Life
Computational Commutative Algebra
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Lecture – 48
Hilbert series – Part 1

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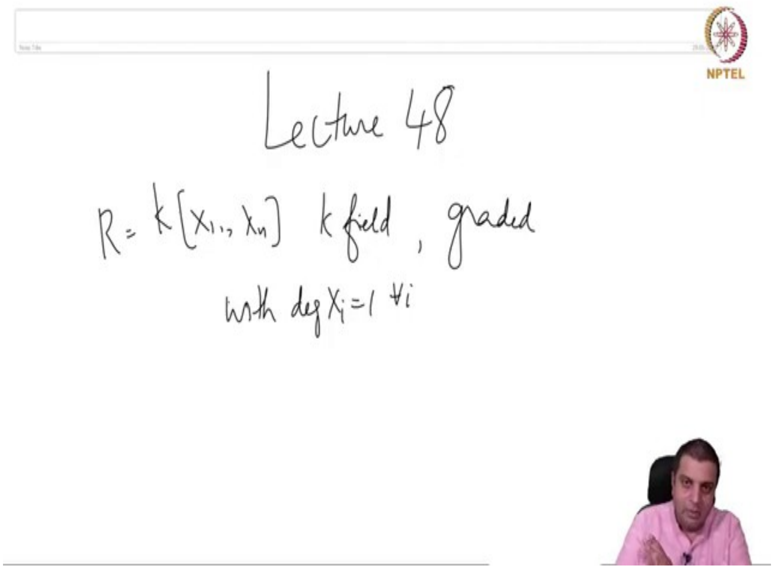

Welcome to lecture 48. In this lecture we look at the polynomial ring $R = k[X_{1,} \ldots, X_n]$ and k is a field and this is graded with $deg(X_i) = 1$ for all i. So, lot of arguments will go through even if you do not assume in case of field we could generalize this to for example, Artinian local rings or Artinian rings in some case we only need that k is Noetherian.

So, this is not the greatest generality that I am writing. I am writing this with so that you know we can revisit our computational techniques with the new I mean with the increased understanding of dimension etcetera.

We can go back to the computational techniques and also because these now have applications in geometry so, we will try to understand various computations that we did etcetera what it means in geometry in little bit more thoroughly or so, this is what we need ok.

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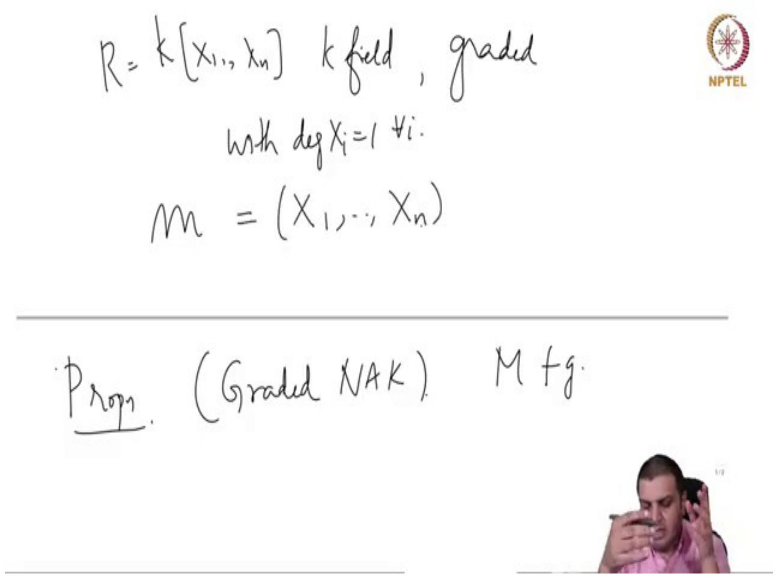

So, here is a graded this is a version of the graded Nakayama's lemma. Suppose that M is finitely generated. So, just maybe I should keep this. So, by $m = (X_{1,}\ldots, X_n)$ the homogeneous maximal ideal .

And, we will see this I mean after this lecture that lot of our intuition about how local rings Noetherian local ring (R, m)would behave sort of comes from or can be tested by working with homogeneous modules, homogeneous ideals, this maximal ideal in that ring .

So, this is a good example not to just for computational setup or geometric setup, but also to for us to sort of understand what would happen in the local case. This is just a meta principle it is not a theorem, but the theory is sort of proceeds in parallel.

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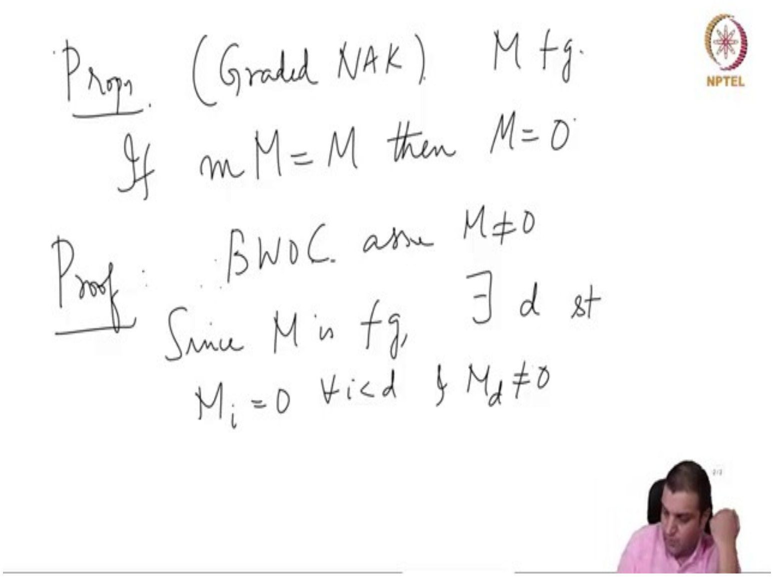

So, M finitely generated if mM=M then M= 0 and the proof is substantially simpler than the local Nakayama case we will see now. M is finitely generated. by way of contradiction assume M is not 0.

Since M is finitely generated there exists a d such that $M_i = 0$ for all i < d and $M_d \neq 0$. Look at a set of homogeneous generators and among them pick the smallest degree that appear there. So, that is this d, this is nonzero.

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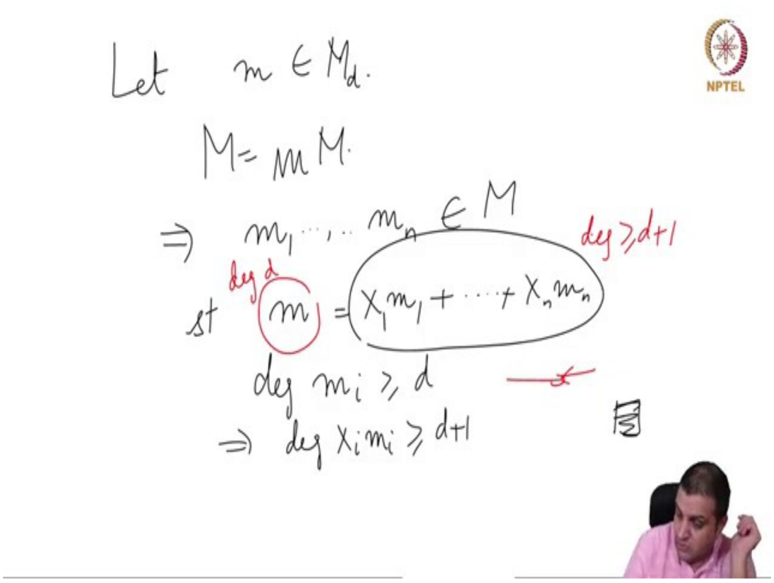

So, let $x \in M_d$, but $M = mM$. So, in other words there exists some $m_1, \ldots, m_n \in M$ such that $x = X_1 m_1 + \ldots + X_n m_n$, but what are the degrees of these? Degrees of these is at least d.

$deg(m_i) \geq d$ this implies that $deg(m_i) \geq d+1$. So, this part has degree greater than or equal to d + 1 while this has degree d and that is a contradiction and that is the end of the proof.

So, we see that the graded Nakayama lemma is substantially simpler than the local version, but the advantage I mean of knowing this result is that we can now redo what we did in the local case which is the following. So, the following many things in the thing that we did (Refer Time: 05:58) this.

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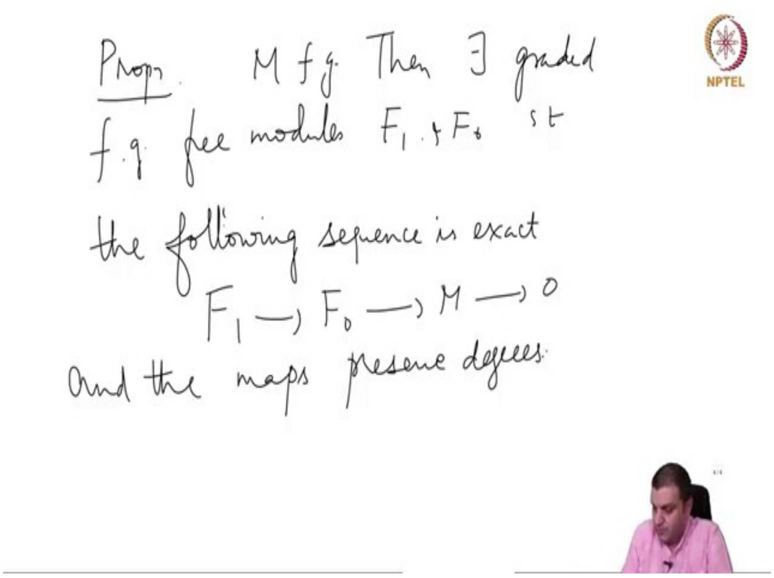

So, with notation as in the so, proposition M finitely generated, then there exists graded finitely generated free modules F_1 and F_0 such that the following sequences exact $F_1 \to F_0 \to M \to 0$. In other words, M is a co-kernel of this map and the maps preserve degrees.

This is the point that we can just take like this we will prove this, but we can take like this so that under this map an element of degree k will go to degree k here and under this map under same thing here these maps preserve degrees.

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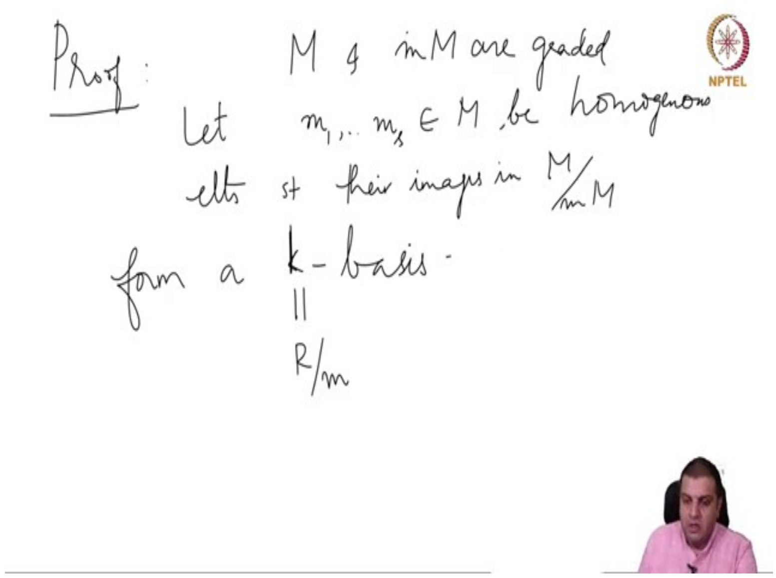

So, proof. So, M and mM are graded. M is graded so, mM is also graded. So, let us say $m_1,\dots,m_s \in M$ be homogeneous elements such that their images in $\frac{M}{mM}$ form. So, the homogeneous this is the key point here.

Otherwise this is the argument as we did the same argument for finitely presented I mean finitely generated is same as finitely presented over an Noetherian ring after that we will refine that thing to say what how we can construct a minimal generating set. It is the same argument, except we can now work with homogeneous elements.

The images will form a k-basis. So, remember k is isomorphic to $\frac{R}{m}$ as a graded k algebra or a quotient of graded quotient of R from a k-basis of this thing.

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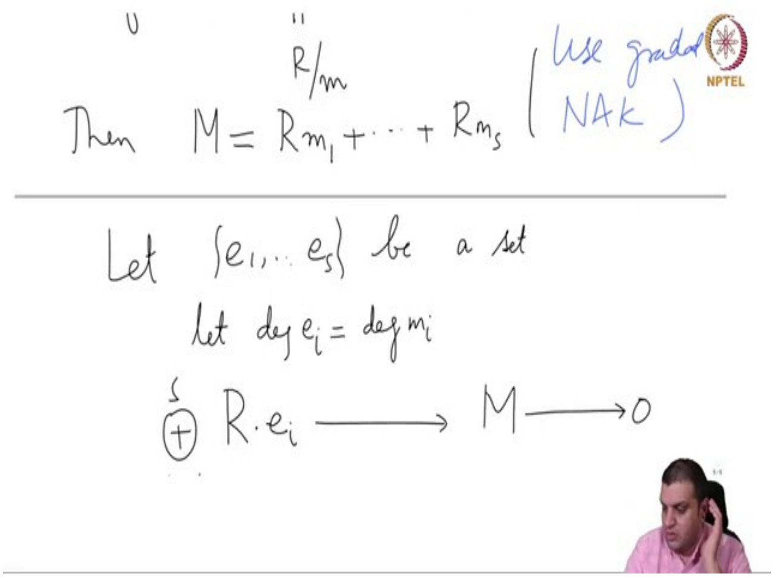

Now, let $\{e_{1,}\ldots,e_s\}$ be a set and let $deg(e_i)=deg(m_i)$. Same as there are things here. So, take the same number of elements here and now, consider the free module generated by these things. It is a free module, but now it is actually a graded module because each of them is not just isomorphic to R, but the generating set itself is homogeneous.

Now, $\cdot\underset{s}{\oplus}R\,e_i\rightarrow M\rightarrow 0$and by graded Nakayama's lemma this would be surjective that is that any generating set for this basis will generate this is the same argument. let me just say it here. So, then M is equal to the submodule generated by them .

Look at the submodule generated by this and then one can show that M is inside the submodule plus mM that is what the generating thing I will say. So, therefore, M is equal to this. So, that argument is exactly that is the part where one has to use graded Nakayama's lemma. So, use graded this is just analogous to what we did in the local case.

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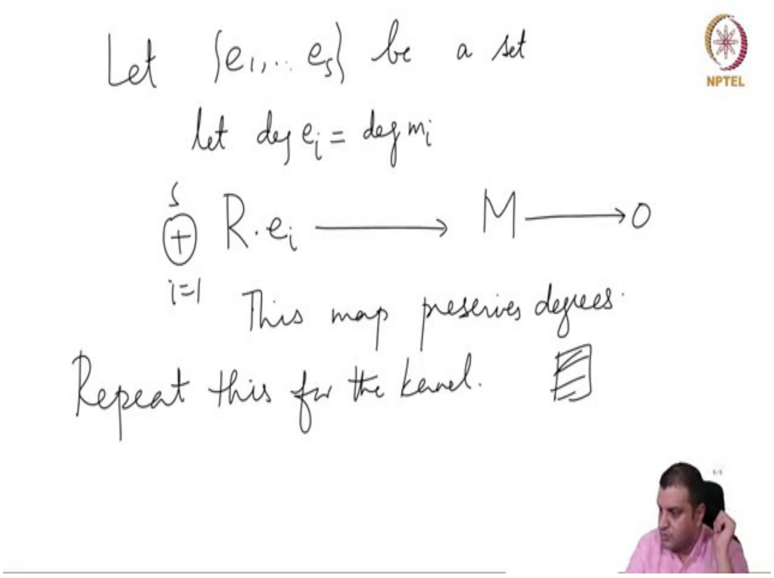

So, we get a map like this and this is this map preserves degrees. So, this is the first observation. Now, repeat this for the kernel and that would give us a map as we asserted finitely generated free modules which are themselves graded modules with a presentation map. So, this is the end of this proof. So, graded Nakayama lemma comes exactly like this.

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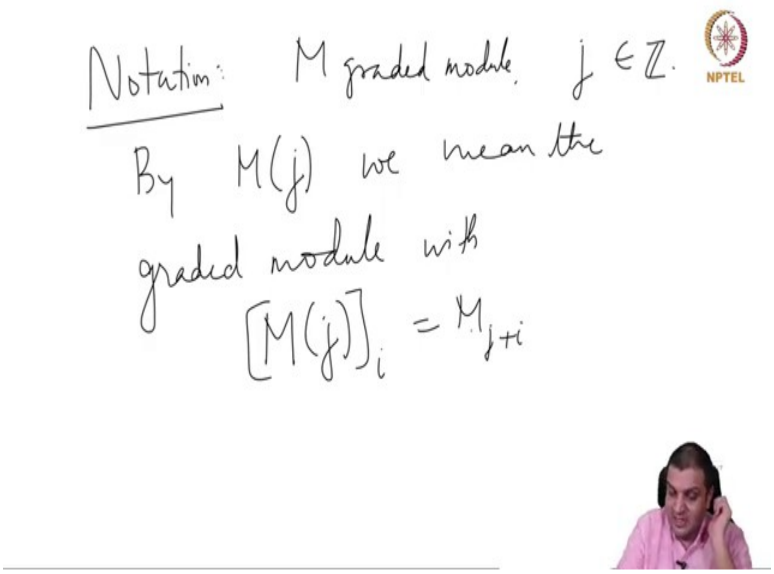

So, now just some notation this is a frequently used notation. So, one should familiarize about the concept that. So, M graded module, $j \in Z$. then by M(j) we mean the graded module with.

So, as abstractly as an abelian group it is same as M, but what we have done is to shift it is degrees. So, with $[M(j)]_i = M_{j+i}$. The reason this notation is slightly counter intuitive and let us check what that means.

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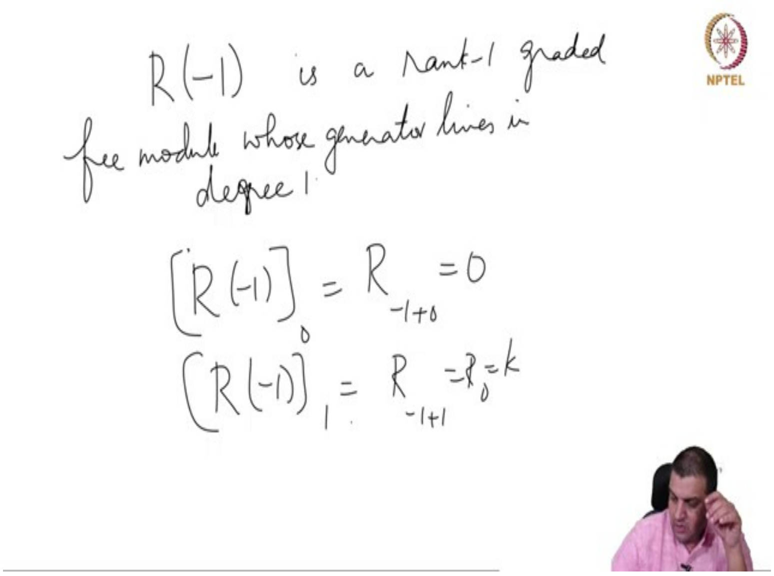

So, R (-1) is a rank 1 graded free module whose generator lives in degree 1 not in degree -1 . So, the notation for the formula this is very convenient and therefore, in applications also it is actually quite convenient, but one should keep this in mind R(-1) means rank 1 graded free module whose generator lives in degree 1 and why is this? I mean why is that the notation.

So, if you take $[R(-1)]_0 = R_{-1+0} = 0$ $[R(-1)]_1 = R_{-1+1} = R_0 = k$ this is where the generator as an R-module comes and then so on.

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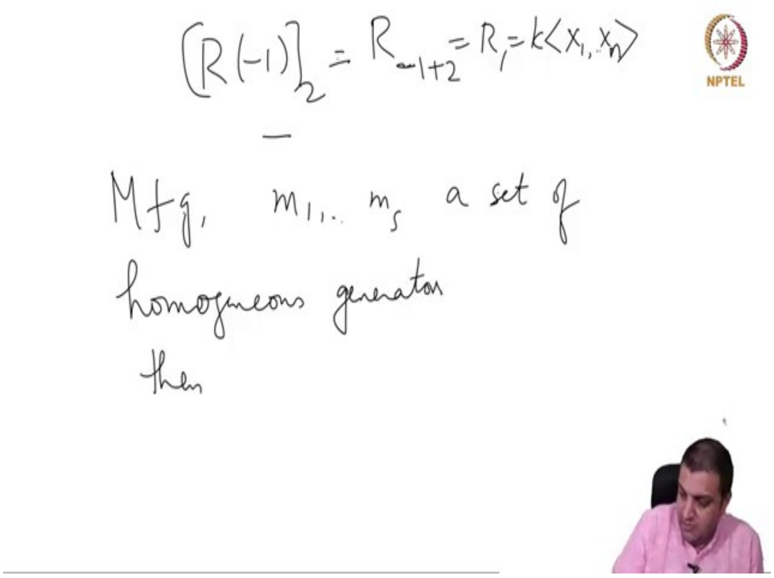

If you look at $[R(-1)]_2 = R_{-1+2} = R_1$ which is the k span of the polynomial ring linear span of the X’S and so on.

So, this is the free module of rank-1; I mean abstractly as an R module it is isomorphic to R except it is not isomorphic it is a graded module because the generators are in different degrees. So, this is one thing that we should keep in mind our notation . So, in other words in the previous presentation matrix; so, back to the so, let us say M is finitely generated $m_{1,} \ldots, m_s$ a set of homogeneous generators.

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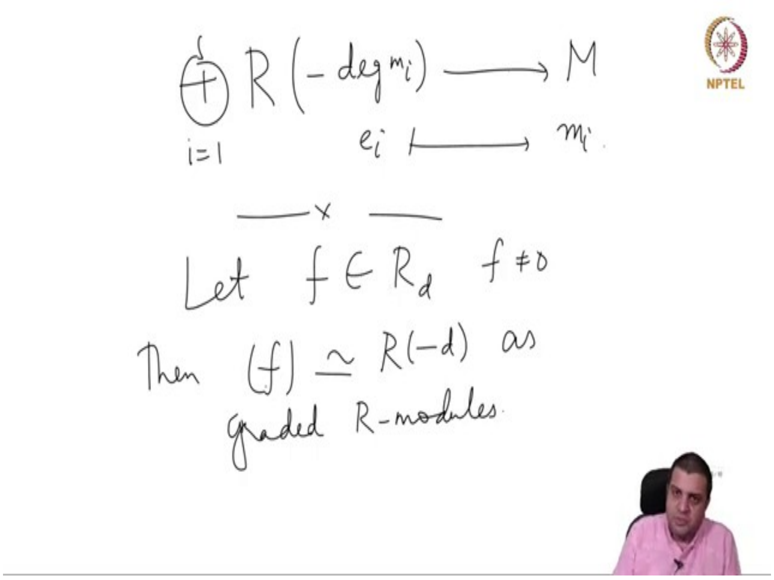

Then, what in that previous how we can restate that previous statement is, if you take R, then we shift by minus the degree of m_i, . $\bigoplus_{i=1}^{s} R(-deg(m_i)) \to M$. $e_i \to m_i$

So, that is what we defined that has this degree. So, if I is a principal ideal. Let $f \in R_d$ homogeneous $f \neq 0$. Then (f) is isomorphic to $R(-d)$ as graded R-modules. Remember, this is homogeneous of degree d. So, and why is that? Well, just use a presentation matrix we use.

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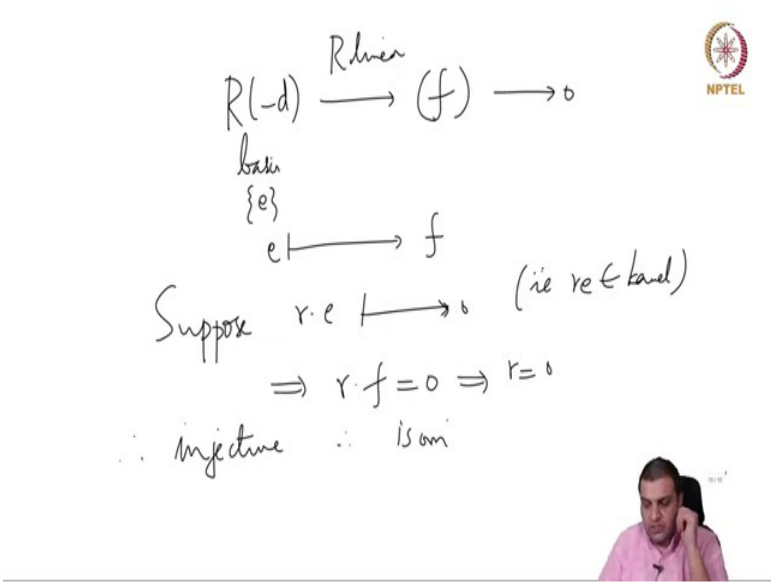

So, we can. So, we want to present we want to find a generating set for that one. It is generated by f itself which has generate which has degree d. So, we take R(-d) and we get a surjective map $R(-d) \to (f) \to 0$.

So, the generator of this one should not call it 1. It is not really 1; the generator is some module element in some degree. So, let us call that basis e. So, $e \to f$ that is what the map is.

Now, $r\,e \to 0$ meaning $r\,e \in ker$. In other words; so, what would that say that would say that r f = 0, but that would imply that r = 0 because now it is actually a multiplication happening inside R. So, in other words this is injective and therefore, isomorphism I mean we defined we started with the surjective map . So, this is one observation.

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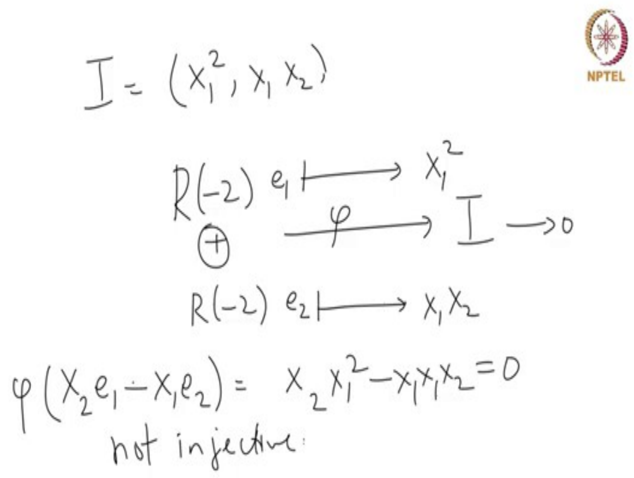

And, let us say that let us just now consider $I=\left(X_1^{2,} X_1 X_2\right)$ this is the ideal So, this has a generating set of two elements both of which have degree 2. So, now, we can map a free module one basis element for each mapping to I and this will be surjective .

So, let us say the basis element here is $e_{1,} e_2$ and we may as well we are free to choose we may choose $e_1 \rightarrow X_1^2$ and this $e_2 \rightarrow X_1 X_2$. We can choose, this is just a choice. So, but this is not an isomorphism because the well the easiest way to well there are many ways.

One for example, so, let us call this map φ. If we choose $X_2 e_1 - X_1 e_2$ if you apply φ to this. So, X_1 has come out because that is in R and this map is R linear we would get $X_1 \varphi(e 1) - X_2 \varphi(e_2) = X_2 X_1^2 - X_1 X_1 X_2 = 0$. So, this is not surjective it is not injective; so, not injective not isomorphism.

And, it is not very surprising at all because another way to think about this whole thing is this is a. If you invert all non-zero elements of R. Now, we are going outside the case of graded things just invert all nonzero elements in R; R is a domain this is a field. So, this is a this would be a rank-2 vector space over the fraction field I will give a rank because elements of I have been now inverted.

So, I will give the full fraction field, but rank 1 vector space and we cannot have an isomorphism from the rank-2 vector space to a rank-1 vector space. So, again so, we say that even after we invert elements the kernel is non-trivial; kernel is already to start with it is a non trivial. So, this is just some notation about this thing and I. So, make a definition and then just one basic property and then more important things we will do in the next lecture.

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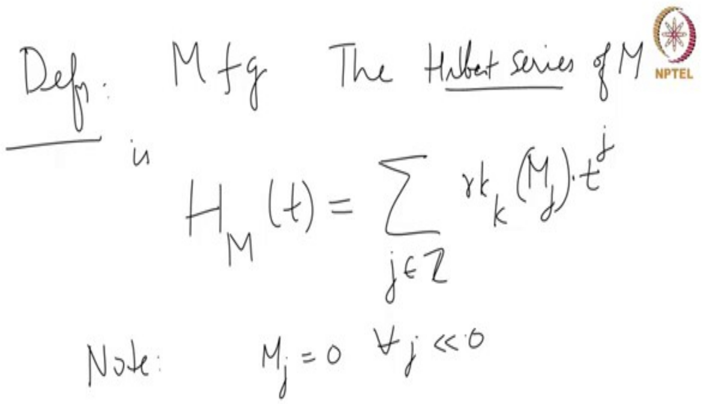

So, definition M finitely generated. So, we are only assuming working for finitely generated modules. The Hilbert series of . So, this is a formal series it is just what is called a generating function. We will not need to worry about these terminologies. What is this .

$H_M(t) = \sum_{j \in Z} rk_k(M_j) t^j$. So, M is finitely generated would imply that in each degree which is finitely generated over the base field over R_0 which is k. So, this is some number an integer for non-negative integer. So, that rank times t^j. So, we can think of this as an infinite sequence.

So, first of all note that $M_j = 0$ for all j < < 0. So, this starts from some value and then goes up and it could be infinite depending on whether M itself is finite or not , but it does not go into minus infinity it goes it might go into plus infinity.

So, this so, we can think of it as an infinite sequence of numbers that is captured together in a formal series. Formal power series well it is not exactly a power series because it might also involve t^{-1}, t^{-2} etcetera, but in some formal object like that. So, that is what this thing it does. So, it sort of captures the Hilbert function.

So, this is the value of the Hilbert function at j. So, it writes that function in terms of it is in terms of a formal series that is all that it has done. Now, but in some sense it is easier to manipulate this than just worry about the function as such, and that is the only reason why this is preferred I mean this is sometimes preferred.

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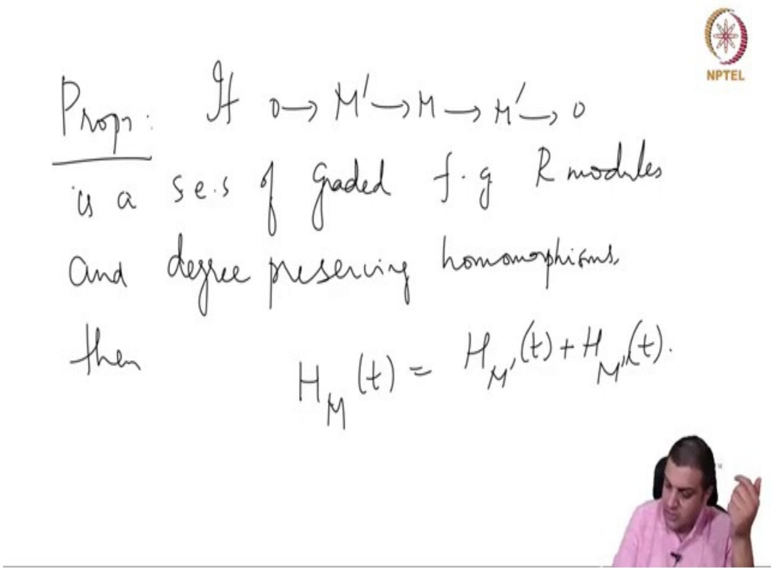

So, an immediate observation relevant here is that if $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$is a short exact sequence of graded finitely generated R modules and degree preserving homomorphisms . Then the $H_M(t) = H_{M'}(t) + H_{M''}(t)$ that is because in each degree if you fix it to a degree the rank of this is the sum of these two ranks and then we can just so proof.

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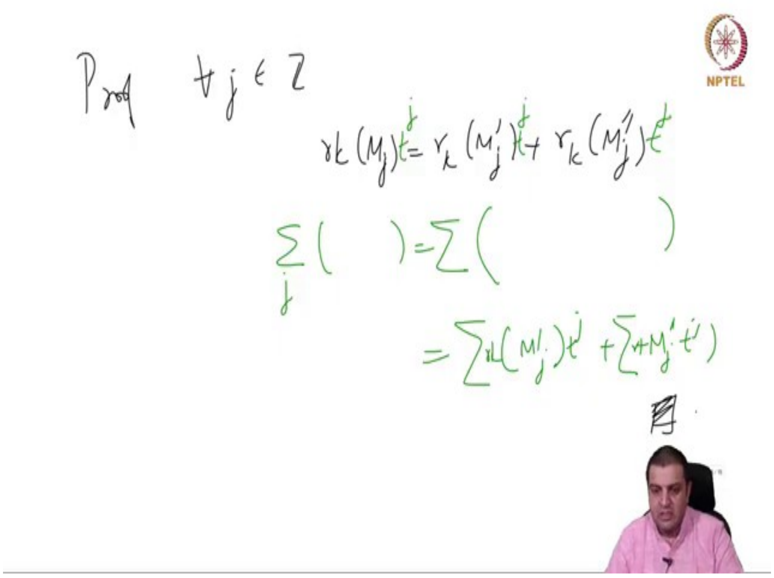

For all $j \in Z$ $rk(M_j) = rk(M'_j) + rk(M''_j)$ and so, then we are just multiplying by t^j here then we take a sum over j of this quantity. So, that is the sum over this quantity but the point is that this is the formal thing.

So, we do not have to worry about convergence or any of those things. So, this is formally $\sum_j rk(M_j) t^j = \sum_j rk(M'_j) t^j + \sum_j rk(M''_j) t^j$

So, we can switch the order in which we are taking the sum it is just a ok. So, this is the; this is the proof.

So, we will stop this lecture here and in the next lecture we will study this Hilbert series a little bit more detail.