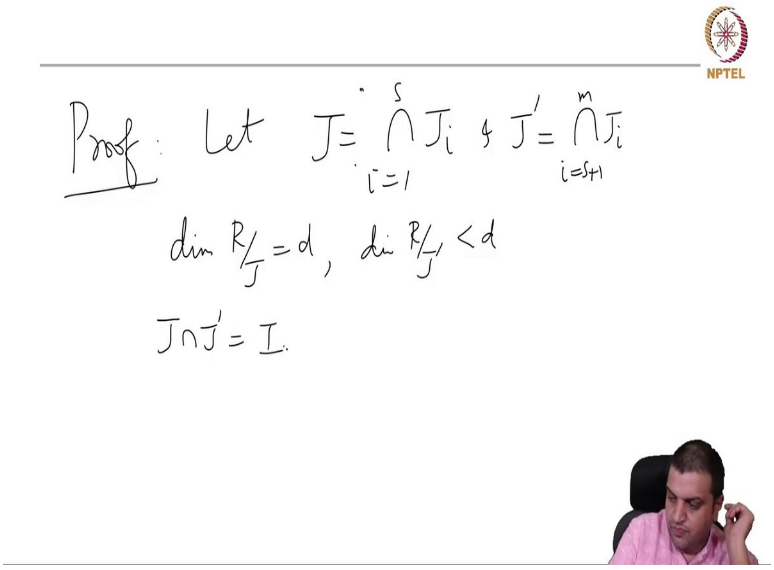

So, this is what we want to prove.   let us split them by the dimension. Let $J = . \underset{i=1}{\overset{s}{\cap}} J_i$. So, this is a primary decomposition of J and all of them have the same dimension, all the components have the same dimension and $J' = . \underset{i=s+1}{\overset{m}{\cap}} J_i$.

So, then $\frac{R}{J}$ and $dim\left(\frac{R}{J}\right) < d$ because all the components there have smaller dimension and $J \cap J' = I$.

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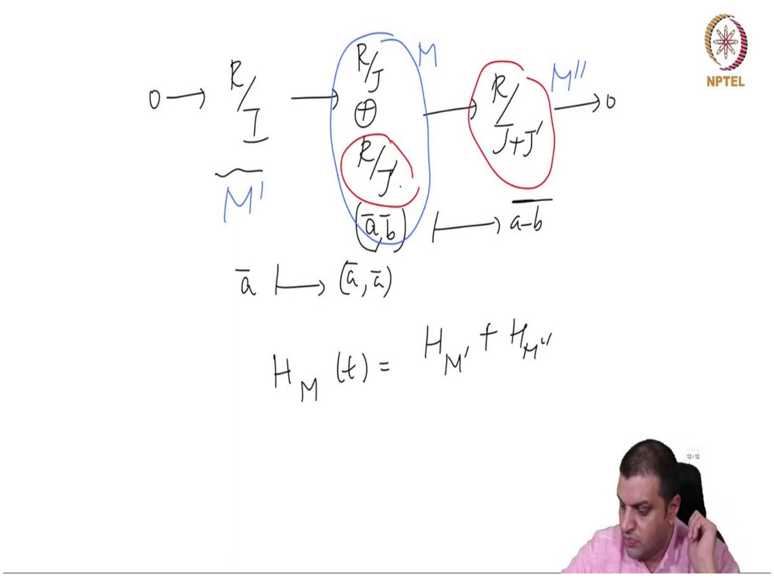

So, now, from exact sequence$0 \to \frac{R}{J} \to \frac{R}{J} \oplus \frac{R}{J}' \to \frac{R}{(J+J')} \to 0$, what do we know? So, we know that this has smaller dimension. So, this one's Hilbert series will grow at a I mean the denominator would be 1 - t to some exponent which is less than d.

For this, the Hilbert series will actually add because it is a direct sum. This one's denominator will be $(1-t)^d$. This will be smaller because this dimension is smaller. I mean dimension of this is less than d, this is a quotient; so, dimension of this is also less than d.

The map is if we take some $(\bar{a}, \bar{b}) \to \overline{a-b}$ And here the map is some $\bar{a} \to (\bar{a}, \bar{a})$. So, this term here and this term here have smaller dimension .

So, if you write the Hilbert series of the middle term So, that for simplicity let us call this

$M' = \frac{R}{J}, M = \frac{R}{J} \oplus \frac{R}{J}'$ and $M'' = \frac{R}{(J+J')}$. Then, $H_M(t) = H_{M'}(t) + H_{M''}(t)$

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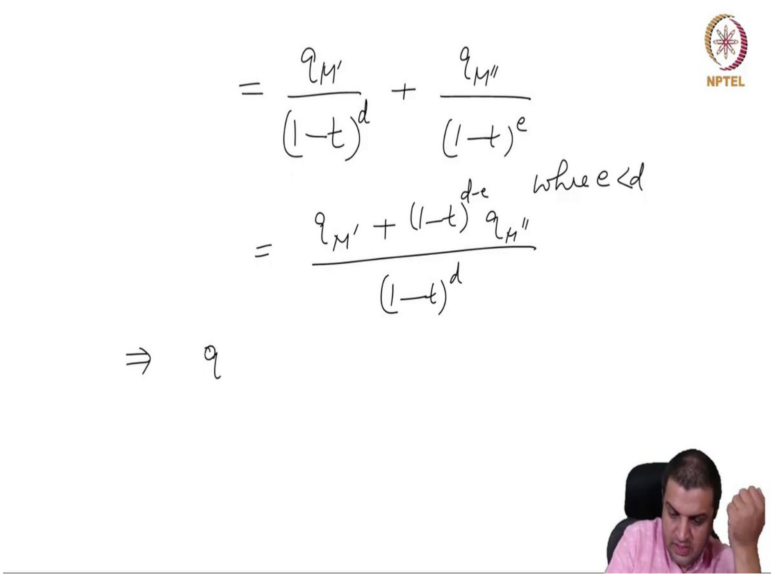

So, this is of the form some $\frac{q_{M'}(t)}{(1-t)^d}+\frac{q_{M''}(t)}{(1-t)^e}$, where, e < d . So, when we rewrite this thing, we would get $\frac{q_{M'}(t)+(1-t)^{d-e}q_{M''}(t)}{(1-t)^e}$ and when we evaluate the numerator at 1, this shows that this thing here for the middle one.

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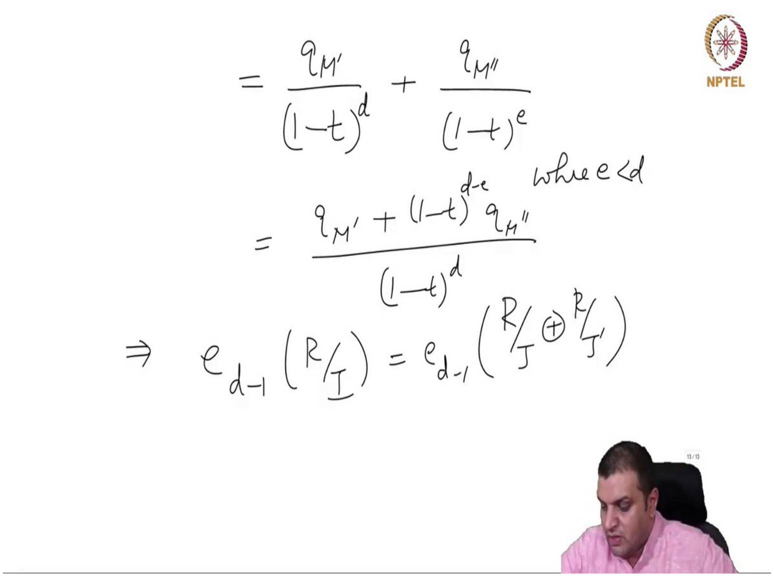

So, the multiplicity of $\frac{R}{I}$ is same as the multiplicity of this sum. This implies that

$e_{d-1}\left(\frac{R}{I}\right)=\sum_{i=1}^{s} e_{d-1}\left(\frac{R}{J_i}\right)$. Now, repeat the same argument for the short for the direct sum. So, that the direct sum give direct sum gives another short exact sequence .

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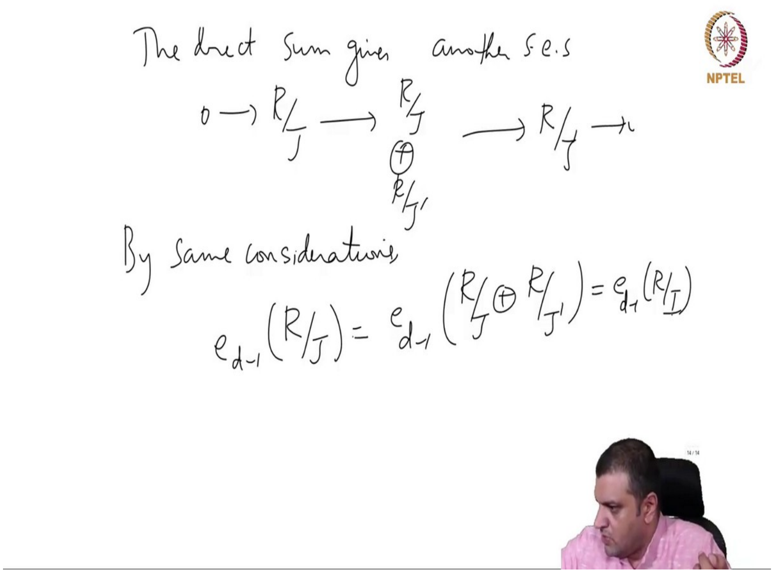

$0 \rightarrow \frac{R}{J} \rightarrow \frac{R}{J} \oplus \frac{R}{J'} \rightarrow \frac{R}{J'} \rightarrow 0$; this is just projection maps. This is a projection map that is an injection map to that factor. Again, for same dimension consideration, this part has smaller dimension . So, by same considerations, $e_{d-1}\left(\frac{R}{J}\right)=e_{d-1}\left(\frac{R}{J} \oplus \frac{R}{J'}\right)=e_{d-1}\left(\frac{R}{I}\right)$

But this is what we had said is we are already concluded is equal to $\frac{R}{I}$. So, in other words, we have removed all components from I of strictly smaller dimension. I mean one could have argued that by removing one component also, this is just; but anyway we have now replaced I, we can now replace I by J if you want and assume that all components of I have the same dimension.

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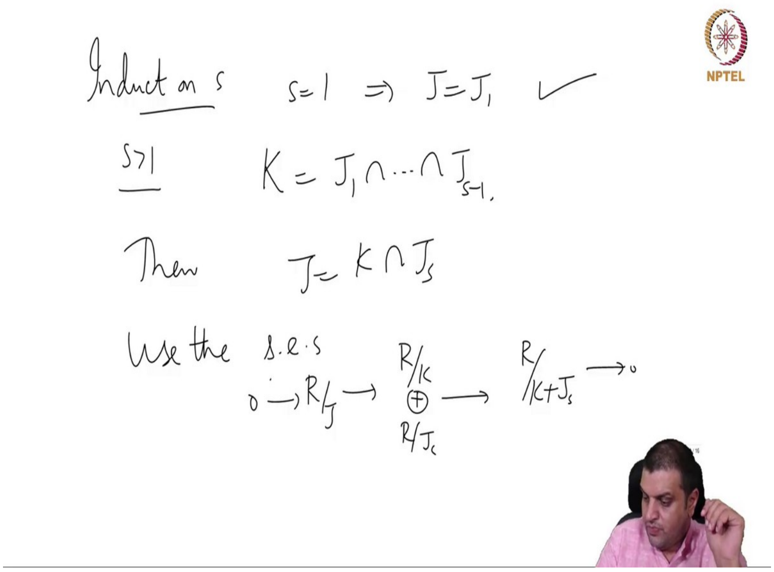

Now, induct on s; s =1 implies that $J=J_1$. So, then there is nothing to prove. s > 1 means write $K=.\bigcap_{i=1}^{s-1} J_i$; then, $J=K\cap J_s$, all of them have the same dimension. Now, use the same short exact sequence and conclude.

Use the same I mean use the short exact sequence $0\rightarrow\frac{R}{K}\rightarrow\frac{R}{K}\oplus\frac{R}{J_s}\rightarrow\frac{R}{K+J_s}\rightarrow 0$. Notice that both of these have the same dimension as $\frac{R}{J}$, but $K+J_s$ has to have smaller dimension.

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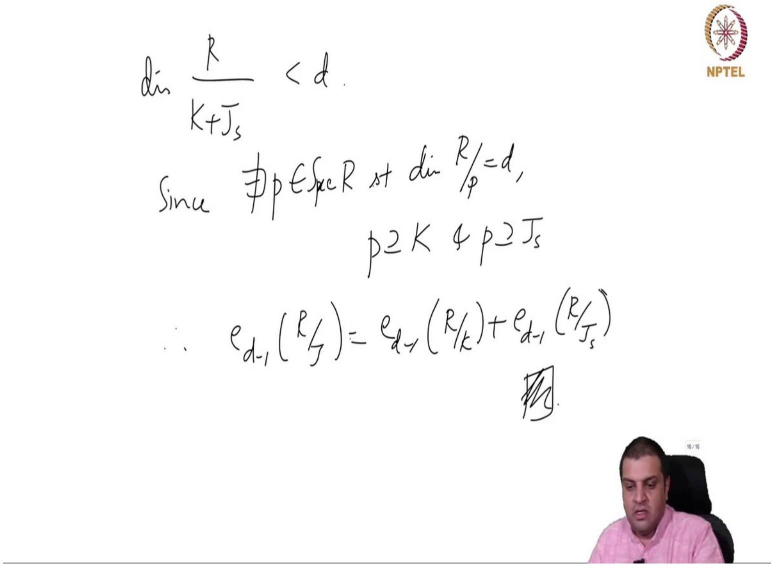

$dim\left(\frac{R}{K+J_s}\right)<d$ since there cannot exist a prime $p\in Spec\ R$ such that $dim\left(\frac{R}{p}\right)=d$, $K\subset p$ and $J_s\subset p$. Because the minimal primes over these things all of them have the same dimension are incomparable. So, if one has to contain at least two of them, then they must have positive I mean it must have smaller dimension So, this is what we need.

So, therefore, now induct. So, what we have now concluded is that therefore, $e_{d-1}\left(\frac{R}{J}\right)=e_{d-1}\left(\frac{R}{J}K\right)+e_{d-1}\left(\frac{R}{J_s}\right)$. So, we have removed we have split one component and added this and now induct another (Refer Time: 28:09). So, this has applications towards what is called degree of a projective variety, what we will define it and we will discuss that in the relation of all of this and to projective geometry in the next few lectures, one or two lectures.