Computational Commutative Algebra

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Lecture – 46
Graded rings – Part 1

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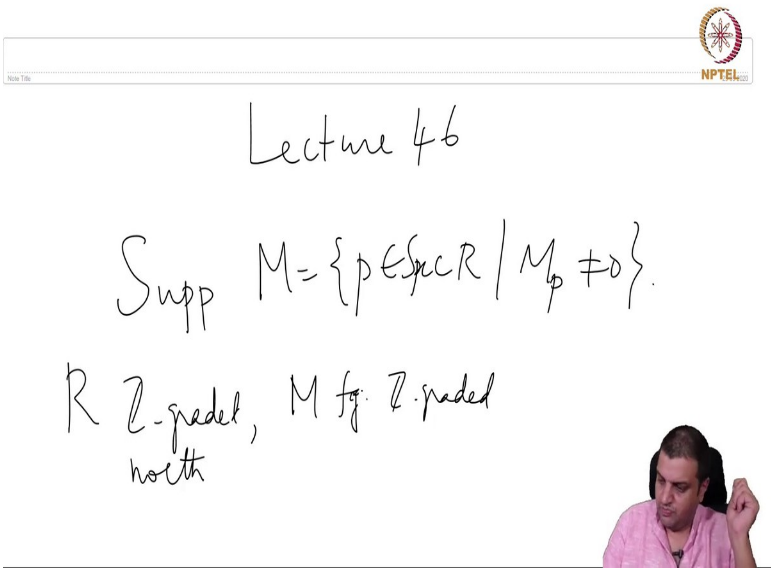

Welcome to lecture 46. We wanted to now describe what is called support of a module. We let me we have seen this earlier, but let me just remind you what this means. So, $Supp(M) = \{p \in Spec\,R : M_p \neq 0\}$. Min(R) is precisely the minimal elements of this and in the noetherian situation associated primes is contains Min R and is a finite subset of support. This is what we know.

So, what we want is R is Z-graded noetherion, M finitely generated Z-graded. We would like some understanding of support of M or at least we want to conclude that where we need to consider if you want to consider chains in support of M we just need to consider chains of homogeneous prime ideals.

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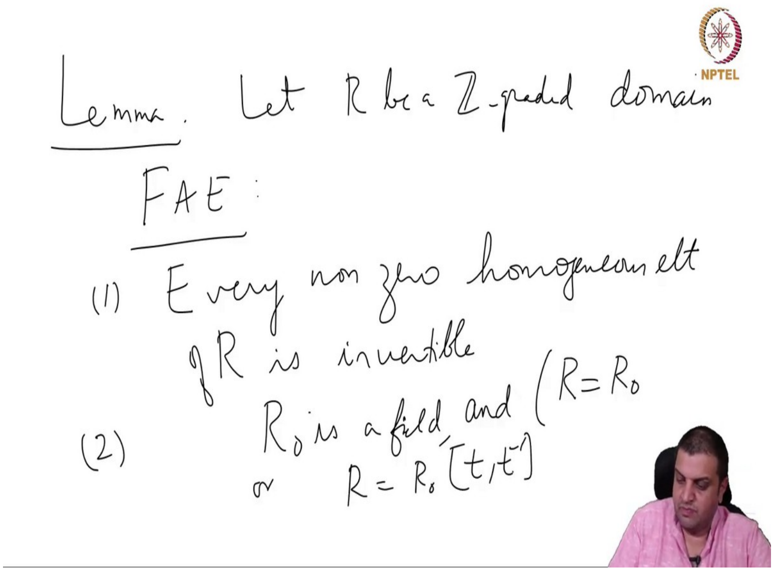

So, we need a lemma first. So, let R be a Z-graded domain. Then the following are equivalent.

1) Every every non-zero homogeneous element of R is invertible.

So, this is not I mean it is not saying it is a field. It is saying only homogeneous elements are invertible and

2) R_0 is a field and $R = R_0$ in which case it is of R itself is a field or $R = R_0[t, t^{-1}]$

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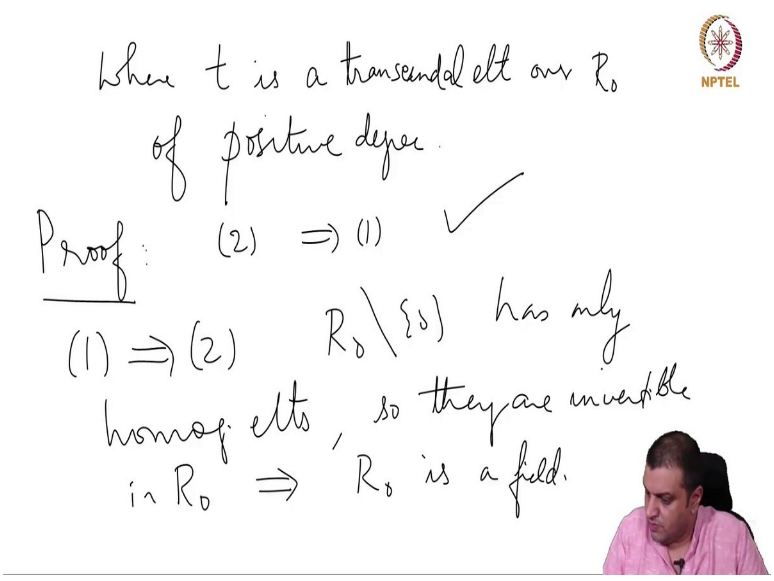

And, what are t? Where, t is a transcendental element over R_0 of positive degree. So, let us look at this; look at this thing. So, in this case this is a field. So, it is some $k[t, t^{-1}]$, t is algebraically independent over k that is what it is meant by saying transcendental element.

It does not satisfy any algebraic relation not just integral relation. Even algebraic relations even relations that involve nonzero, non-trivial leading coefficient even those not satisfied and t^{-1} is just the inverse of that. So, $k[t, t^{-1}]$ where t is a variable is precisely this situation and where we give this to a positive degree.

Positive degree is not a big restriction because if this had negative degree t^{-1} would have positive degree. So, you can choose the one of positive degree as t. So, this is the result that we would need. Proof: 2 implies 1 is immediate right. If R_0 is a field then and one of these conditions hold then every non-zero homogeneous element of R is invertible.

Because, if you take $k[t, t^{-1}]$ homogeneous elements are αt where α comes from the field. So, α has an inverse power of t has an inverse which is just t^{-1} So, they are all invertible . So, 2 implies 1 is immediate.

Now, every non-zero element of R_0 is homogeneous. So, it is now 1 implies 2. $R_0 \setminus \{0\}$ consist of homogeneous elements. I mean every element of this has only homogeneous elements.

So, they are all invertible, but if the inverse of a homogeneous element is also homogeneous of minus that degree they are all invertible in R_0 not just inside R . They will have because they have degree 0 inverse will also have degree 0. So, they are invertible in R_0 which now implies that R_0 is a field. So, that proves the first part.

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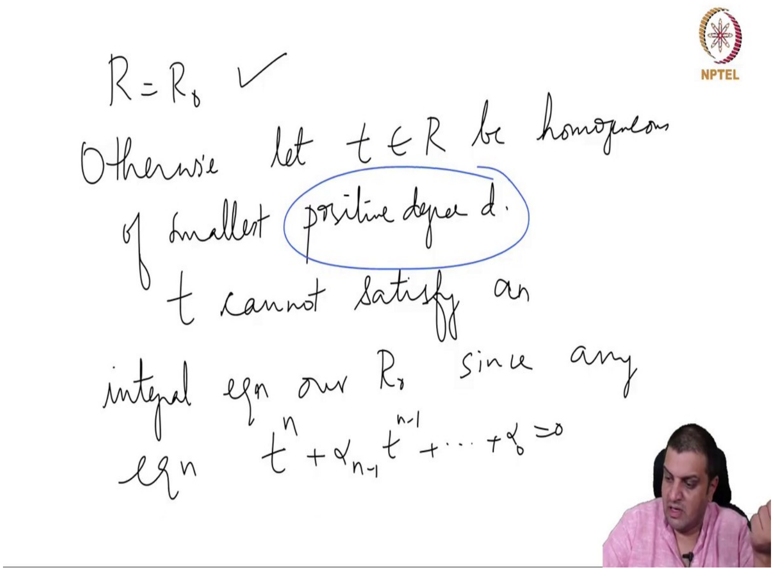

Now, if $R=R_0$, then there again nothing to prove we are done. Otherwise let $t \in R$ be homogeneous of smallest positive degree d. Now, t cannot satisfy n algebraic equation over R_0; R_0 is a field algebraic equation is same thing as saying integral equation, cannot satisfy an integral equation over R_0.

Integral or algebraic is the same because it would say something like $t^n+\alpha_{n-1}t^{n-1}+...+\alpha_0=0$. All of these have different degrees. So, and the right side is 0 which is homogeneous. So, they must all split individually and such a thing cannot exist because t^n cannot be 0.

So, let me just write down what I was saying. Here this thing since any equation like this splits into its homogeneous parts. And that would give us $t^n=0$ which is not true, which is I mean that t would be 0. So, therefore, t has to be t is transcendental over R_0 .

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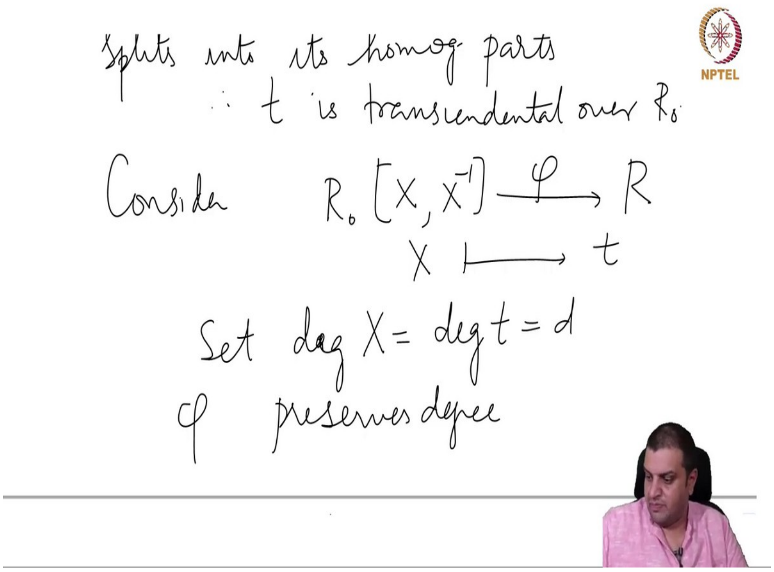

Now, we wanted to show that. So, this is just the first step. So, now, consider $R_0[X, X^{-1}]$ So, this is just inverting the multiplicatively closed system given by powers of X. One is for example, 1 + X has not been inverted, but X has been inverted .

Consider this map $\varphi : R_0[X, X^{-1}] \to R$ which sends R_0 to R_0 and X to t . Now, if again sort of similar argument; if something is in the kernel, then its individual pieces would be in the kernel which means some power of X would be in the kernel.

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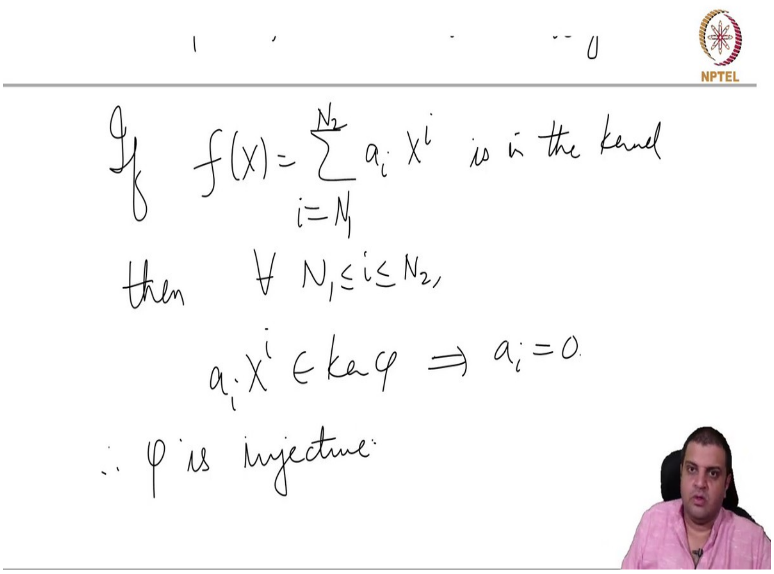

If $f(X)=\sum_{N_1}^{N_2} a_i X_i$; is in the kernel sorry, I need to make this homogeneous. So, let degree(X) =degree(t) =d which we called smallest which is this d .

So, we this element t has a smallest possible positive degree in that ring. So, this to be set this to be d. So, then φ is homogeneous. I mean φ preserves degrees. If you apply φ to a homogeneous element, it would have the same degree on the image will also have the same degree.

So φ preserves degree this would imply that $ker(\varphi)$ is a graded ideal. So, if any element is in the kernel, then for all $N_1 \le i \le N_2$, $a_i X^i \in ker(\varphi)$, but this is the only possibility therefore, is that $a_i = 0$.

Because X^i will go to something non-zero and we are in a domain, this is the one domain is used. X^i goes to t^i, which is non-zero and so, the only way the product can be 0 is that a_i is 0. Now, let us yeah. So, therefore, it is injective.

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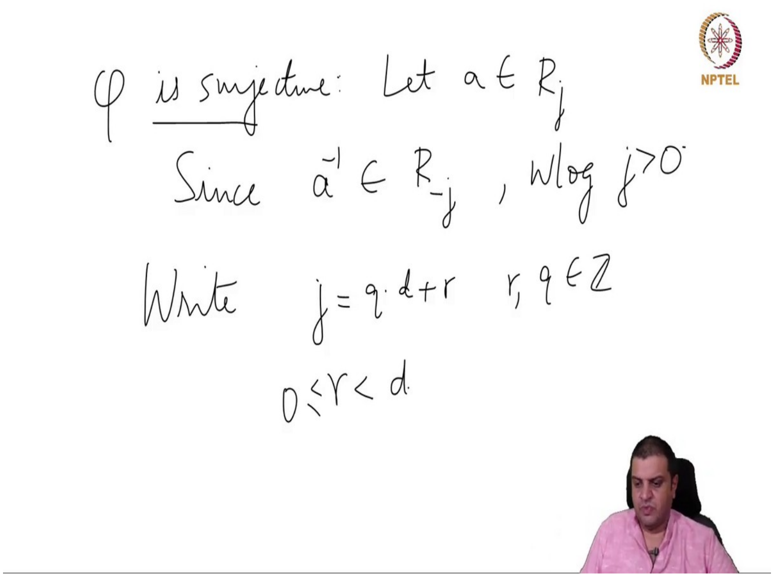

Now, to show a φ is surjective. So, suppose not. So, let $a \in R_j$, a homogeneous element of degree j. it is homogeneous and we are proving 1 implies 2. So, we are assuming that every homogeneous element is invertible 1 plus 2.

Since $a^{-1} \in R_{-j}$, we may assume without loss of generality that j is positive . Now, consider the following. So, write remember the degree of a is j, write $j=qd+r$ where $q,r\in Z$ and d is the degree of t just division algorithm .

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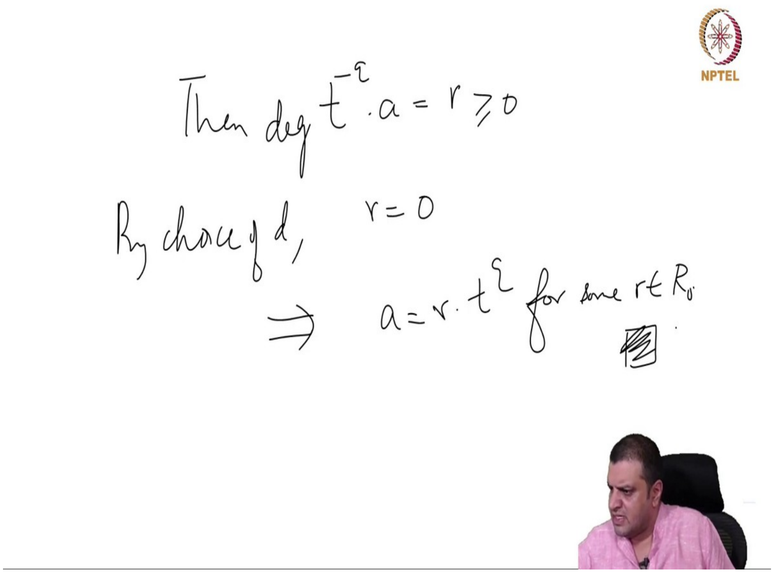

So, now so, then we consider $t^{-q}a$; a has degree j, t^{-q} has degree -qd. So, this thing has degree equal to r , but and this is non-negative right. So, by choice of d, r = 0. In other words a is some element of R_0, $a=rt^q$ for some $r\in R_0$. So, this is the proof.

So, this is somewhat technical looking statement, but once we come down to lots of induction steps in chain of primes a homogeneous primes, one would have to use some uses this.

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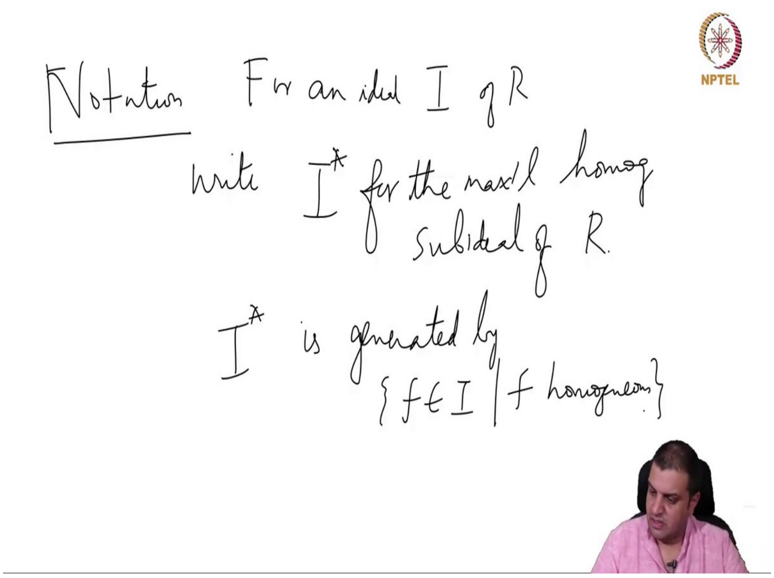

So, the next is notation for an ideal I of R write I^{star} for the maximal homogeneous subideal of R. . So, this is I^{star} is the set of all is generated by all $f \in I$, f homogeneous.

And, so, we so, remember where we are headed? We are headed to find the relation between chains of primes in the support of a graded module and chains of primes of homogeneous ideals. So, that is why we need this we need this notation.

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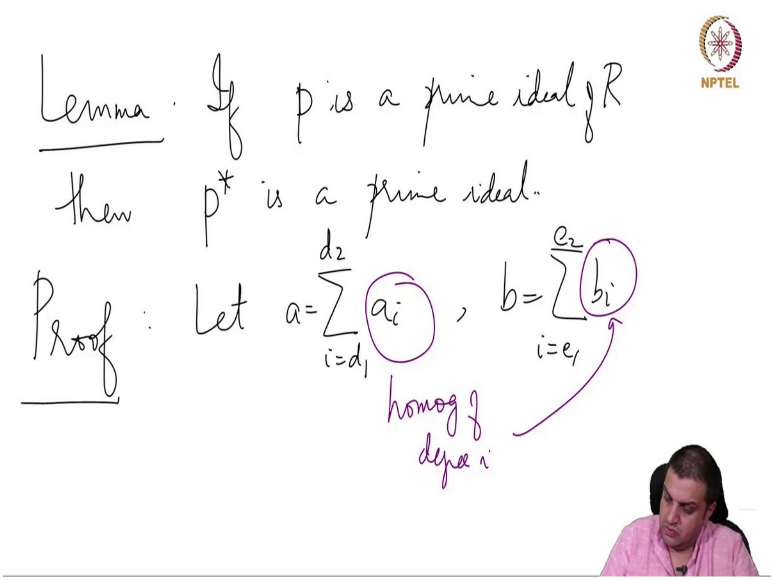

So, now lemma if p is a prime ideal of R, then p^{star} is a prime ideal. So, using this we will replace some arbitrary chain of primes with chain of homogeneous primes and we will be able to conclude something. So, let us prove this.

This is not I mean this is not a very difficult proof if one make this definition of I^{star} and gave this definition gave this claim; we will be able to prove it. But, let us just nonetheless try ok.

So, let $a=\sum_{i=d_1}^{d_2} a_i,\ \ b=\sum_{i=e_1}^{e_2} b_i$So, the by this we mean we are splitting this into its homogeneous parts . It is not just some summation. These are I mean a_i is homogeneous of degree i. So, this is homogeneous of degree i and similarly b_j. So, write like this. so both are elements of R.

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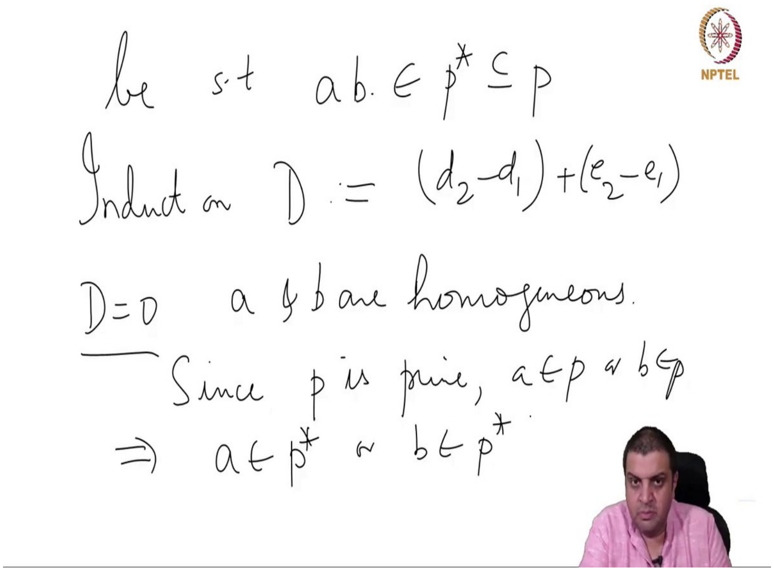

Be such that $a b \in p^{star}$, so which is inside p. So, we will need to use that one of them is inside p. So, we will induct on this difference of degrees. So, we can think of this as the difference between the lowest degree and the I mean lowest degree nonzero term and the highest degree of a nonzero term in both a and b.

So, let us induct on some integer on D. $D:=(d_2-d_1)+(e_2-e_1)$. So, we want to induct on this. If this is 0, which means that there is exactly one term here and one term here; a and b are homogeneous; a and b are homogeneous.

So, yeah we do not need to induct on the sum we could induct on the minimum number of this thing, I mean but the induction will have to do that we can replace that sum by something which has fewer terms that is the only idea.

So, we could probably induct on the minimum of the 2; a and b are homogeneous and since p is prime, either $a \in p$ or $b \in p$, but one of both are homogeneous yeah both are homogeneous which now implies that $a \in p^{star}$ or $b \in p^{star}$. So, if both a and b are homogeneous, then we have a conclusion.

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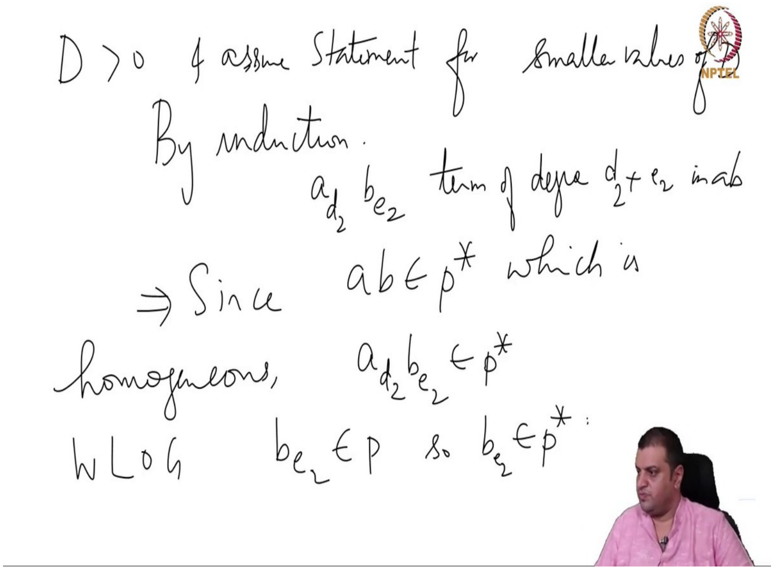

So, now assume that D is positive and assume the statement is true for than smaller values of D. So, let us look at what is the top degree term in here in the product well. So, we can assume without loss of generality that a_{d_2} is nonzero and b_{e_2} is nonzero. Otherwise we would we can lower these things and then we will have a smaller d.

So, by induction we can assume that a_{d_2} and b_{e_2} are nonzero because if they are 0 there will not be anything to prove and this is a term of degree $d_2 + e_2$; . And, this is the only thing that appears in that degree, the rest of it will have smaller degree.

Now, this implies now that since p^{star}. So, remember the product is inside p^{star}, if we have a if we have an element inside p^{star} all its homogeneous parts term of degree of in ab the product.

Since $ab \in p^{star}$ its every degree term will be in p^{star} this will be since $ab \in p^{star}$ which is homogeneous, $a_{d_2} b_{e_2} \in p^{star}$. So, without loss of generality b_{e_2} this is in p, but it is homogeneous. So, $b_{e_2} \in p^{star}$.

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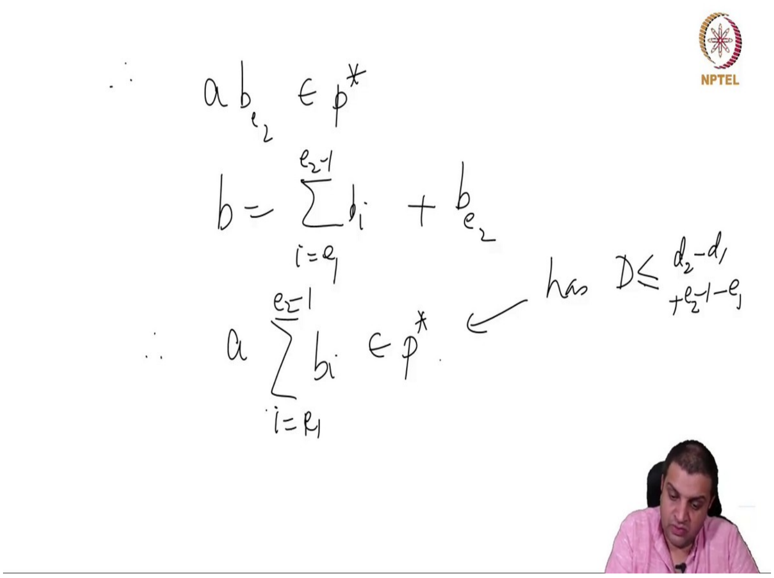

Therefore, $a b_{e_2} \in p^{star}$, it is an ideal. So, it is inside. Now, we can replace this by . So, replace b by this. So, $b = \sum_{i=e_1}^{e_2-1} b_i + b_{e_2}$. Now, therefore, $a \sum_{i=e_1}^{e_2} b_i \in p^{star}$. So, this is in the p^{star} , but now this has a smaller d.

This has smaller $D < d_2 - d_1 + e_2 - 1 - e_1$. So, this is the less than equal to 1.

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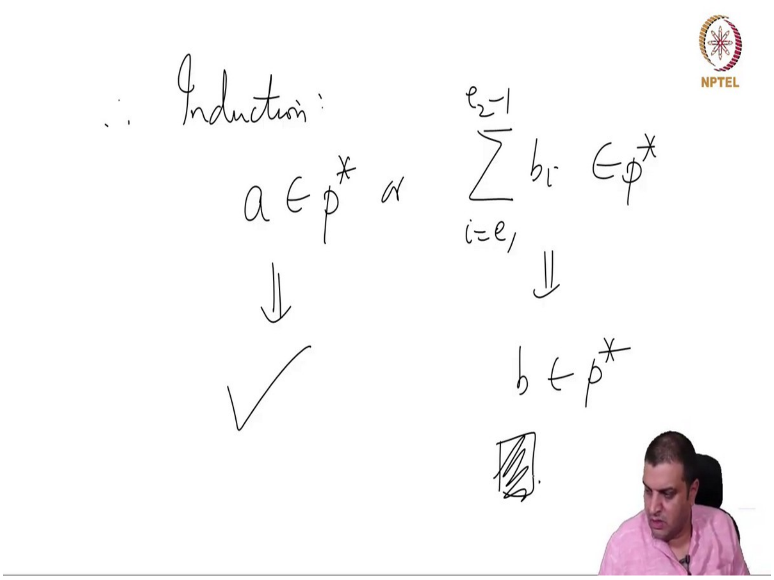

So, therefore, by induction $a \in p^{star}$ or $\sum_{i=e_1}^{e_2-1} b_i \in p^{star}$. If $a \in p^{star}$ then we do not have to do anything if $\sum_{i=e_1}^{e_2-1} b_i \in p^{star}$, then this implies that $b \in p^{star}$ because that final term that we are already said in this I mean we had without loss of generality we had assumed $b_{es} \in p^{star}$. So, without, so, this is the end of proof ok.

So, we will stop, we will end this lecture here and in the next lecture we will continue looking at Spec of support of a finitely generated module. And, we will prove the relation between non homogeneous ideals and their corresponding star homogeneous prime ideals and then we will try to understand chains and primes the chains of primes and support.