Computational Commutative Algebra
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Lecture – 49
Hilbert series – Part 1

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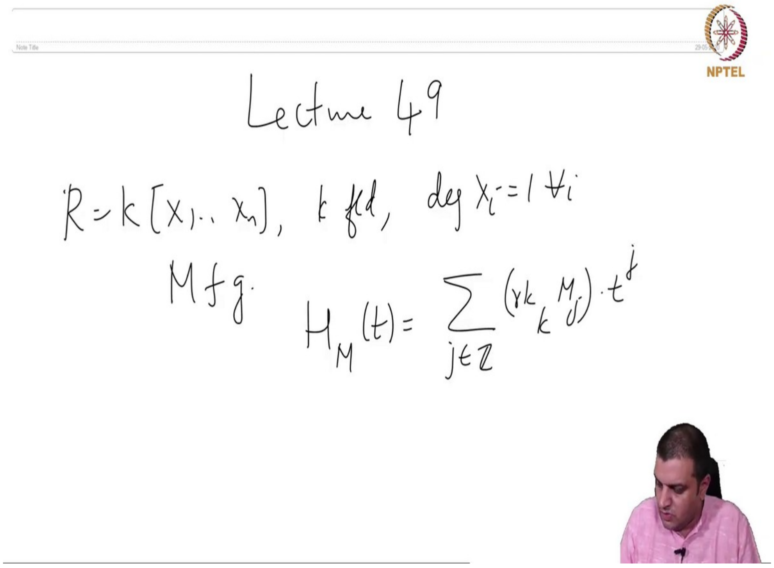

So, the point of this lecture and eventually is to understand what does the Hilbert series look like and how is it related at a course level to the Hilbert polynomial and we will make a precise when we get there ok.

So, recall that. So, $Ri=[X_{1,}\ldots,X_n]$ k field and graded with $deg(X_i)=1$ for all i and M finitely generated, then there is a Hilbert series of M $H_M(t)=\sum_{j\in Z} rk_k(M_j)t^j$.

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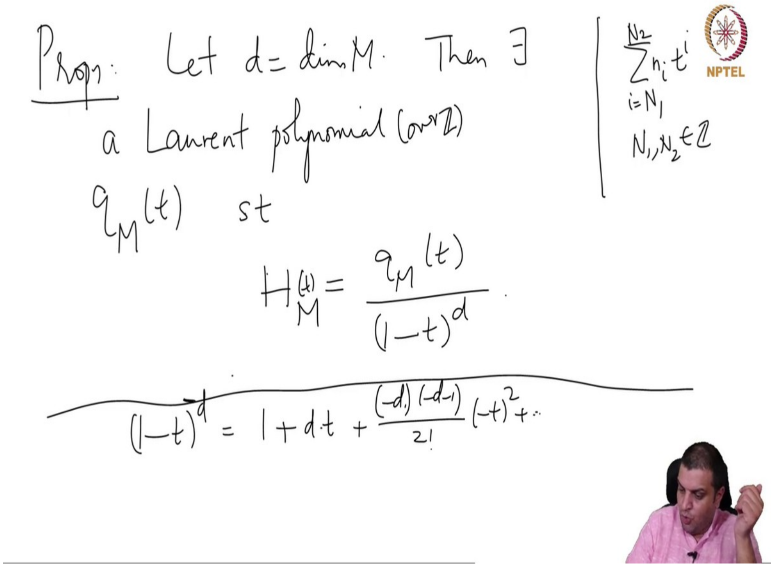

So, then proposition; what does the Hilbert series look like? So, let d = dim(M); it could be less than n then there exist a Laurent polynomial. So, what we will what this is mean this means? So, this means that something of the form over. So, the Laurent polynomial over Z .

So, $\sum_{i=N_1}^{N_2} n_i t^i . N_{1,} N_2 \in Z$. So, from sum exponent N_1 to sum exponent N_2 both of them could be negative or positive or something they are just integers.

So, it is not a polynomial that is why it is called a Laurent polynomial and in this particular case they are integers. So, it is . So, this is over Z, a Laurent polynomial which we will denote by $q_M(t)$. So, point is that there are only finitely many terms it might involve negative exponents, but just only finitely many terms such that $H_M(t) = q_M \frac{(t)}{(1-t)^n}$.

So, what does this mean? So, this one can be expanded as a formal as a power series . So, what does this mean? $(1-t)^n = 1 + dt + \frac{(-d)(-d-1)}{2}!(-t)^2 + \ldots$. So, one can expand it like this.

o, then $q_M$is a Laurent polynomial finitely many terms that times this will give I mean if d is positive this will give an infinite term and that is what H_M will look like. So, this is just sorry just is explain about that notation means.

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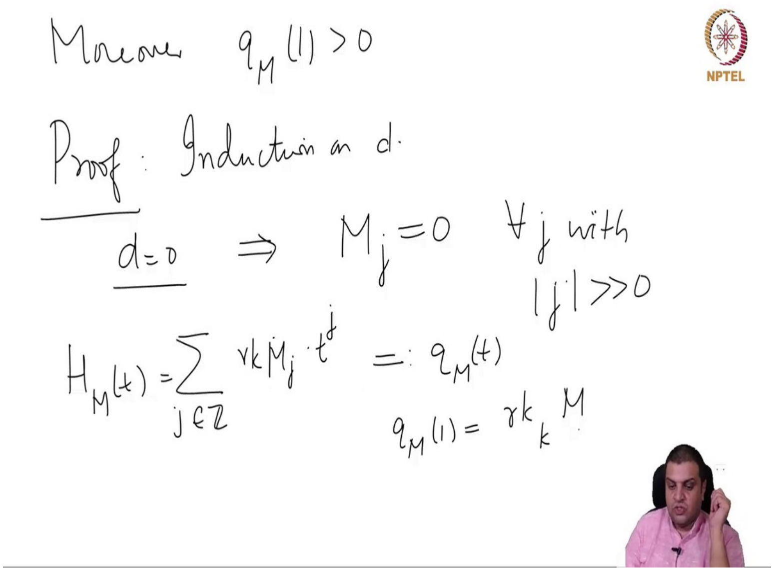

And, moreover $q_M(1)>0$. We will see in the next proposition what does it mean ok. This is the, this is the statement. So, let us just read it again. d is the dimension then it can be written as some Laurent polynomial divided by $(1-t)^n$.

So, we have to allow for negative things here because M could be nonzero in some finitely many negative degrees . So, we would allow for that. So, we can write it like this, but this has the property that $q_M(1)>0$.

So, in other words we cannot factor out $(1-t)$ from this again that is what it says because if this was 0, $q_M(t)$= (1-t)times something else and (1-t)would cancel one more term. So, it says it cannot factor (1-t) from it anymore .

So, proof is induction on d. If d = 0 which means we are talking about a 0-dimensional module. So, this now says that $M_j=0$ for all j with |j|> >0.

In other words, if you write the sum $H_M(t)=\sum_{j\in Z} rk_k(M_j)t^j=q_M(t)$

there are only finitely many terms in this one . $q_M(1)=rk_k(M)$ So, it is the rank of M as a k vector space that itself is finite because of this condition . So, this is the d= 0 case. So, yeah.

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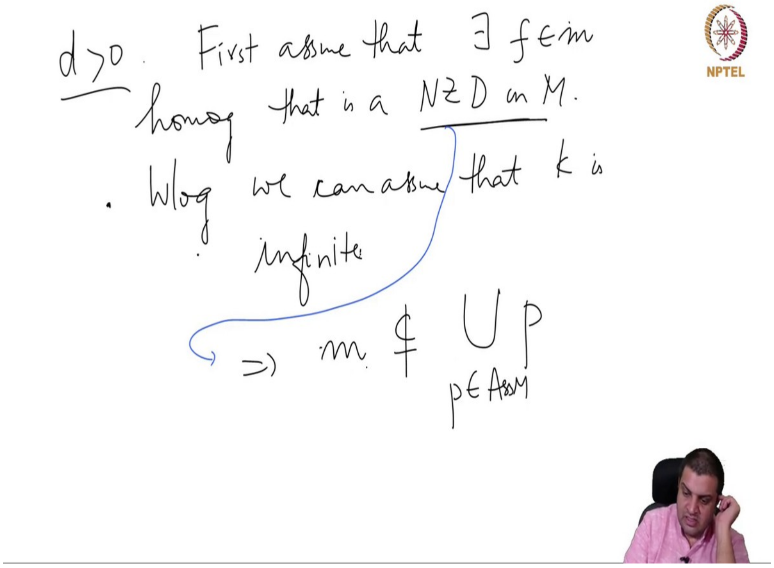

So, d > 0. So, we will. So, address this in two steps first some special M and then the general case will be reduced to it. There will be two ways to reduce the general case and I will explain both. I mean they are independent proofs I will explain both because they involve ideas that could be otherwise applicable. So, d > 0.

First assume that there exist some $f \in m$, maximal ideal homogeneous that is a nonzero divisor on M. So, we have done the following argument without loss of generality we can assume that k is infinite. So, this I will put.

So, this has to do with so many problems in which we study about Hilbert functions we can reduce this to this problem. So, this is similar to an argument that we had used earlier where we had a polynomial ring over k and we had the polynomial ring in the same number of variables over k bar and then we went back and forth using an argument.

So, with a similar sort of argument one can also reduce this that when we study when we study Hilbert functions etcetera we can always replace k by $\bar{k}$ for instance and assume that it is infinite. Not all problems are where you replace k by $\bar{k}$ there may be other ways where one has to replace and still get infinite there are various techniques, but in this particular case we can just take k equals to this $\bar{k}$.

So, then so, this I will explain the exercises it is a slightly longest calculation. So, but if $k = \bar{k}$, so, notice that what does what do we mean by there is a nonzero divisor this implies that. So, this implies this just this step implies that $m \not\subset \bigcup_{p \in Ass(M)} p$.

We observed earlier that the nonzero divisors are all in some associated prime. I mean we use prime avoidance for slightly stronger statement the reason is every nonzero; let if f is a nonzero divisor, then the ideal generated by f kills something. And, so, that is contained in the annihilator of some element inside M and maximal things with that property are exactly the associated primes. So, therefore, f is contained in some associated prime.

So, that is and then conversely every p itself kills something. So, every element of p is a zero divisor. So, to say that there is a nonzero divisor is to say that there this is true, but now we use k is infinite.

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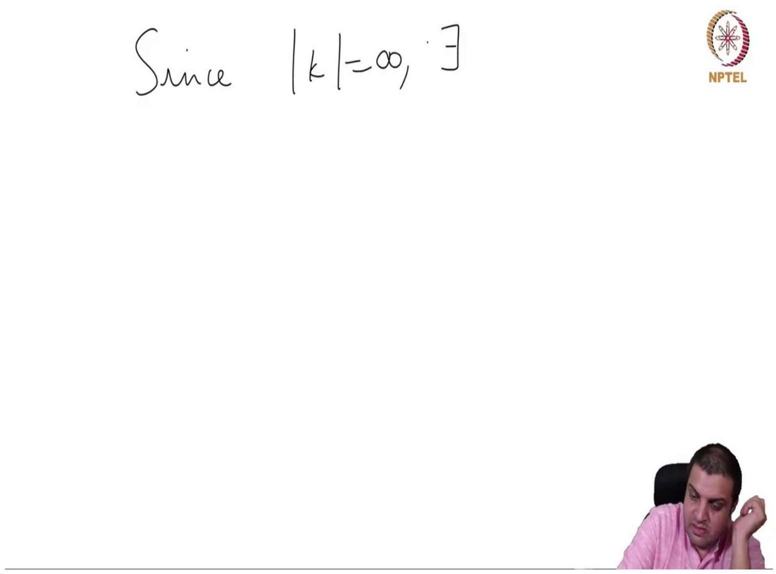

Since k is infinite. So, this is a the slightly general form of prime avoidance. $m \not\subset m^2 \cup \bigcup_{p \in Ass(M)} p$

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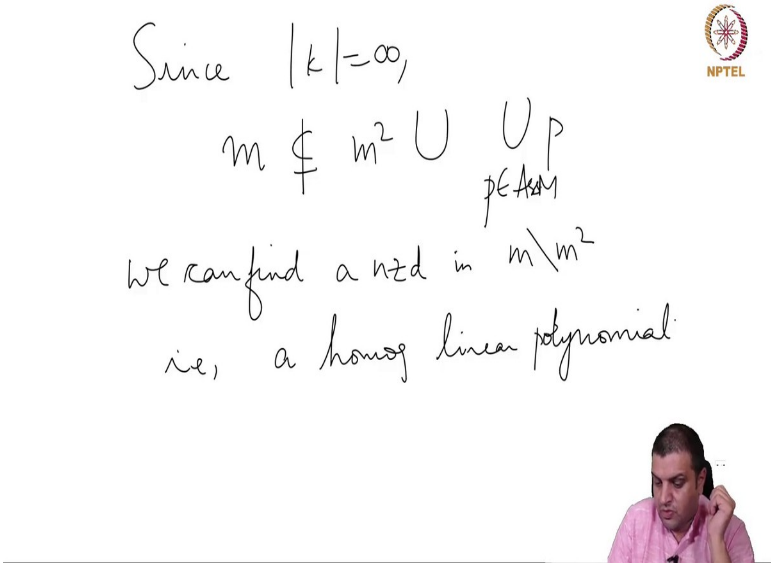

In other words, we can find a nonzero divisor in $m \setminus m^2$ that is a linear form that is a homogeneous linear polynomial sometimes called form..

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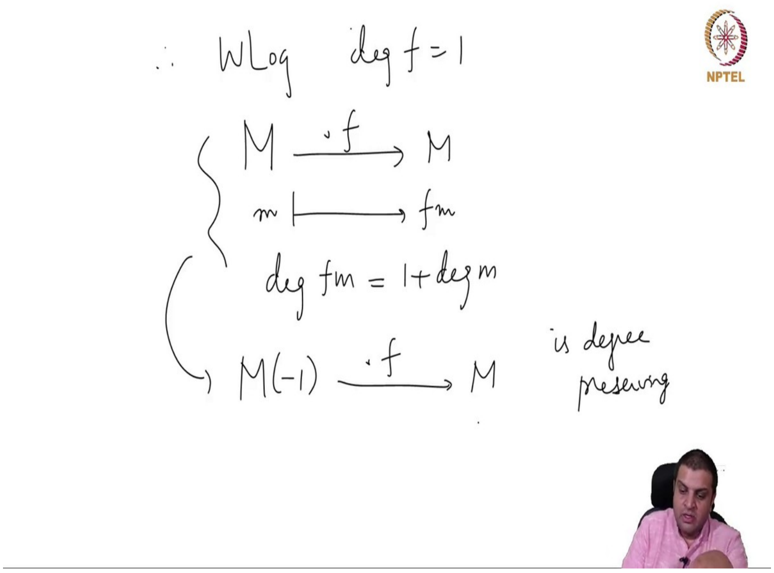

So, therefore, without loss of generality deg(f) = 1 . So, now let us we want to consider multiplication by f. So, if you take map $M \to^{f} M$ multiplication by f some homogeneous some element $m \to fm$. So, the deg(fm) = 1 + deg(m).

So, in other words, this is not a degree preserving map, but we can do one thing. We can shift we can replace this by M(- 1). So, instead of doing this $M(-1) \to^{f} M$ multiplication by f. So, now, this is degree preserving. The element of M here would now live in the next degree I mean sort of artificially moving into the next degree and then m will go to fm, but deg(fm) = deg(m).So, this is degree preserving.

So, we will keep this because remember for it to Hilbert series to add we need this we need degree preserving maps.

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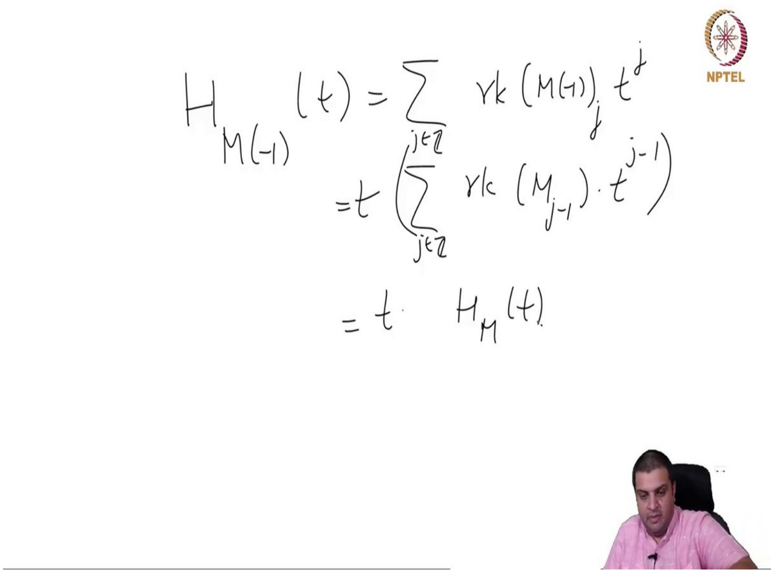

So, what is the Hilbert series of M(-1)?. $H_{M(-1)}(t) = \sum_{j\in Z} rk_k(M(-1)_j) t^j = t \sum_{j\in Z} rk_k(M_j) t^j = t H_M(t)$ So, in general if you shift the module degrees by some a, we so, if you if you replace M by M(-a) we would get t^a . So, that is what we have.

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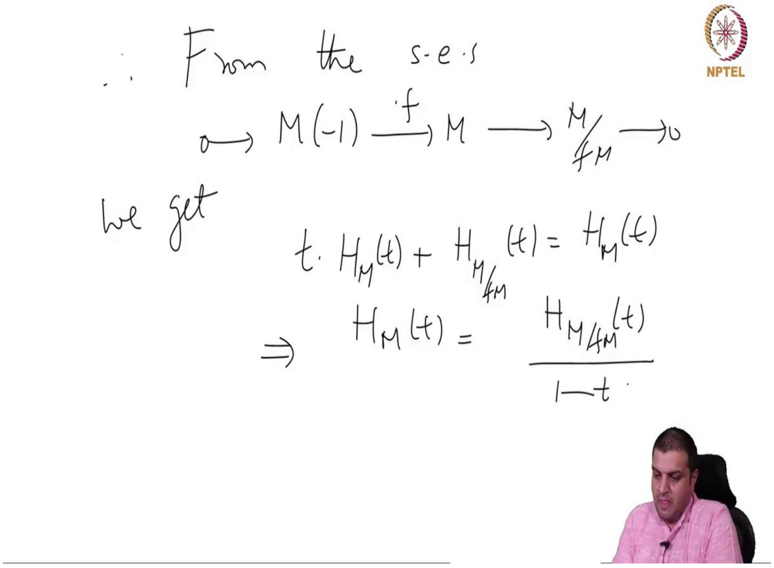

So, therefore, from $0 \to M(-1) \to^{f} M \to \frac{M}{fM} \to 0$ short exact sequence. Now, we did this, this is multiplication by f. Now, from the short exact sequence now, it is degree preserving we get from the short exact sequence of degree preserving maps

We get that $t H_M(t) + H_{\frac{M}{fM}}(t) = H_M(t)$ this is the middle one this is the outer two. So, from this we conclude that $H_M(t) = \frac{H_{\frac{M}{fM}}(t)}{(1-t)}$. So, we get this thing.

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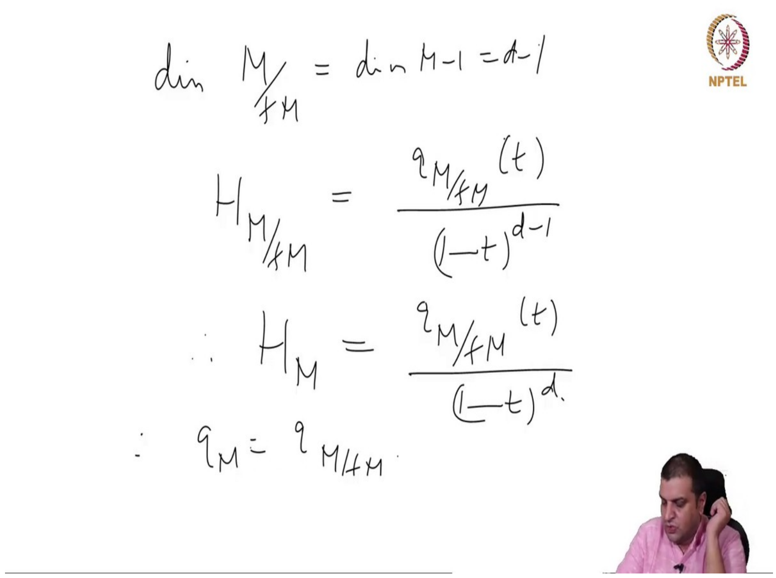

So, we have checked earlier $dim\left(\frac{M}{f\,M}\right)=dim(M)-1=d-1$. So, therefore, $H_{\frac{M}{fm}}(t)=\frac{q_{\frac{M}{fm}}(t)}{(1-t)^{d-1}}$

from here if you just divide it by further (1-t) t therefore, we get $H_M(t)=\frac{q_M(t)}{(1-t)^d}$ Laurent polynomial did not change if you went modular nonzero divisor.

And, then so, therefore, $q_{\frac{M}{fm}}(t)=q_M(t)$ and this proves that it does not vanish at 1 because by induction this does not vanish here. I mean by induction if you substitute 1 here we get something positive. So, here also we will get something positive. So, this does the higher dimensional case assuming that there is a nonzero divisor.

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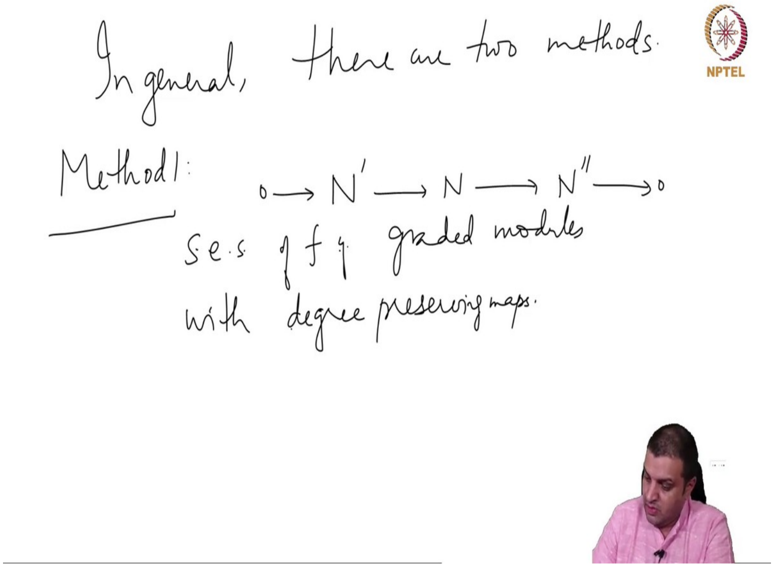

So, now let us get to this case. In general, we want to reduce to this case. So, we want to reduce to the case that there are two methods. So, method 1. So, I will do both because they are just sort of useful techniques that you might use it later or some other place.

So, suppose we have similarly $0 \to N' \to^{f} N \to N'' \to 0$ short exact sequence of finitely generated graded modules of the polynomial ring with degree preserving maps. So, hereafter I might just say short exact sequence of graded modules and this will be understood that we want we will keep by default we will shift these modules by some degree, so that maps are all degree preserving suppose.

So, we need we want to say that if you know the theorem for N’ and N’’, then we will know it for N, how we will know it for N’ and N’’ is a different question. Let us let us do that. So, if we take so, we need so, there are various quantities to be compared we need. So, what do we know?

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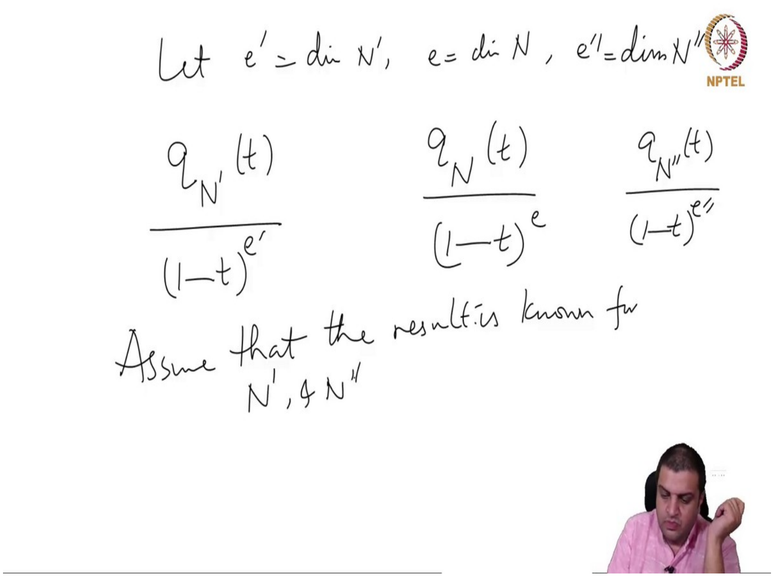

We know that so, let e' = dim(N'), e = dim(N), e'' = dim(N''). So, what do the three Hilbert series look like? One which would look like, $H_{N'}(t) = \frac{q_{N'}(t)}{(1-t)^{e'}}$

. The middle one will look we do not know what it is, but we are trying to prove that $\frac{q_N(t)}{(1-t)^e}$ we do not know that this is true. This is the expected form and this we assume that the theorem is known for N' and N''.

We know that the Hilbert functions are additive this we saw in the last lecture that degree preserving maps means that the Hilbert series of M. So, therefore, so we let us assume that we know the theorem for N' and N''. Assume that the result is known this is not a correct argument because we have to check we are doing an induction on dimension. So, we have to check that dimension is that is preserved, but right now I am just suggesting that if we can check that how will we proceed. Assume that the result is known for N' and N''.

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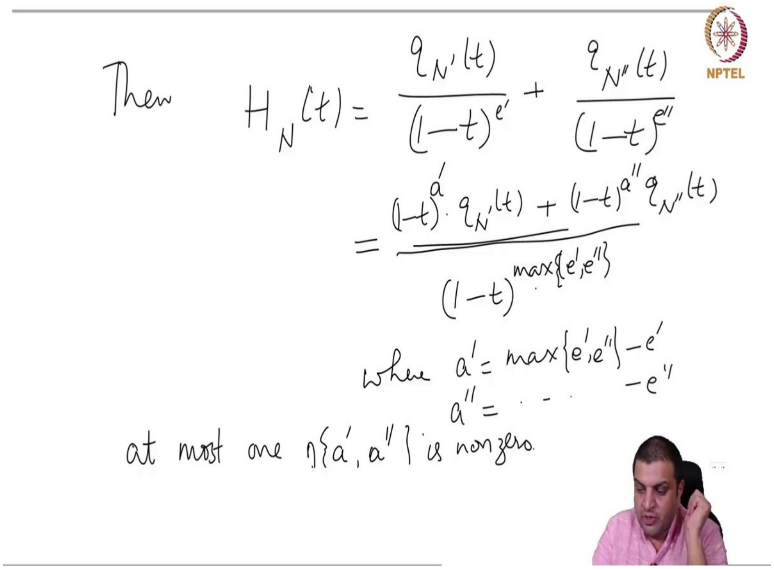

So, then $H_N(t) = \frac{q_{N'}(t)}{(1-t)^{e'}} + \frac{q_{N''}(t)}{(1-t)^{e''}}$ and this is so, we have to adjust the exponent in the denominator.

So, we can say $(1-t)^{\max\{e',e''\}}$ then depending on what whether this is the max or not it would be some $H_N(t) = \frac{(1-t)^{a'} q_{N'}(t) + (1-t)^{a''} q_{N''}(t)}{(1-t)^{\max\{e',e''\}}}$

where this where $a' = \max\{e',e''\} - e'$ and $a'' = \max\{e',e''\} - e''$ So, we would need to adjust that so that the denominators are the same. So, we get something like this.

So, now let us go back. So, this is what we get. So, now, we can complete the proof if we know something about dimension. Well, we know something about dimension that dimension of N is equal to this number and going back to the problem yeah. So, dimension of N is equal to this number, then we can go backward.

So, note that at most one of a', a'' is nonzero because we are adjusting this to get to the max. So, only one has to be adjusted . So, at most one of them is nonzero. So, therefore, so, we can rewrite this is the potential candidate for q_N and this is the candidate for dimension. So, we will keep this approach in mind.

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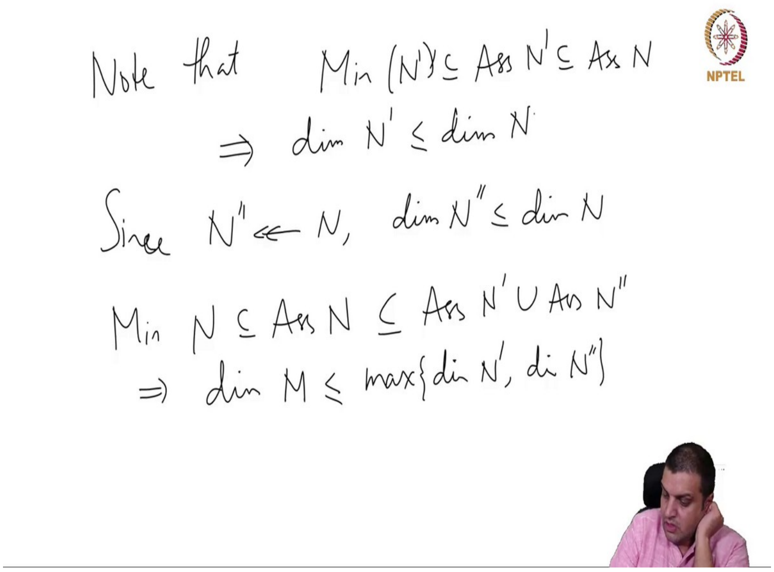

So, note that the $Min(N') \subset Ass(N') \subset Ass(N)$. In other words, $dim(N') \leq dim(N)$ because dimension is length of a chain be from some such prime to the to a maximal ideal and definitely that will include this is because of this look at this condition.

And, N'' is a quotient of N since N double prime is a quotient of N $dim(N'') \leq dim(N)$ Now, remember dimension is always counted by something from here.

So, $Min(N) \subset Ass(N) \subset Ass(N') \cup Ass(N'')$.

So, now, this implies that $dim(M) \leq max\{dim(N'), dim(N'')\}$ because any chain from here must appear either in this either in the supp(N') or in the supp(N'').

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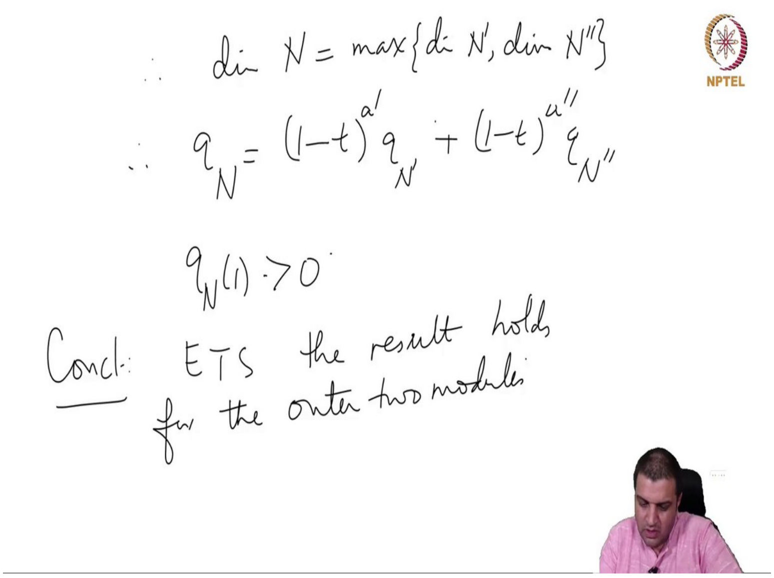

So, in other words our conclusion is that therefore, $dim(N) \leq max\{dim(N'), dim(N''), dim(N)\}$ So, that we can actually use this thing here . So, once this is max then we this is the potential candidate for q_N . So, then we can do that.

So, therefore, now we can conclude that $q_N = (1-t)^{a'} q_{N'} + (1-t)^{a''} q_{N''}$

and at most one of them is positive. So, therefore, q_N(1) is also positive because you will not get 0 on both sides.

If this is if a' is positive then you will get 0 here, but you will get 1 here and so, this is positive assuming the statement for N' and N''. So, it is enough to prove the theorem for elements in for the outer two of a short exact sequence So, the conclusion is that enough to show that the result holds for the outer two modules maybe I should be more precise.

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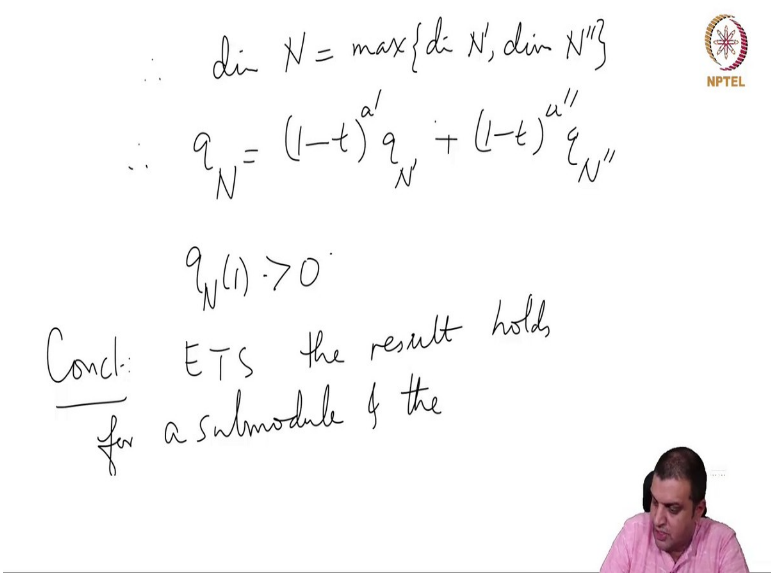

Result holds for a submodule of M and the corresponding quotient of M.

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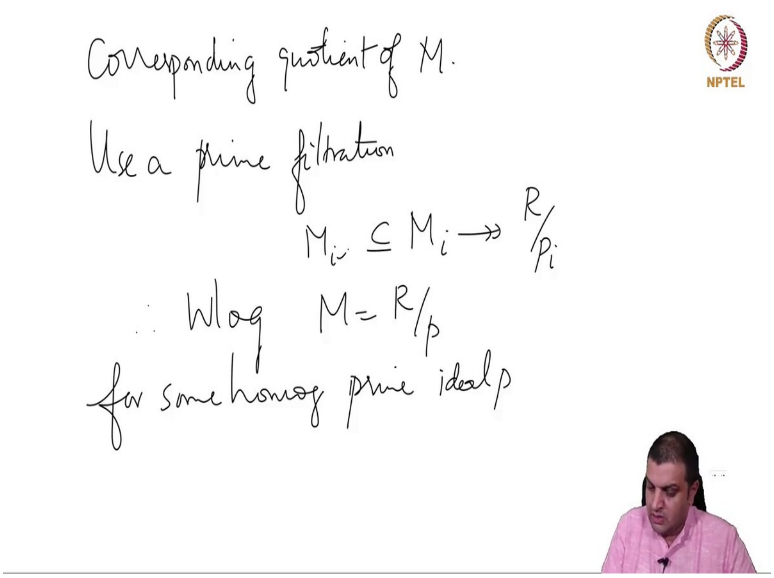

So, now use a prime filtration. So, we will get some $M_{i-1} \subset M_i \to \frac{R}{p_i}$ we can adjust the degrees to make this a short exact sequence of degree preserving maps. So, therefore, without loss of generality and this every module has a finite prime filtration.

So, without loss of generality we can assume that $M = \frac{R}{p}$ for some homogeneous prime ideal p , but now there is a nonzero divisor nonzero prime ideal p which is in a positive dimension .

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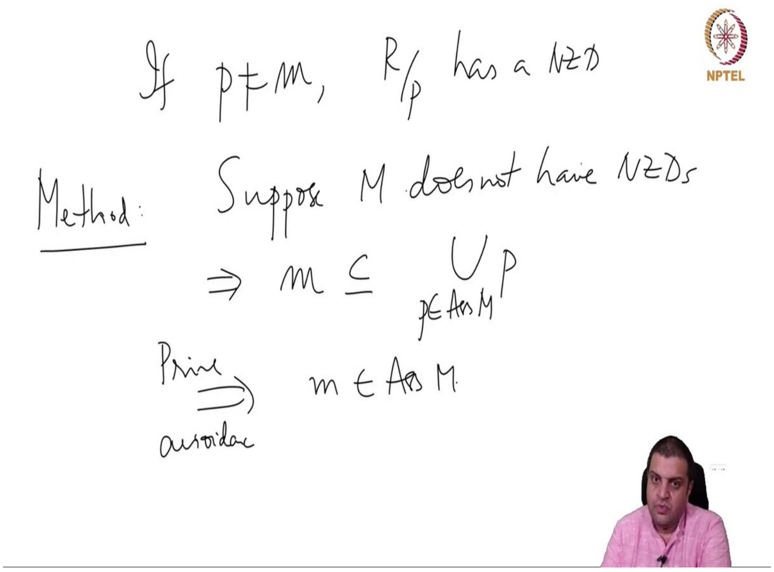

I mean if $p \neq m$ $\frac{R}{p}$ has a nonzero divisor and then we have reduced to the previous case which we have already addressed in the case where the module has a nonzero divisor we have already addressed.

So, another method so, this finishes the proof, but I just want to illustrate another method which goes in a slightly different, but still useful technique which is the following. So, suppose M does not have a nonzero divisor.

this implies this we observed earlier $m \subset . \bigcup_{p \in Ass(M)} p$ by prime avoidance prime avoidance. $m \in Ass(M)$ contained in some associated prime therefore, m is associated.

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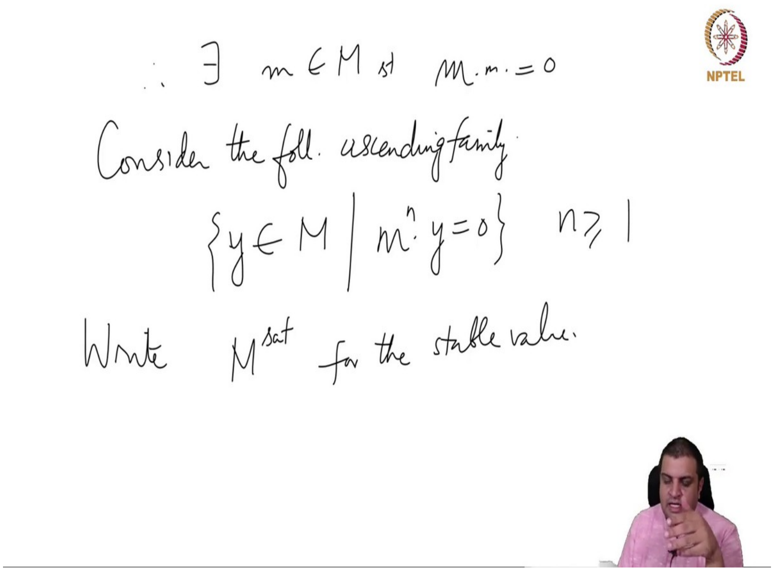

So, therefore, there exists some $x \in M$ such that $xm = 0$. So, now we can consider the following ascending family.

What do we want? We want to take all elements $\{ y \in M : m^n y = 0 \}$ and this for $n \geq 1$. So, if m^n kills then m^{n+1} also kills. So, this is an ascending family M is noetherian. So, write M^{st} for the stable value. So, it has to be every element in M^{st} would be killed by some high large enough power of m.

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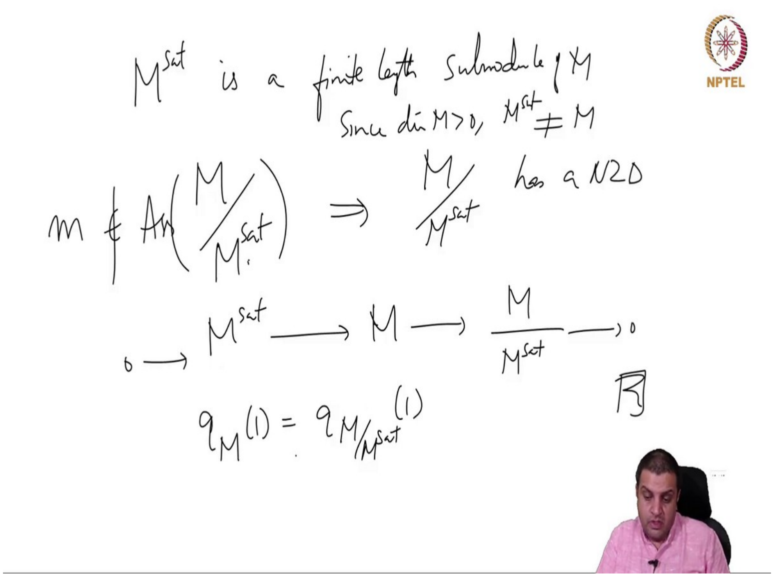

So, M^{st} a finite length submodule of M and we will see saturation when we look at what is called projective geometry yeah in a later lecture. So, M sat is a finite length sub module of M which and moreover $m \notin Ass\left(\frac{M}{M^{st}}\right)$ because if this is associated then there is something in this quotient which is killed by the maximal ideal.

Therefore, something in M which is multiplied into M^{st} by the maximal ideal, but then that would have been multiplied it to 0 by some even larger power of maximal ideal. So, it would have already been counted. So, m is not associated. So, which means that this implies that $\frac{M}{M^{st}}$ has a nonzero divisor.

So, we take this one M^{st} this is a finite length module. since dim(M) >0, M^{st} is not equal to M . So, the quotient is. So, this is a finite length module. Its Hilbert series is just a Laurent polynomial.

If you substitute t = 1 we will definitely get something positive. We have $0 \to M^{st} \to M \to \frac{M}{M^{st}} \to 0$. So, the result is true for this and then we can just write some Laurent polynomial and then use a sum value here and if you substitute equals 1 we will get something just.

So, in fact, we will you will see that when you work it out $q_M(1) = q_{\frac{M}{M^{st}}}(1)$. There is a reason for this which we have not yet seen which we will see you know in the next proposition, at least we will see in the next proposition for ideals not for all modules, but it is the same reasoning will apply for all modules. So, this is the this is what we will conclude .

q_M is not equal to this, but when you substitute 1, the contribution from M^{st} will go to 0 because of some terms. The H_M for this will have some denominator. $H_{\frac{M}{M^{st}}}$ will have some denominator you have to multiply by that this is the Laurent polynomial.

I mean to make the denominators common you have to multiply by a power of 1 - t. So, this will not contribute in this calculation I mean this will not contribute in this calculation. It will contribute to q_M, but not to $q_M(1)$. So, that is the end of this yeah.

So, in the next lecture we will look at some geometric applications called proj of a ring then use it to see some few cases of few basic situations of projective geometry. We will revisit saturation at that point. And, then that the lecture after that we will try to convert arbitrary non graded situation to a graded situation which also has a geometric meaning in terms of spec and proj, it has a geometric meaning.

And, there are certain advantages of working with in the geometric in the graded case ok. So, we will try to discuss them in the next two to three lectures.