Computational Commutative Algebra
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Lecture – 47
Graded rings – Part 2

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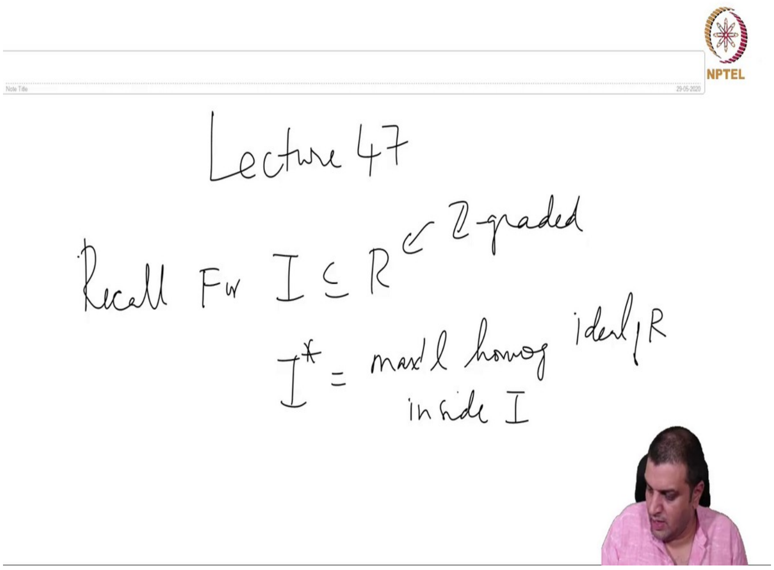

Welcome this is lecture 47. We wanted to continue looking at this maximum homogeneous sub ideal so, we recall. R this is Z-graded and for I inside that, define I^{star} to be the maximal homogeneous ideal of R inside I.

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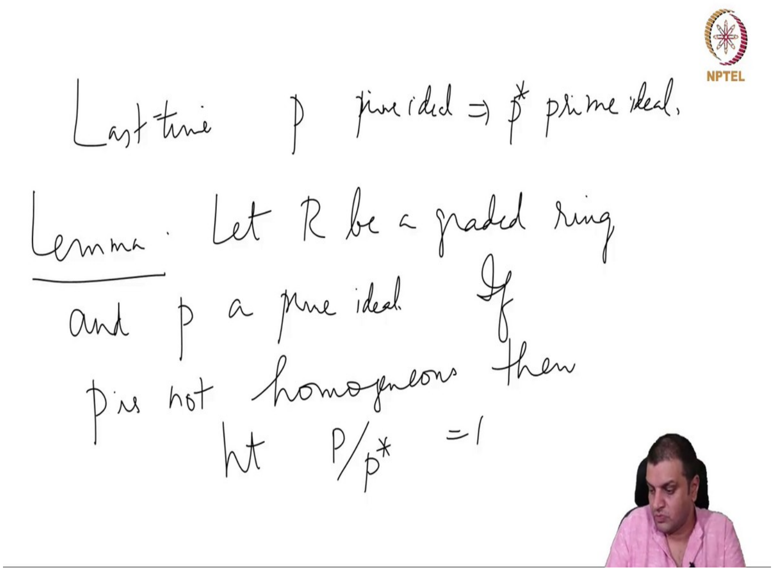

And, we saw last time that p prime ideal implies that p^{star} is also a prime ideal. So, now, we wanted to make two observations lemma write them. Let R be a graded ring Z-graded as and P a prime ideal.

If p is not homogeneous, then $ht\left(\frac{p}{p^{star}}\right)=1$. Then p would not be equal to p^{star} and, but they cannot be any ideal, the homogeneous or not any ideal between strictly between them . So, height of this is 1 proof.

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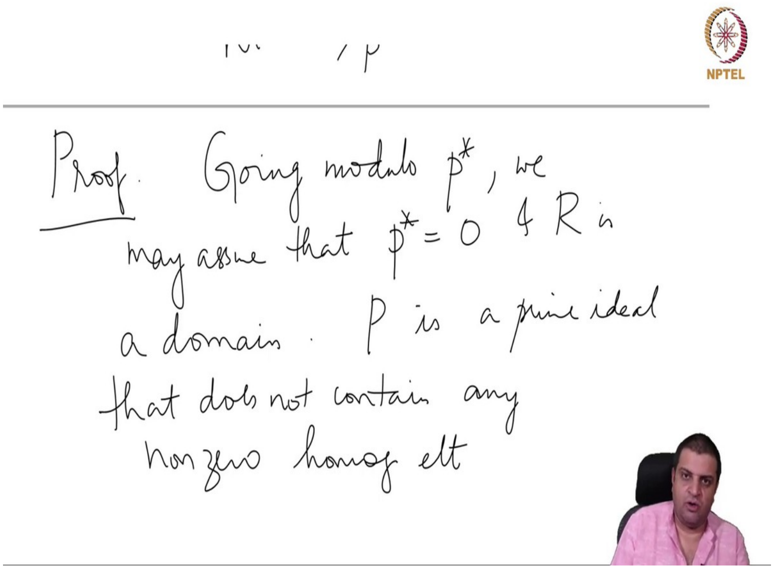

So, of course, this is a statement about chains going from p^{star} to p So, we may as well just kill p^{star} and which is a prime ideal. So, we may as well kill p^{star} and then look at chains going from 0. So, going modulo p^{star} we may assume that p^{star} is 0.

R is a domain, p^{star} is a prime ideal this we established last time. So, now if we think about this we have a prime ideal p, which does not contain any homogeneous ideal. So, that is what p is a prime ideal, that does not contain any non-zero homogeneous element does not element.

So, if we invert all the non-zero homogeneous elements p would still be a prime ideal there and more importantly any chain between 0 and P will be preserved there of the same length .

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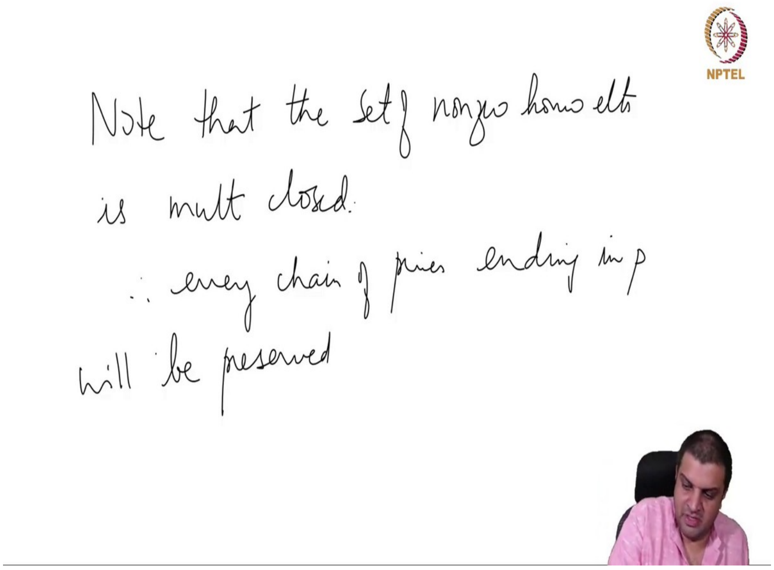

So, note that, the set of non-zero homogeneous elements is multiplicatively closed. , if you have 2 homogeneous elements their product is homogeneous and non-zero we are working in a domain. So, it is non-zero, so it is multiplicatively closed. Therefore, every chain of primes ending in p will be preserved. preserved in the sense that it will give an chain of the same length or maybe.

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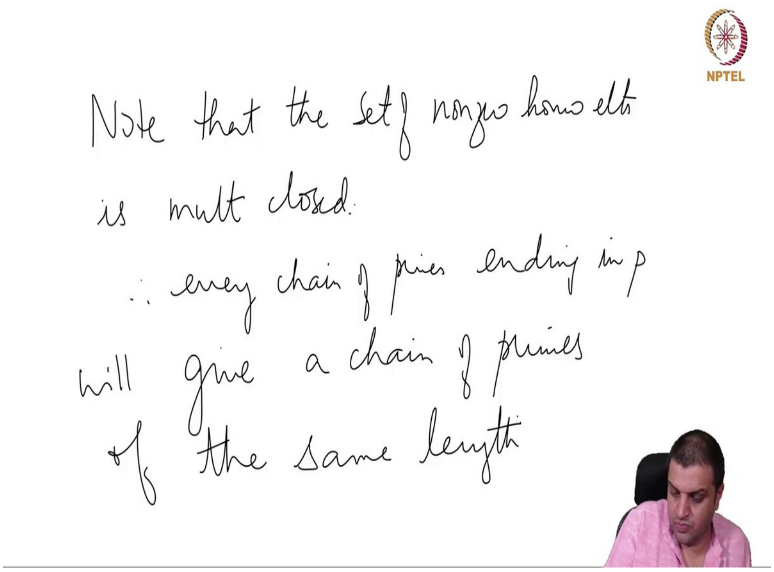

We will give a chain of primes of the same length.

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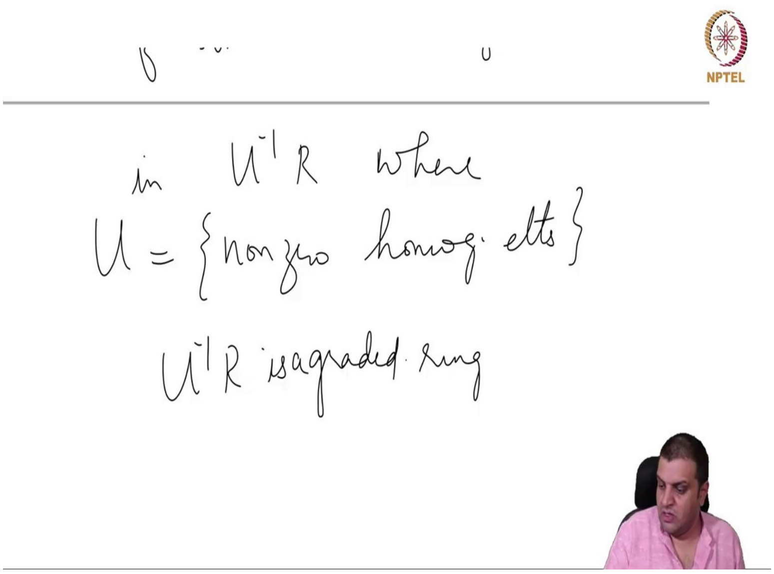

In $U^{-1}R$, where U is the that set of all nonzero homogeneous elements. So, in the ring $U^{-1}R$, there are no every homogeneous element is invertible in $U^{-1}R$.

So, this is a $U^{-1}R$is graded. It is decomposition is it can be written every element can be written uniquely as some homogeneous element over some element inside U, which are also homogeneous. So, it is a graded ring.

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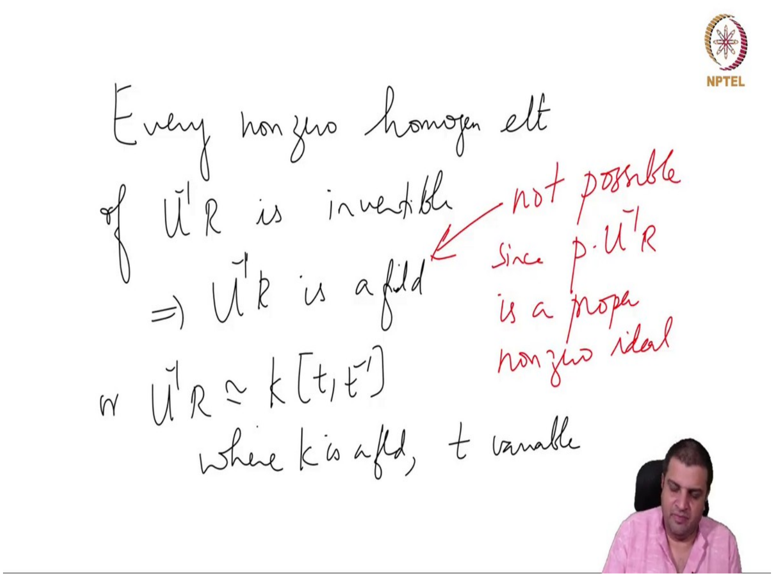

So, in particular, so what do we know and every non-zero homogeneous element of $U^{-1}R$ is invertible. Which means that, $U^{-1}R$ is a field or $U^{-1}R$ is isomorphic to $k[t, t^{-1}]$ where k is a field and t is a variable. but this is not possible because we know that the prime p gives a non-zero non-trivial prime inside $U^{-1}R$. So, this is not possible.

Since, $pU^{-1}R$ is a proper non-zero ideal so, it $U^{-1}R$ could not be a field. So, this is not possible. Therefore, it is of this form . So, and therefore, but this ring has only two kinds of primes. There is the 0 prime ideal and then there is all the maximal ideals, which is not t.

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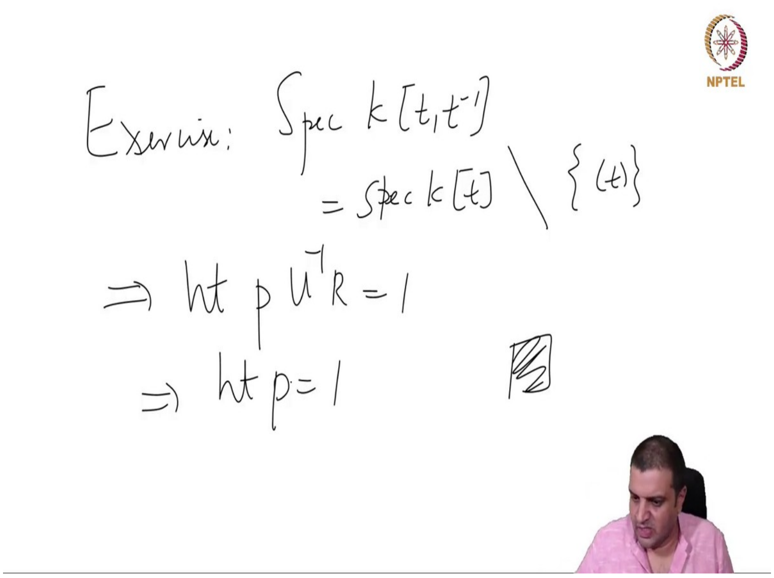

So, exercise $spec(k[t,t^{-1}])=spec(k[t])\setminus\{(t)\}$. The set containing the maximal ideal generated by t. every other prime ideal k[t] has a it is image here is a prime ideal, proper prime ideal except for this this is the case there. So, which now implies that $ht(pU^{-1}R)$

is 1, it is not 0.

Therefore, it can only be 1, which now implies that ht(p) is 1, you know this is it could not have been bigger than 1. So, this is the proof which is what we wanted to prove we reduced to the case where p^{star} is 0, then we wanted to show that p was height 1 prime. So, this is what we needed. So, this is where the lemma about the structure of such rings, every non-zero element is was going to be used.

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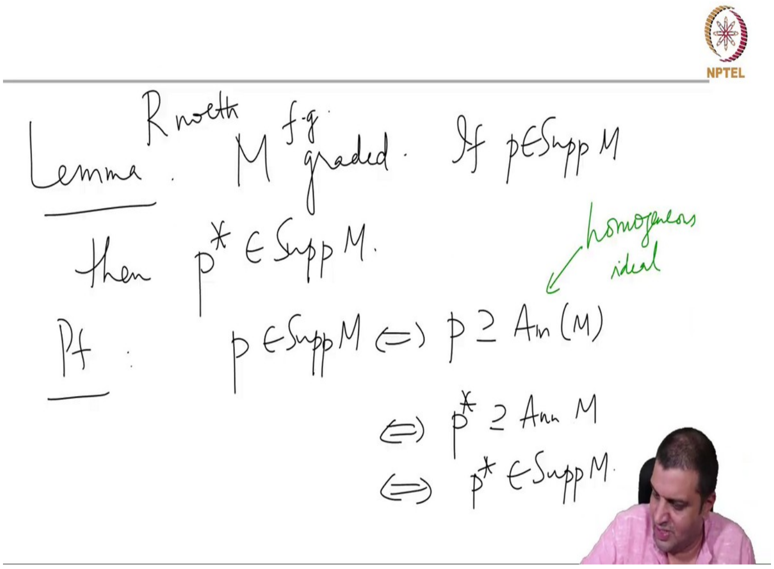

Another lemma , M is graded if $p\in supp(M)$then $p^{star}\in supp(M)$. So, let us just assume for this case that it is finitely generated, because the argument is.

So, the observation is that $p\in supp(M)$ if and only if $ann(M)\subset p$ this is where the finitely generated is used . But, annihilator of M is the intersection of the annihilator individual elements of a generating set and we already observed that annihilators of homogeneous elements are homogeneous ideals.

So, intersection is also homogeneous ideal. So, this is the homogeneous ideal. But, p contains homogeneous ideal and p^{star} is a maximal homogeneous ideal containing inside p. And, so, therefore, this is equivalent to say p^{star} contains annihilator of M, which is equivalent to saying that p^{star} is in the support of M.

For non-finitely generated modules this first equivalence and third equivalence may not hold, that support of M could be very small, but annihilator could be 0 that is possible, but for finitely generated modules this is true yeah. So, now, we come to description of the support of a finitely generated module theorem.

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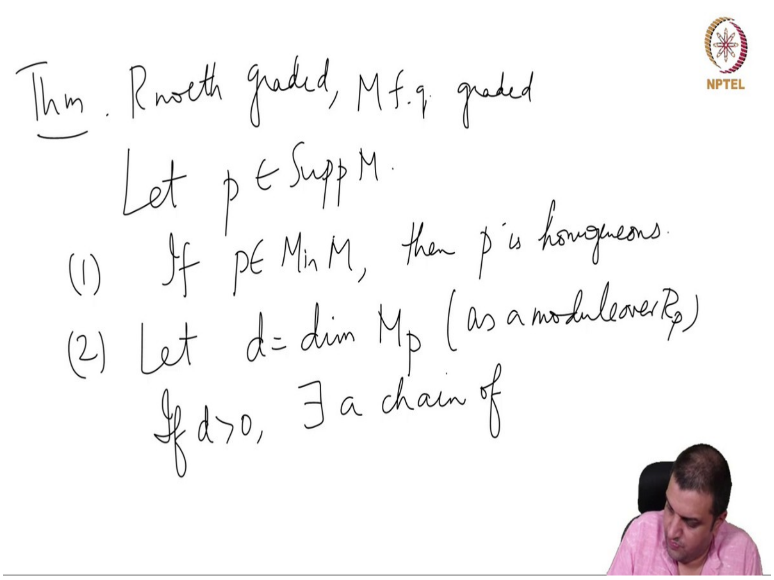

sorry in this lemma also I in this proof I assumed R noetherian. Let us anyway we will worry we need to worry about this only in the noetherian case R noetherian graded, M finitely generated graded module. Let $p \in supp(M)$.

We will make 4 statements successively building up

1) If p is a minimal prime of M, then P is homogeneous. This is really the previous lemma if p is a minimal prime then p^{star} also contains annihilator of M and is inside p. So, p has to be equal to p^{star}. So, I will write down, but this is just previous lemma .

2) Let $d = dim(M_p)$. So, now, we have lost all grading it is just a local ring and I mean when we say this this mean as a module over R_P. So, over R_P, M_P is a finitely generated module and it is just an noetherian local ring and a finitely generated module over it no sense of grading and just well we make the statement , if d is positive .

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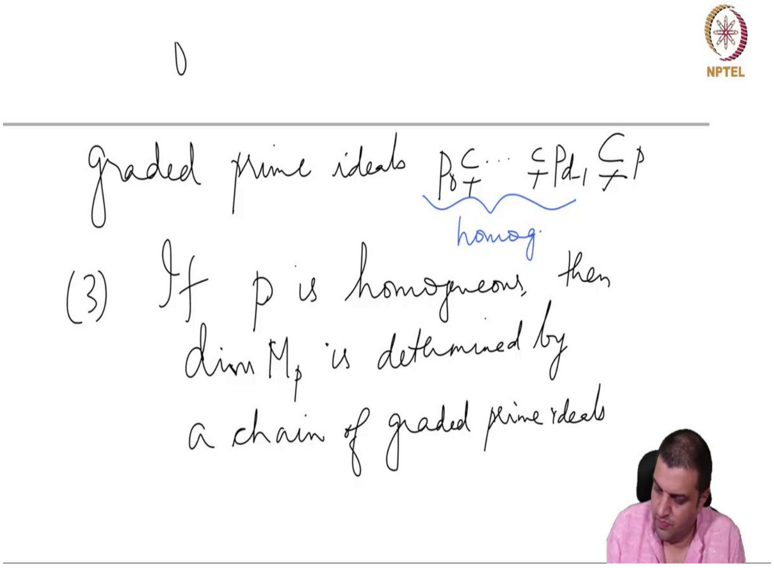

There exist a chain of graded prime ideals $p_0 \subset P_{d-1} \subset p$. So, all of these are homogeneous ideals inside p. So, sorry just to clarify, these are the homogeneous ones not p. p itself is not given to be homogeneous just to clarify we are not asserting anything about p.

Now, let us assume 3 if p is homogeneous, then $dim(M_p)$ as an R_P module is determined by a chain of graded prime ideals. So, in other words, if d is the dimension of this dimension, then there is a chain of primes in which p is also homogeneous.

So, sorry in this case we can assume strict inequality here also. So, there we if p is homogeneous we can take this p also to be homogeneous I mean this is that same p . So, then we get a chain of length d and hence that determines the dimension .

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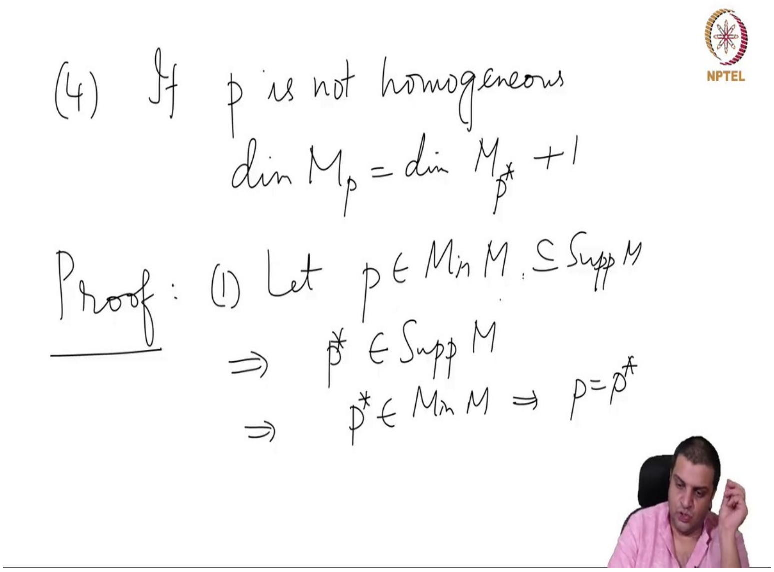

And, 4 if p is not homogeneous, $dim(M_p) = dim(M_{p^{star}}) + 1$. So, proof I will prove 2, 3 is immediate from 2, 4 requires a little bit of thought, but I will leave that as an exercise. So, that 1 becomes familiar with this sort of calculation. So, 1 is immediate from the previous lemma, previous lemma said that anything if p is in support p^{star} is also in the support.

So, let P be minimal prime over M yeah let me write the proof using previous lemma 1 could prove it slightly differently also, but we will do that. So, let $p \in Min(M) \subset supp(M)$. So, this implies now that .

So, this is inside support this is the minimal elements in support p^{star} is inside support of M, but by definition of minimality this 1 says that, p^{star} is also in minimum of some elements of M, which the only way it is possible is $p = p^{star}$.

So, another argument could be that minimals minimal elements of this minimal elements over M minimal primes over M are associated, but we are already observed while ago that associated primes are homogeneous. Because, associated primes are annihilators of homogeneous elements in this particular case 1 can show that and so, this is homogeneous. 2 so 2 is the technical part.

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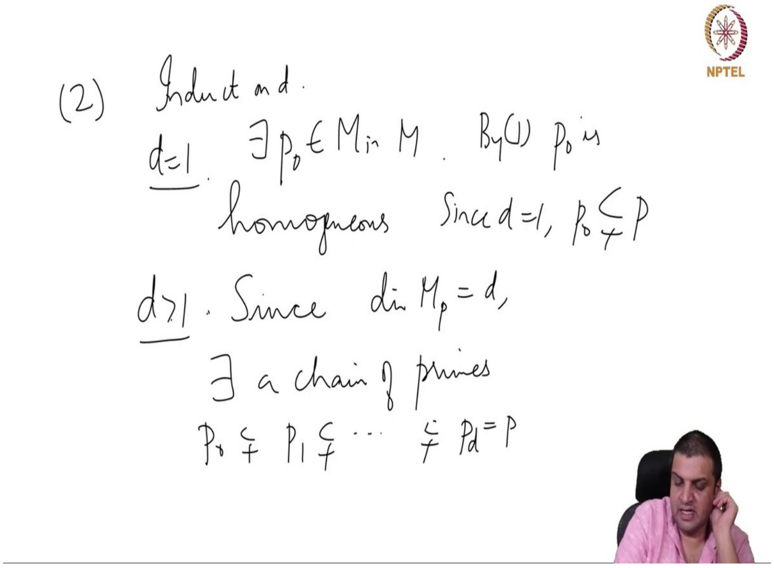

So, I will prove that and the rest 3 is immediate and 4 is I will leave it as an exercise. So, 2 so, we induct on d we assume that statement only for positive d. So, if d = 1, then there exists $p_0 \in Min(M)$ by 1 p_0 is homogeneous. And, also since d = 1 p_0 is properly inside p. So, therefore, we got if d = 1 we just need to find 1 prime homogeneous prime ideal, that we can take as 1 in the minimal primes itself.

So, now assume d > 1 and assume it is true for smaller d. since $dim(M_p)=d$, there exists a chain of primes $p_0 \subset p_1 \subset \ldots \subset p_d = p$. Because, that is somehow the dimension has to has to be has to come from a chain of primes. So, this is true let us look at P_{d-1}, d > 1. So, we could d -1 is still positive .

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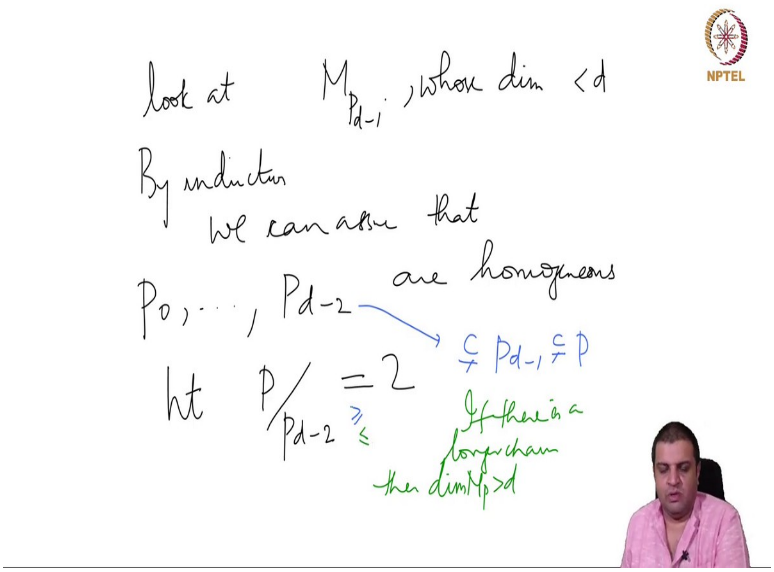

So, look at $M_{p_{d-1}}$ ok. this one's dimension is less because it is inside you know 1 shorter chain of primes. So, this by induction look at this whose dimension is less than d . So, now we can use induction. So, by induction we can assume that $p_{0,} \ldots, p_{d-2}$ are homogeneous. now nothing about P_{d-1}.

Because, in the theorem there is no nothing about p_d, which is P sorry maybe I forgot the yeah. It is only about $p_{0,} \ldots, p_{d-1}$ that we are asserting, that there it is homogeneous prime ideals last 1 there is no assertion. So, same thing we get this is a homogeneous. What about

$ht\left(\frac{p}{P_{d-2}}\right)$?

It has to be 2, because it is at least 2 because it is a P_{d-1} probably not homogeneous and then there is p. So, it is at least 2, but it cannot be bigger than 2, because you have already covered d - 2 elements and if after this there is chain longer than 2, then dimension would have been greater than d, but since dimension is d this is equal to 2 itself.

So, let me just say this again it is at least 2; because after this there is a p_{d-1}, which is which is there because and then there is p. So, there is at least 2. So, this is therefore, this is at least 2.

So, this gives at least. And then we observe that if there is more there is a longer chain .

Then, $dim(M_p)$ would be bigger than d, but we said d is the dimension. So, that says it is less than and together it is equal. So, dimension of this is 2.

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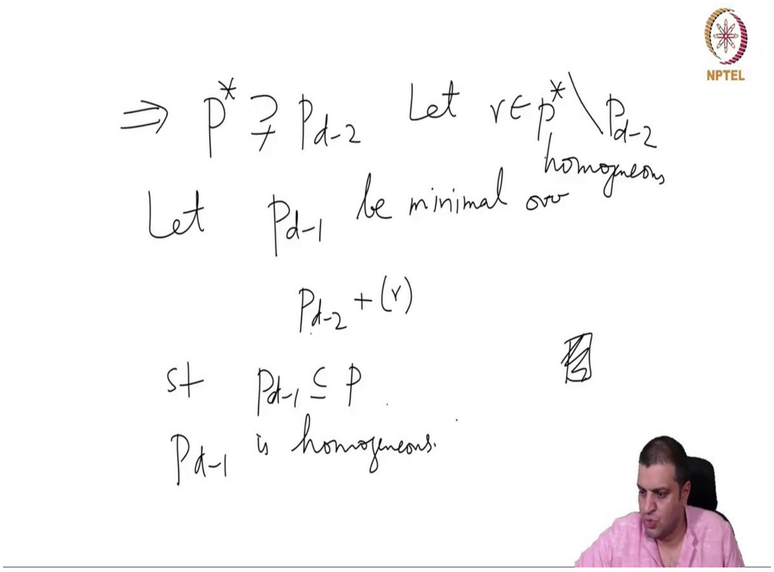

This implies that, p^{star} must properly contain p_{d-2}, because $\frac{p}{p^{star}}$ has height only 1, but $\frac{p}{p_{d-1}}$ has at least 2 and p_{d-2} is homogeneous. I mean it is possible that p is homogeneous . So, we are not asserting that p is not equal to p^{star}, but in a in either case p^{star} cannot be equal to p_{d-2}, because the difference in height is 2 not 1.

Now, let $r \in p^{star} \setminus p_{d-2}$ and let p_{d-1} be minimal over $p_{d-2}+(r)$, such that p_{d-1} is inside p. So, this is a homogeneous ideal a minimal prime will be homogeneous and we choose a minimal prime inside this or let us argue you know we do not need to worry about homogeneity.

Now, I mean this is some ideal inside p, I mean $ht\left(\frac{p}{p_{d-2}}\right)$ is equal to 2. So, there must be a prime which contains this bigger ideal and p, there must be a prime ideal containing those 2 among them take 1 that is minimal that would be homogeneous.

And, therefore, we have got p_{d-1} also and this is inside P. So, this is so, p_{d-1} is homogeneous, because it is minimal over a homogeneous prime ideal, homogeneous ideal, yeah sorry yeah this r has to be taken to be homogeneous. So, this ideal is homogeneous; therefore, a minimal prime is also homogeneous and that is the end of the proof.

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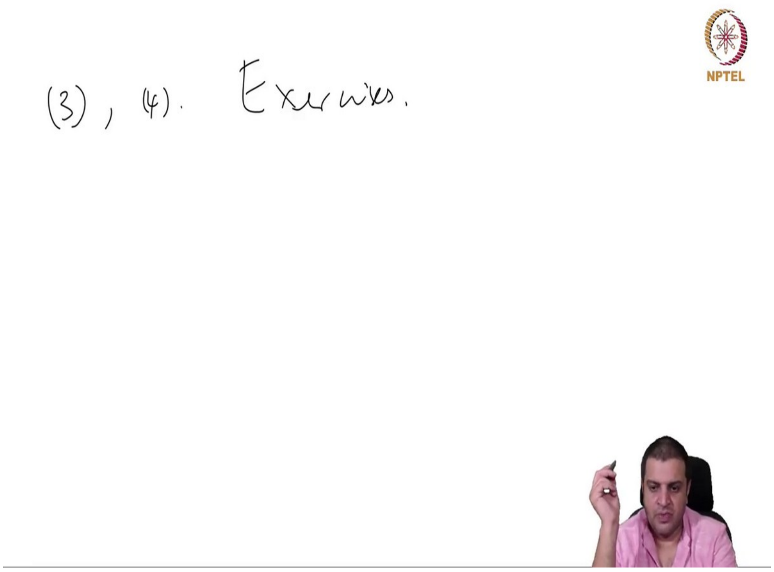

So, this is 2, 3 and 4 I leave as an exercise. 3 is immediate 4 requires a little bit of thought if p is not homogeneous, then the dimension is smaller; because there are any chain from up to p^{star} in the support of M can be extended by at least 1. But, it cannot be smaller by more than 1. So, just think a little bit and 1 one can prove this. So, these are exercises follows from 2 really.

So, this is this is the general structure of graded rings Z-graded noetherian rings and finitely generated module over them, how chains of primes will look. So, therefore, when we want to look at graded rings. Now, we will restrict ourselves to polynomial rings in 1 in finitely many variables over a field. And, when we want to look at them we will worry about graded ideals and there are applications to; so, here after we will worry about these things.

So, in the first lecture maybe little bit of an in the next lecture and little bit after that we will discuss some one thing called the Hilbert series of a graded module. It builds on what we did earlier about Hilbert polynomial in Hilbert function and Hilbert polynomial.

It is just another way of representing the same data has certain other certain advantages. So, we will learn that, then we will sort of you worry about applications of these graded concepts in geometry. And, we will also look at some things I mean tie these things with back with Grobner basis ok. So, that is that will be the content in the next few lectures.