Computational Commutative Algebra
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Lecture – 45
Algebras over a field

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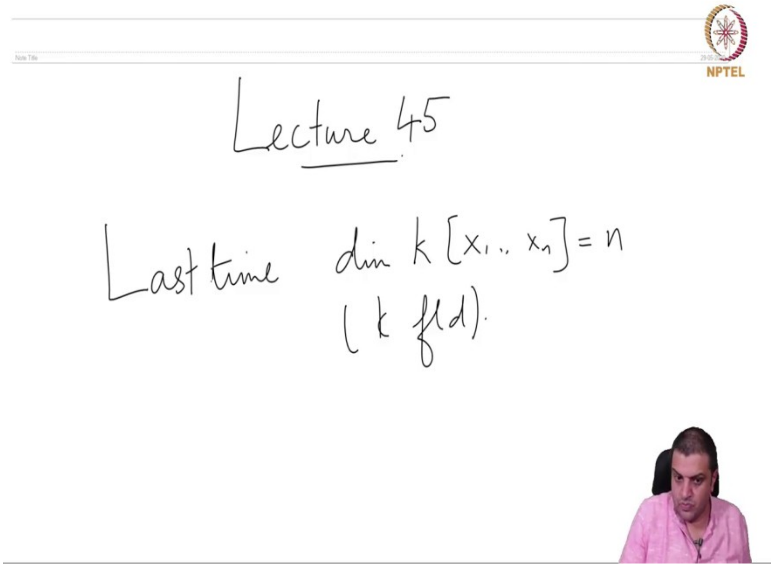

So, we apply what we learned about the dimension theory of Noetherian local rings to understand polynomial ring over a field. And we concluded that the dimension of a polynomial ring in n variables is n, where k is a field. This says that there exists a chain of prime ideals.

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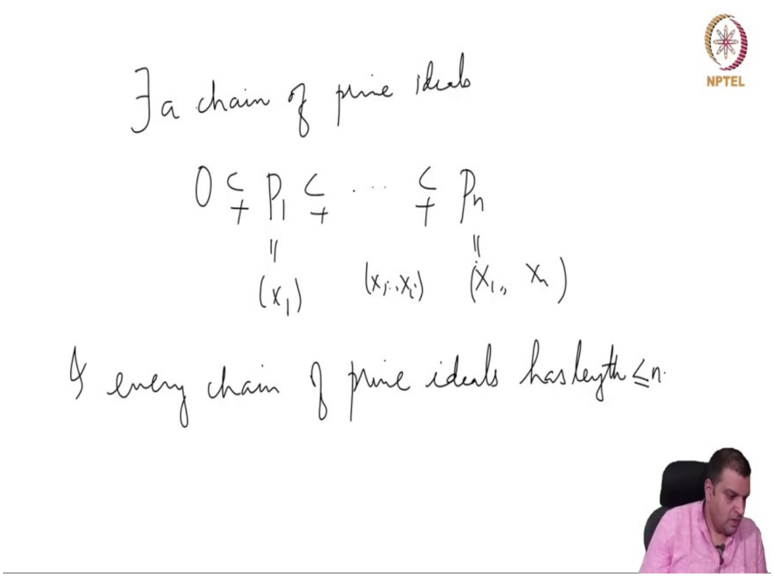

Recall, we are in a domain, so 0 is a prime, and then $0 \subsetneq p_1 \subsetneq p_2 \subsetneq \ldots \subsetneq p_n$. And we know one choice which is just the variables. So, we could take this one to be x_1, and then $x_1, \ldots, x_i$ and then the whole domain all the variables here. So, there exists such a chain of prime ideals, and every chain of primes has length less than or equal to n.

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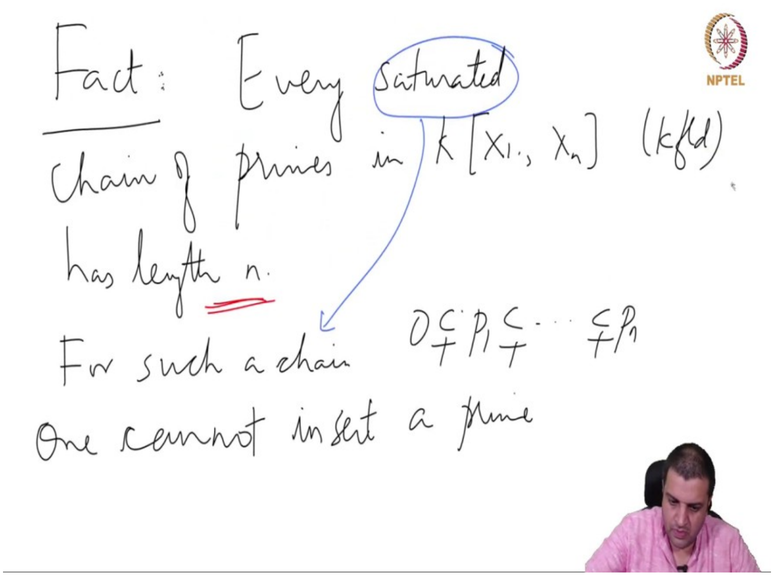

But in fact, one can prove, so fact meaning this is a result that I am going to state I will only sketch a proof and not sketch a proof in the sense that I leave the details as exercise to you.

The details themselves require proofs which and theorems which we could have proved, but we have not. So, take this just as a general discussion.

So, in fact, every saturated chain of primes in $k[x_1,\ldots,x_n]$, again k is a field has length n. What does saturated mean here? So, this means that for such a chain $0 \subsetneq p_1 \subsetneq \ldots \subsetneq p_n$ one cannot insert a prime between successive stages between p_i and p_{i+1} for any i, so, that is what we mean by saturated in other words we cannot add more elements into the chain. So, it has to start from the 0 prime ideal, and end at a maximal ideal and no gaps in between in the sense that.

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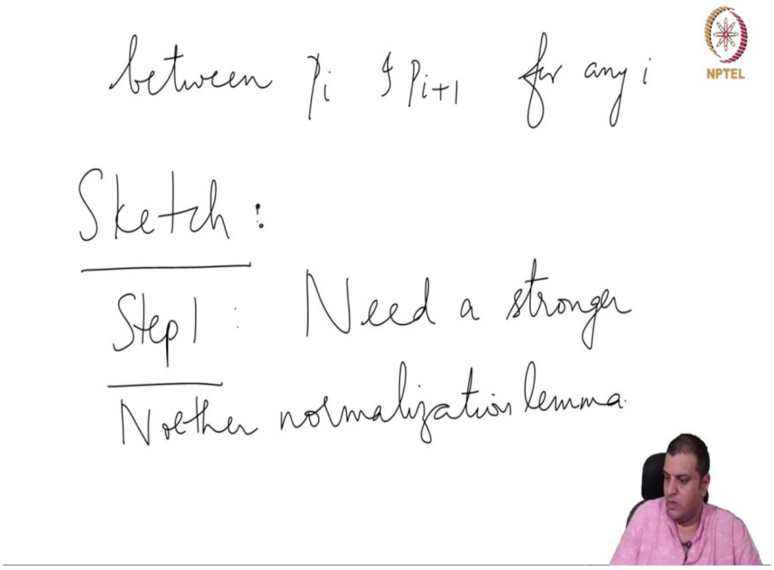

So, every saturated prime ideal has length has length n. So, this is what theorem this is a property about polynomial rings over fields. And we will not be able to prove this because its proof uses two other theorems.

So, let me just briefly sketch what is behind the proof. It is a good thing to know even if you have not seen the proofs of those results it is a good thing to know that those results exist.

So, the first step is, we need a stronger version of Noether normalization lemma. What does, what do we need?

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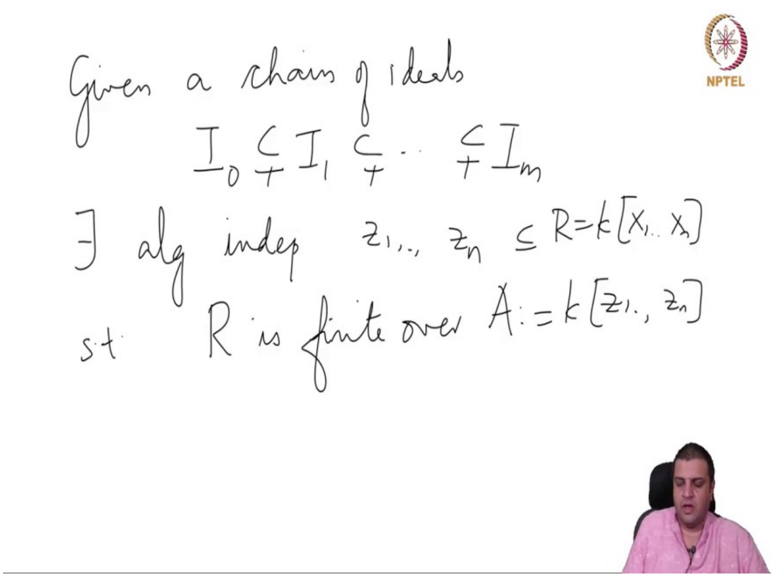

So we need the following which is that. So, given chain of ideals not necessarily prime ideals $I_0 \subsetneq I_1 \subsetneq \ldots \subsetneq I_m$ not necessarily maximal or not necessarily saturated, there exists algebraically independent $z_1, \ldots, z_n$ inside R, R is a polynomial ring in n variables. Such that R is finite over the usual statement about Noether normalization. So, let us call that ring $A = k[z_1, \ldots, z_n]$.

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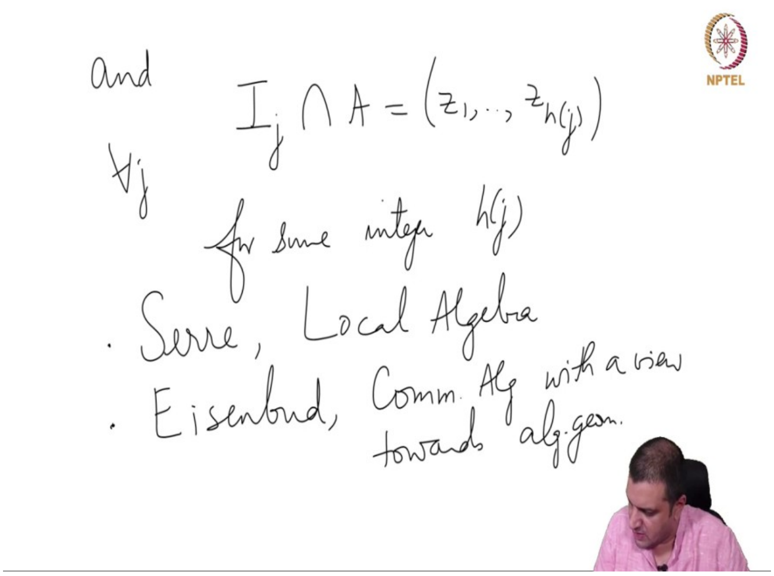

And $I_j \cap A = (z_1, \ldots, z_{h(j)})$ for some integer $h(j)$. So, this is for all j. For all j, for all ideals in that family when we contracted to this sub ring, we get the ideal generated by these initial some number of variables z_1; z_1, z_2; z_1, z_2, z_3; need not be need not go one step at a time, it can go I mean this $h(j+1)$ need not be $h(j)+1$. So, it is just some function.

The way we have defined it, it is clearly weakly increasing because I_{j+1} contains this. So, the contraction also has to contain, but this is something that we need. So, this stronger version of Noether normalization is the one that is you that would that is this is one step of proof of this theorem. So, this thing is proved in two sources I know.

So, for a proof of this version of Noether normalization lemma it is proved in Serre, local algebra. And it is also proved in Eisenbud, the one that we had mentioned at the beginning commutative algebra with a view towards algebraic geometry. So, it is a proof of that statement is given in at least these two books. So, that, so we need to use this version.

So, let us assume that theorem. So, then what do we know. So, let us look at this statement it says that every chain of prime ideals in this has length n. So, now, we can think of it in other way which is the way we will try to use now.

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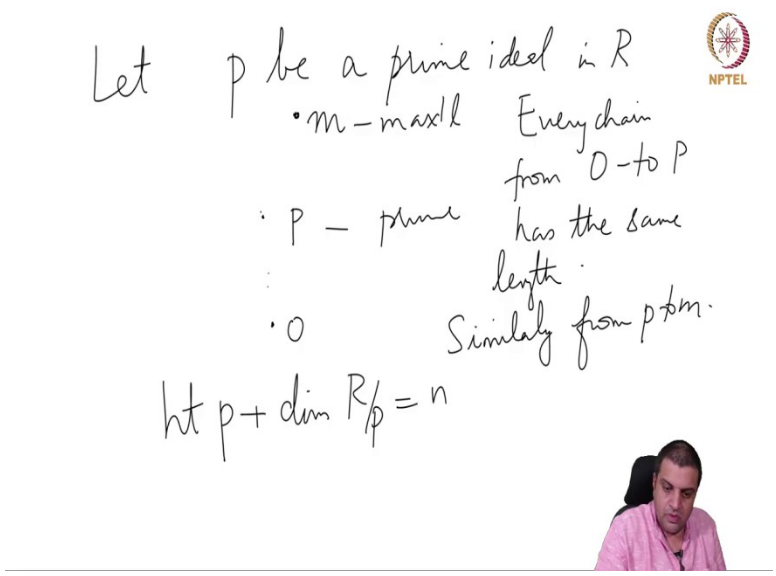

So, let p be a prime ideal in R. So, let us just draw a picture this is a 0 prime ideal p contains that and p contains some maximal ideal m. So, this is containment. So, I have drawn increasing to say larger and larger ideals. So, this containment like this. Now, what we are trying to sketch is a proof of the fact that every chain from 0 to m.

So, this is the maximal ideal, this is just prime. Every chain from 0 to m has the same length n that is what we are trying prove. Suppose, it is true, then it would say that every chain from 0 to p, followed by a chain from p to m the length would be equal to n.

For every chain because if there is some chain which was short here then you append that chain to this side, and then you will get a shorter chain which is smaller than of length less than n or the same thing here if there is a chain between p and m which is shorter.

Then first start a chain from 0 to p, and then use the shorter one here we will get something which is less than n. So, the conclusion of this thing is that every chain from 0 to p has the same length, and every chain from p to m has the same length. In other words, the conclusion is that so every chain, so let us write this. So, I will write here every chain from 0 to p has the same length. Similarly, from p to m, so these two are ok, now this is true for every p.

So, one observation that we get immediately is that height of p which is the length of the longest chain from 0 to p, they are all the same length. Dimension of p which is the length of the longest chain from p to a maximal ideal containing p, but they are all the same length because the total length is n.

So, $ht\, p + dim \frac{R}{p} = n$. So, a consequence of this thing that we are trying to prove is that this is true for a p. Now, if this is true for every p, then it would say that every chain from 0 to maximal ideal will have the same length, because suppose you take a shorter chain then take a prime in between its height. So, this you will have to you have to go up one step at a time.

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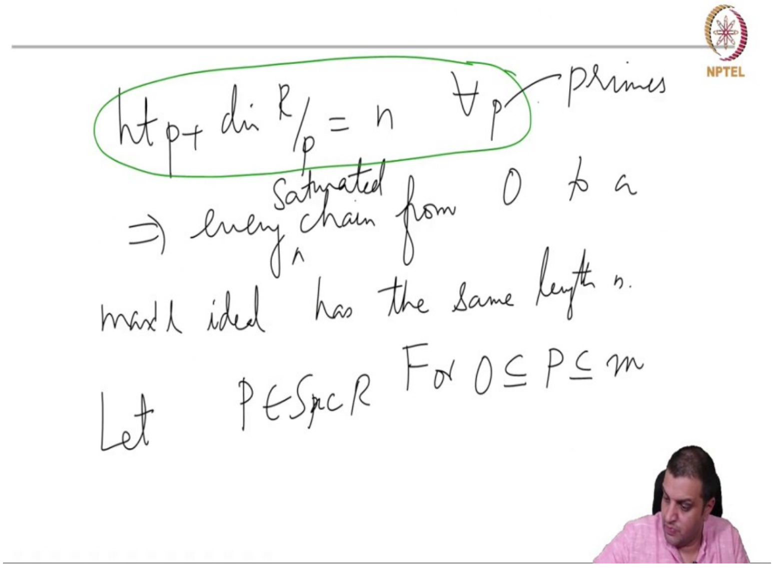

So, the other direction $ht\, p + dim \frac{R}{p} = n$. So, R is a polynomial ring for all p will imply that every chain from 0 to a maximal ideal every saturated chain maximal ideal has the same length n. So, this statement here is equivalent to what we are trying to prove.

So, we will prove this statement. We will also see it in a slightly different way after we finish the sketch of this proof. So, it is this equivalent version that we are trying to prove. So, now let us apply. So, let p be in spec R.

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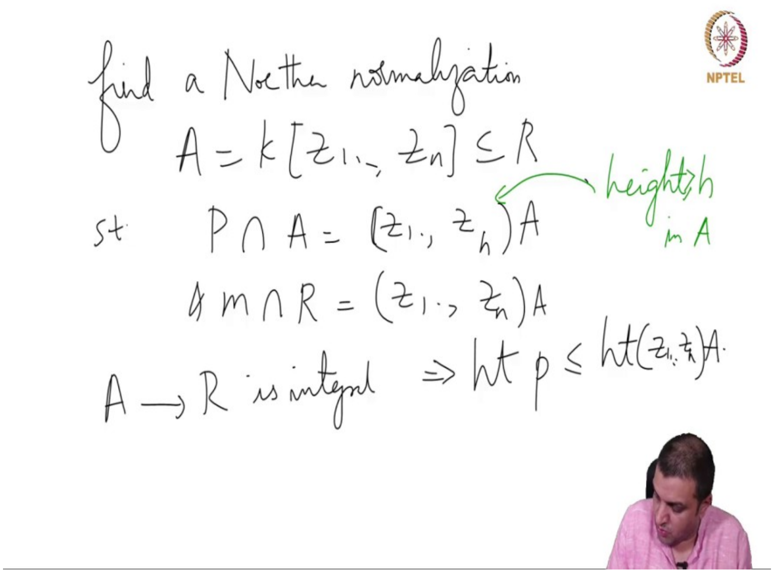

Now, for the chain 0 to p to maximal ideal for this find a Noether normalization A which is some $k[z_1, \ldots z_n] \subseteq R$ such that $p \cap A = (z_1, \ldots, z_h) A$ and $m \cap R = (z_1, \ldots, z_n) A$. And 0 contracted to A is anyway 0. So, we get such a relation.

Now, since A to R is integral finite. Notice that this one is easy to determine. This one has height h in A, R is integral or maybe we do not need equality here just $e \geq h$ in A because clearly there is a chain of primes $(z_1) \subsetneq (z_1, z_2) \subsetneq \ldots \subsetneq (z_1, \ldots, z_h)$. So, there is at least this ideal has a height at least h. R is integral, so which means that actually we do not particularly (Refer Time: 15:16) ok.

Height of p is less than or equal to height of $(z_1, \ldots, z_h)$ inside A. So, this we proved in you know when we discussed integral extensions. So, if p contracts to A, A to R is integral, p contracts to A then height of p is less than or equal to height of the contraction. So, this we proved.

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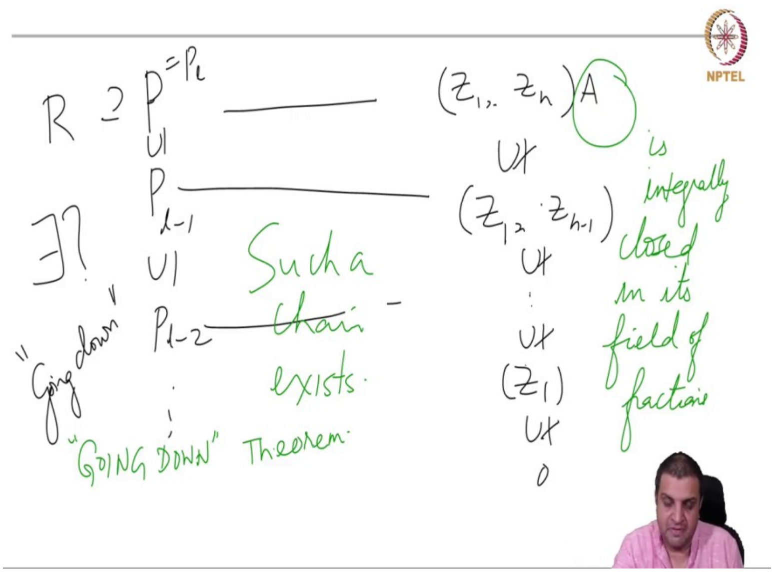

So, now what is the picture? We have a p here and we have $(z_1, \ldots, z_h)$ here inside A. So, this is inside R. And A to R is an integral map. Ok this contracts here. We wanted to show that height of p is at least h. How would we prove that? Let us look at the chain of primes here. So, this one contains $(z_1, \ldots, z_{h-1})$ and so on contains (z_1) and contains 0 is all strict containments.

We have seen the going up theorem which said that if an ideal in the extension ring contract, if let us draw the going up theorem. So, let us use slightly different notation.

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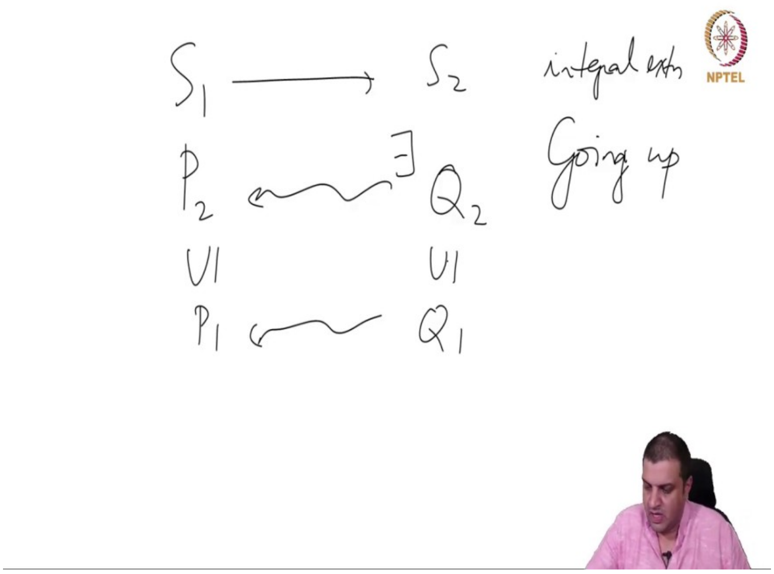

Let us call the rings S_1 and S_2 . This is integral extension of ok. And we have some $p_2 \supseteq p_1$ primes here. Some Q_1 contracting to this, then so this is going up.

We have a Q_2 then there exists a Q_2 such that $Q_2 \supseteq Q_1$ and Q_2 contracts to P_2. So, the given information is this chain here, and something in S_2 that maps on to P_1, then the chain goes up that is what going up in this case.

So, now let us go back. what is the situation that we have here? We have R over A integral. Here is a prime which contracts to this prime here. But here is a chain going down, can we extend it to a chain going down here? So, does there exist? let us call this thing $P_l \supseteq P_{l-1} \supseteq P_{l-2} \ldots$. So, that this contracts to here this is whatever here and then so on ok. So, this is going down.

So, this is if we can prove this then we will know that this has a length larger than this larger than or equal to this. And hence we know that height of p is greater than the height of this. So, this is what we need.

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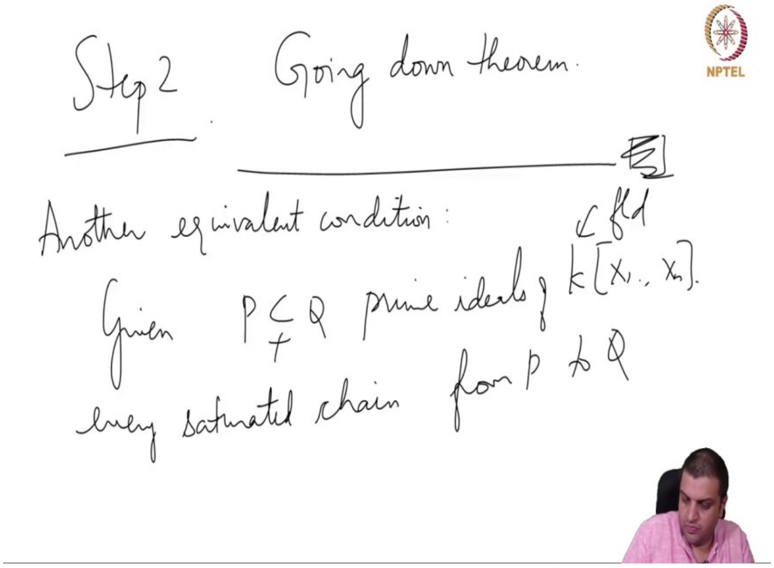

So, what we need is step 2 is called going down theorem. So, I will not state the theorem it require some hypothesis etc. So, what it says is given a chain like this; given a chain like this and prime ideal here mapping to the first one the top most one, there is a chain going down mapping successively to the lower ones.

So, such a chain exists and what we need to use in the theorem so such a chain exists, so this is the going down theorem. So, this is going down theorem for integral extensions there are other going down theorems.

And what one needs to use in addition to the statement of going up theorem, one has to use the fact that this thing here is integrally closed in its field of fractions. So, that is necessary I mean that is part of the hypothesis it would fail if we do not assume that the smaller ring is integrally closed in its field of fractions.

So, using that hypothesis, one has a going down theorem, and it says precisely such a chain exist. Therefore, height of p is greater than the height of this. And therefore, it will give an inequality in the other direction, and that would prove the statement I mean that will proof for this p. So, this is the proof. So, this will prove the statement.

So, what I gave is just a very rough sketch of this argument. We need to prove the stronger version of stronger version of Noether normalization lemma. We need to prove the going down theorem.

And after we prove these two statements, one can put the argument together one can fill in the details of this argument and conclude that every chain of primes in every saturated chain of primes and spec of a polynomial ring over a field is has length same length, length equal to dimension of R.

This is the general theorem that applies not to just to polynomial rings or algebras over fields. It is sort of in the same generality as the going down going up theorem, you need some extra hypothesis on the ring. But it is generally in that context ok, while the Noether normalization lemma is about algebras that are finitely generated over fields, so that is just a brief because this is a.

So, this is sorry I mentioned that I will discuss about some condition that is equivalent to this that every saturated chain of primes has the same length. One condition we saw in the proof which is this $ht\, P + dim \frac{R}{P} = n$ for all primes P it is equal that we saw.

Another condition its equivalent to this is the following which is that given $P \subseteq Q$ prime ideals of k adjoint n variables k field every saturated chain from P to Q has the same length.

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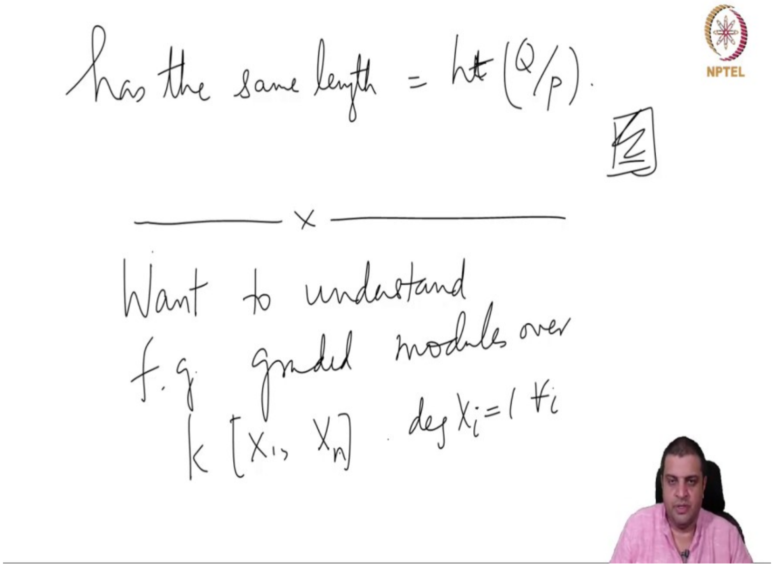

Which is equal to $ht\left(\frac{Q}{P}\right)$ in the quotient ring $\frac{R}{P}$ the height of $\frac{Q}{P}$ in there. So, this is another way of saying this, this same property of polynomial rings and often this nice property of about spectrum that all chains between two points at the same length is sometimes useful in various problems.

Although we are in this course we are unlikely to have to use it, but it is an important geometric property of some Noetherian rings, not all Noetherian rings have this property.

So, now I want to change topic a little bit, and want to look at graded rings, I mean, mostly we wanted to understand and use modules over. So, we wanted to understand. So, this is ok.

We are not discussing polynomial rings over fields or generality of Noetherian rings in anymore we will concentrate finitely generated graded ideals I mean modules over $k[X_1,\ldots,X_n]$. And when we say graded we have to actually say what is the grading of the ring in which degree of X_i equals 1 for all i.

We can develop this slightly more generally without assuming this. We can just assume that all of them are positive, but it is sort of and I mean it complicates the arguments, but ultimately even in that case although we would not discuss it many of problems can be reduced to some other ring in which this is true all the generators have the same degree. So, I think as a first round of understanding this is good. So, in order to do this we will right now discuss a little bit more general. So, we will assume that for now initially.

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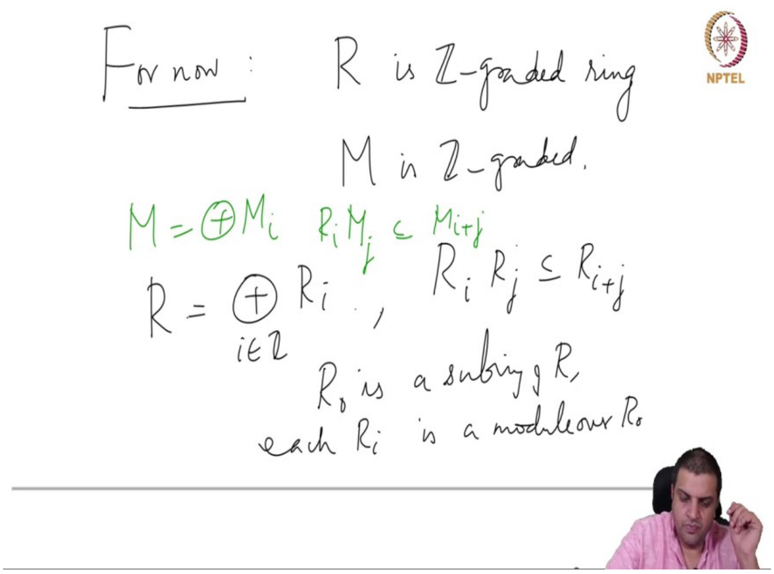

We will assume that R is a Z-graded ring. So, the polynomial ring with that grading I specified is actually non negatively graded k in degree 0, and all polynomials of positive degree in positive degree, the natural degree notion of degree of a polynomial. But we may have to work occasionally with Z-graded rings just in the initial setup.

So, we will allow the initial parts we will develop in this generality. And in even over this we will have to assume that M is Z-graded just for now.

So, this means that $R = ¿ i \in Z R_i$. So, this is R_0. It decomposes like this with $R_i R_j \subseteq R_{i+j}$. So, this says that R_0 is a ring sub ring of R. Each R_i is module over R_0.

This is not the first time we are seeing graded rings, but just to remind ourselves, what we are assuming. This is not the first time we are seeing this, but just to remind ourselves what where we are standing.

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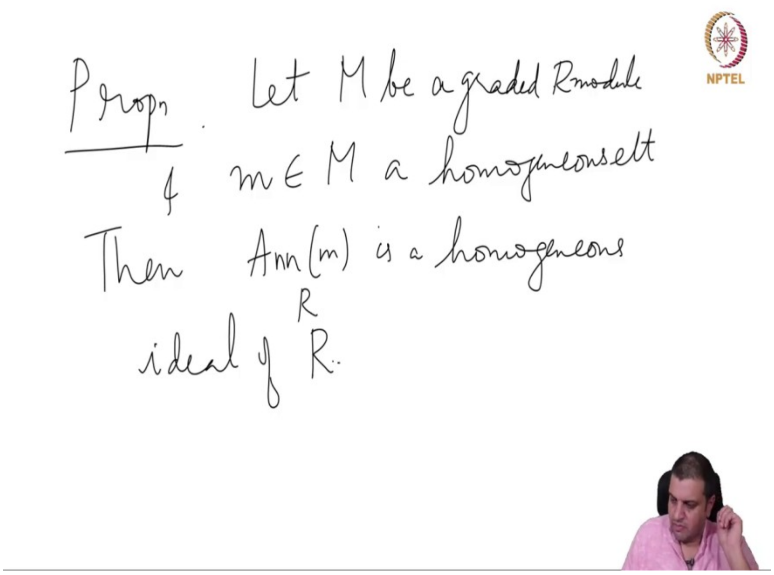

So, now proposition let M be a graded R module, then and $m \in M$ a homogeneous element, in other words m belongs to one of the summand. So, similar to this there is M also decomposes, and $R_i M_j \subseteq M_{i+j}$. Homogeneous means it is in one of these pieces. So, let homogeneous element. Then the $Ann_R(m)$ is a homogeneous ideal or a graded ideal of R.

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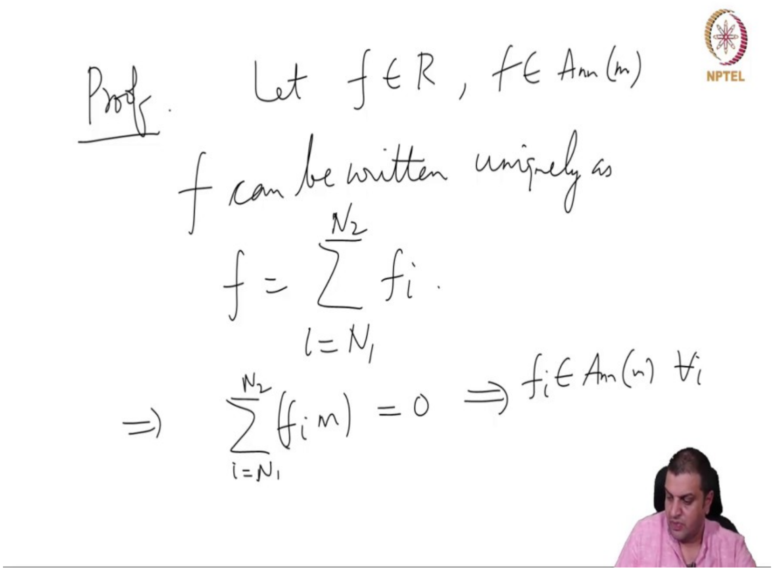

So, this is proof. So, let $f \in R$ be in the annihilator of m. The point about a graded ring is that it can be written uniquely as $f = \sum_{i=N_1}^{N_2} f_i$ in some finite range. So, let us call it N_1 to N_2. So, this implies that $\sum_{i=N_1}^{N_2} (f_i m) = 0$. But each of these live in different degrees, the only way can be 0 is that individually they are all 0.

And if for every f in this its homogeneous parts are in the ideal, then this would imply that we can do this for every generator, and generator itself breaks up into its homogeneous parts. So, we can actually get a generating set based on our homogeneous parts.

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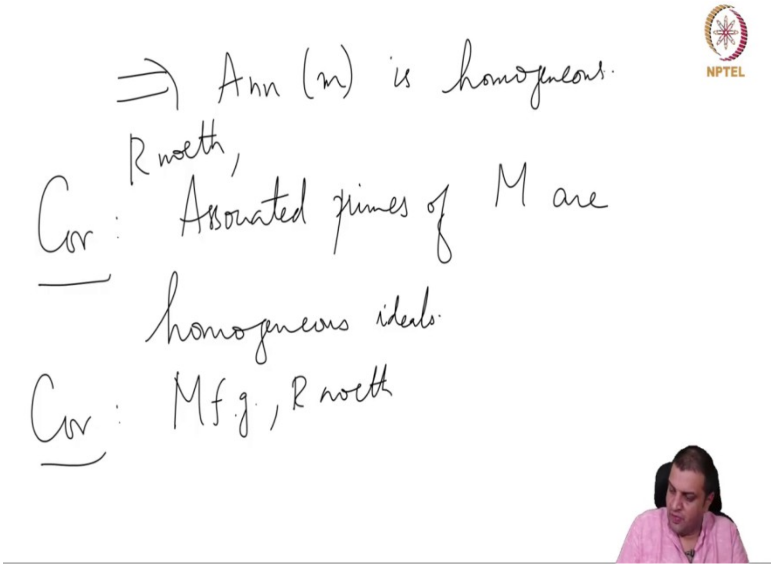

So, in other words annihilator of m is homogeneous. So, we do this argument for every take a generating set and then do this for every generator and then we would get this ok. So, an immediate corollary is that associated primes of M.

So, let us assume that R noetherian and R noetherian means M has an associated prime. Associated primes of M are homogeneous ideals that is this because a little bit of working.

But if there is an annihilator of some element, we can actually break it up into annihilator of homogeneous elements, and then maybe I should not have said it is an immediate corollary it one can use the same argument to prove this statement.

And another corollary is the following which is that M. So, now, let us assume M is finitely generated in addition to R being noetherian.

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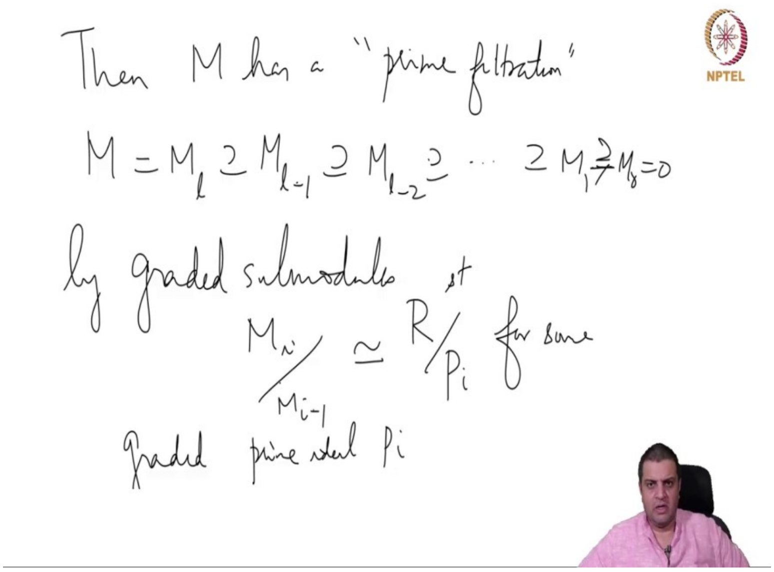

Then we saw that then M has a prime filtration. $M = M_l \supseteq M_{l-1} \supseteq \ldots \supseteq M_1 \supseteq M_0 = 0$ which is filtration by graded sub modules such that $\frac{M_i}{M_{i-1}} \cong \frac{R}{P_i}$. Now, this is a graded module its isomorphism for some graded prime ideal P_i.

So, this is slightly loose and we will see that a little later. We will not refine this statement. All that at this point, I want to emphasize is that we could choose the sub modules to be graded and so that this is true. This is not what is called a graded isomorphism. It is just an isomorphism as modules, not isomorphism as graded modules which we will discuss just a little later.

So, we will end this lecture now. And in the next lecture, we will continue discussing more about the first topic that we want to discuss is the dimension theory of graded rings which we will build from the dimension theory of noetherian local rings, what exactly do chains of primes and support of M look like, and then we will use that, and then we will sort of specialize ourselves into the polynomial ring over a field situation.