Computational Commutative Algebra

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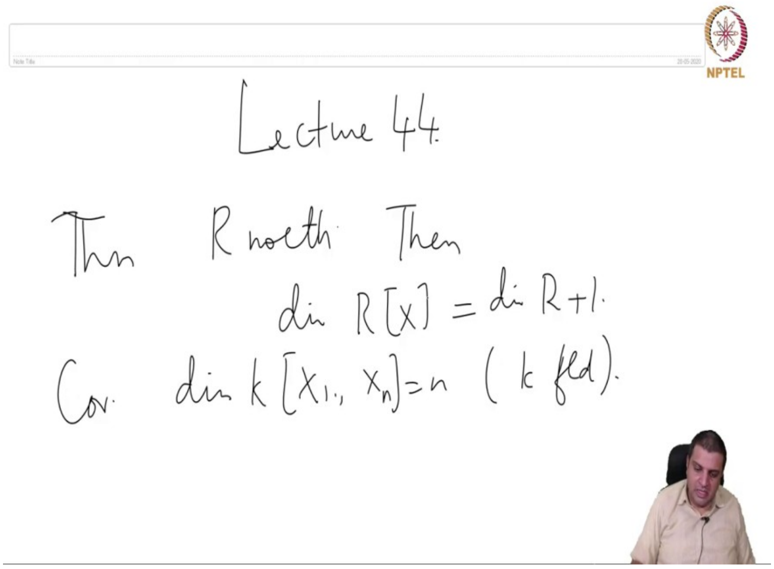
Lecture – 44
Dimension of polynomial rings



Welcome. This is lecture 44 and in this thing we prove the following theorem. R noetherian, then $dim\,R[X]=dim\,R+1$. So, as a corollary we would then immediately see that dimension of $k[X_1,\ldots,X_n]=n$ where k is a field. This is a statement that we had wanted to prove a little while from a little while ago.

But, we needed the statement that the previous the theorem for which we will need to use things that we learned so far. We need to use Krull principal ideal theorem and its consequences. So, we did not need to develop the dimension theorem in it is in the way that we are done, but this is one way of proving it.

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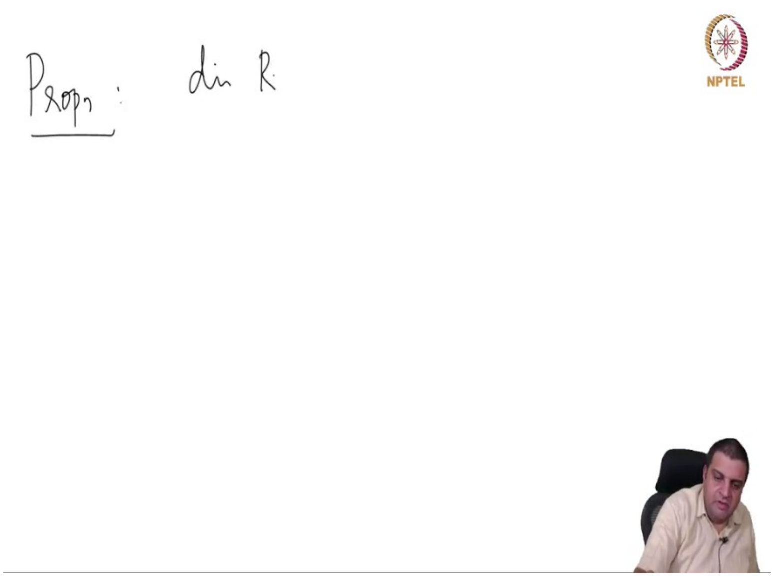

So, first before we go to the theorem let us start with some easy propositions. We will prove the theorem at the end.

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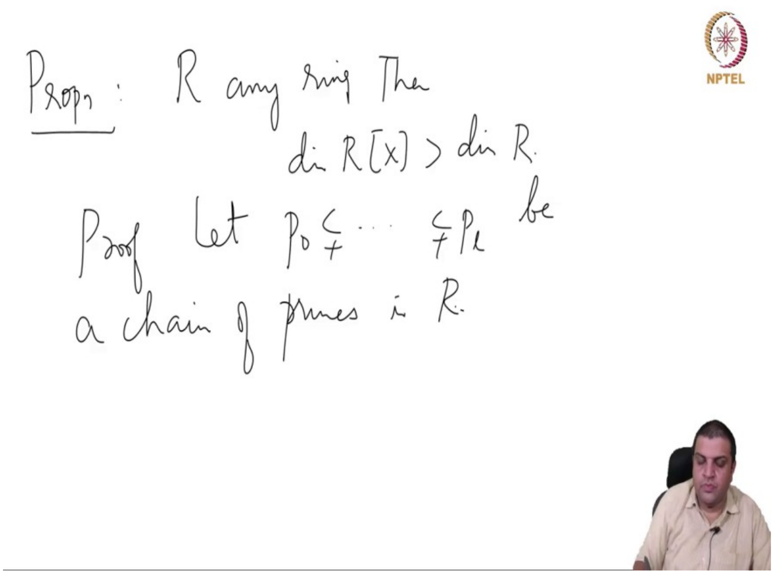

Dimension of $R[X]$ even if R is any ring not necessarily noetherian any commutative ring, then dimension of $R[X]$ is at least I mean it is bigger than dimension of R. Why is that? Proof let $P_0 \subsetneq \ldots \subsetneq P_l$ be a chain of primes in R.

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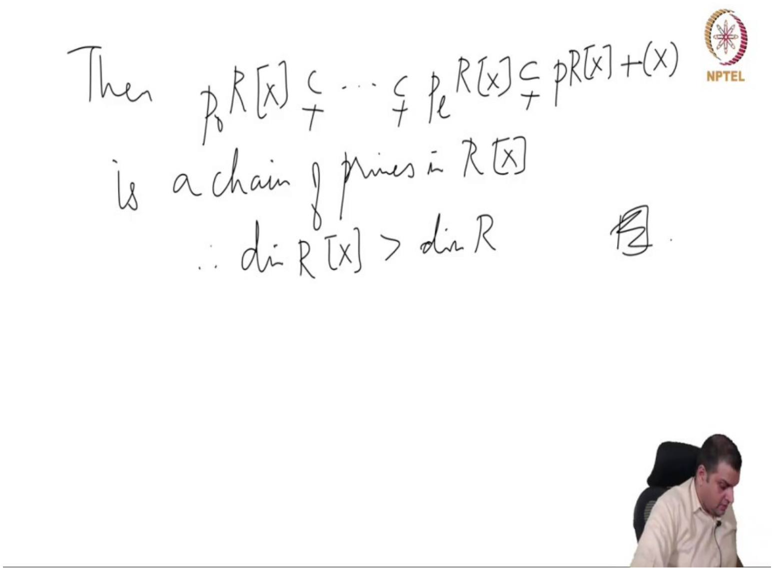

Then it is easy to check that $P_0 R[X] \subsetneq \ldots \subsetneq P_l R[X] \subsetneq P_l R[X] + (X)$. So $R[X]$ modulo this ideal would be $\frac{R}{P_l}[X]$ or one can prove that they are isomorphic. So, $\frac{R}{P_l}$ is a domain; therefore, $\frac{R}{P_l}[X]$ is a domain. Therefore, $\frac{R[X]}{P_l + (X)}$ is also a domain. So, this is a prime ideal and after this prime ideal we can put at least one more which is take this and the variable X .

So, this is a chain of primes in $R[X]$. So, for every chain of chain there is a chain of length at least one more. So, therefore, $dim\, R[X] > dim\, R$. So, this is the general situation.

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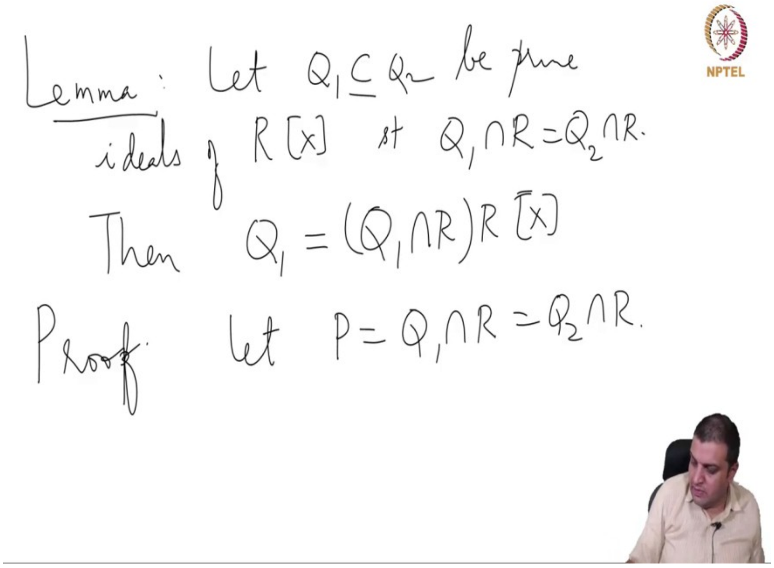

So, we need one more lemma. Let $Q_1 \subseteq Q_2$ be prime ideals of $R[X]$ such that when we contract them to R we get the same prime ideal of R. So, it is possible for example, we could take, R could be a field and then $R[X]$ is a PID and if you take any irreducible polynomial when you contract.

So, that is a nonzero ideal in $R[X]$ when you contract it you just get 0 because k is a field. So, but of course, the 0 ideal of $k[X]$ also contracts to 0. So, if it is possible that there is a pair of ideals with this containment with this property, this is not unusual.

Then the smaller one Q_1 is the extended ideal. This is the prime, it is extended ideal is also a prime in this case I mean extending to polynomials is still prime. So, if you have a pair like this, then the smaller one with if you have a pair with this property, then the smaller one is the extended ideal.

Proof: let $P = Q_1 \cap R = Q_2 \cap R$.

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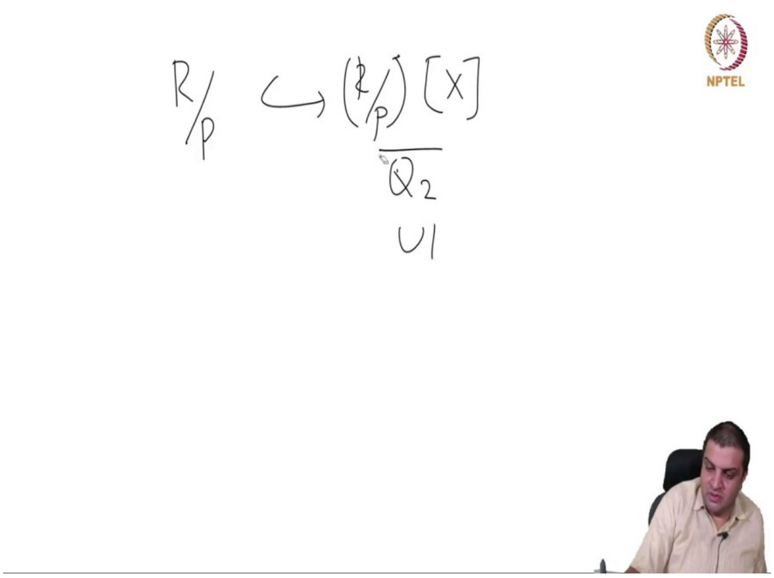

Now, if we go modulo P then ok.

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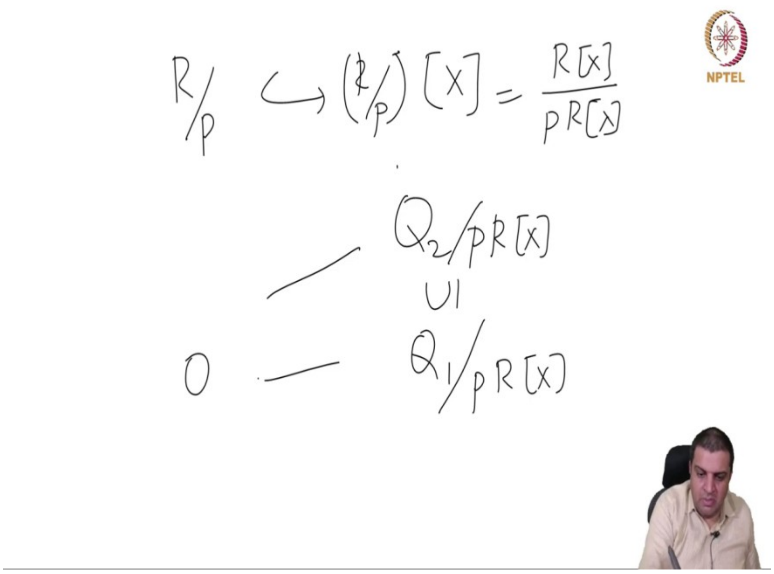

So, then $\frac{R}{P}$ is we can look at it this way. Inside here is $\frac{Q_2}{PR[X]}$ and $\frac{Q_1}{PR[X]}$. So, remember it is $\frac{R[X]}{PR[X]}$. So both are from here and here this contracts to 0. That is the picture, but this is a domain.

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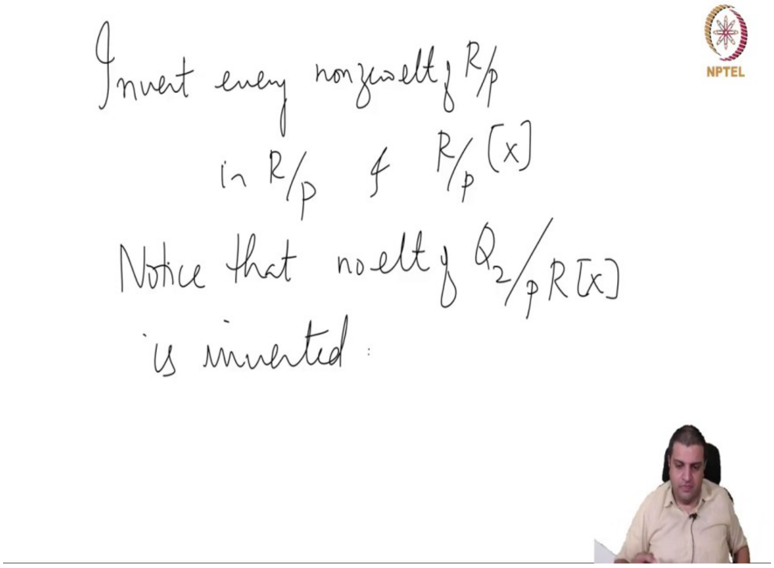

So, invert every nonzero element of $\frac{R}{P}$. So, we get the fraction field there of $\frac{R}{P}$. So, we do this in $\frac{R}{P}$ and $\frac{R}{P}[X]$. If you invert nonzero elements inside here they do not intersect the images of Q_2 and Q_1.

So, notice that no element of any way we just have to worry about Q_2 because that is the bigger one no element of Q_2 is inverted I mean this thing contracted to $\frac{R}{P}$ is just 0. So, nothing has been inverted in Q_2 we are only inverting elements of R, but on both sides.

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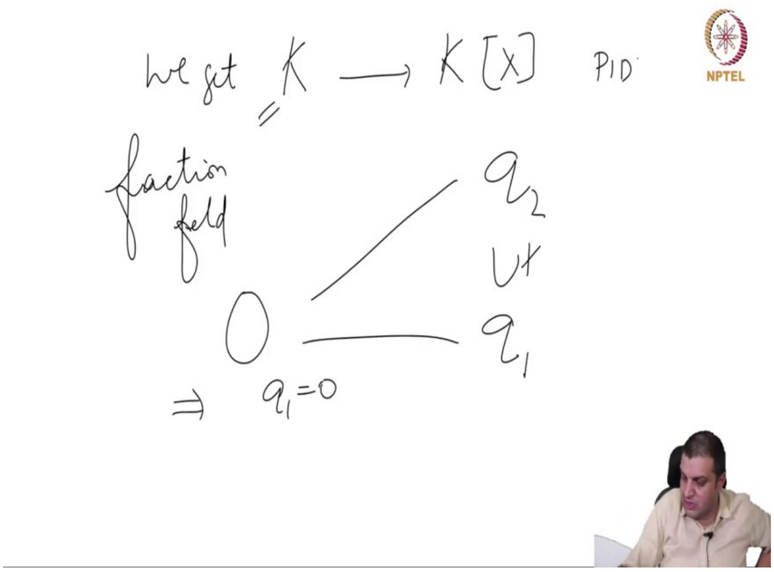

So, then we get k which is a field. So, this is the fraction field $k \rightarrow k[X]$ and then let us just site a different symbol q_2 and q_1 both of them contracting to 0 here. This is the fraction field and there is a containment like this.

But, then it is, but this is a PID, I did where did I say I am sorry one minute it has been equal. This must be strictly smaller sorry, all these containments is strictly smaller. So, this $q_2 \subseteq q_1$ any will also be like this and, but this is the PID. If you have two prime ideals in a PID then the smaller one must be 0. So, this one says that $q_1 = 0$.

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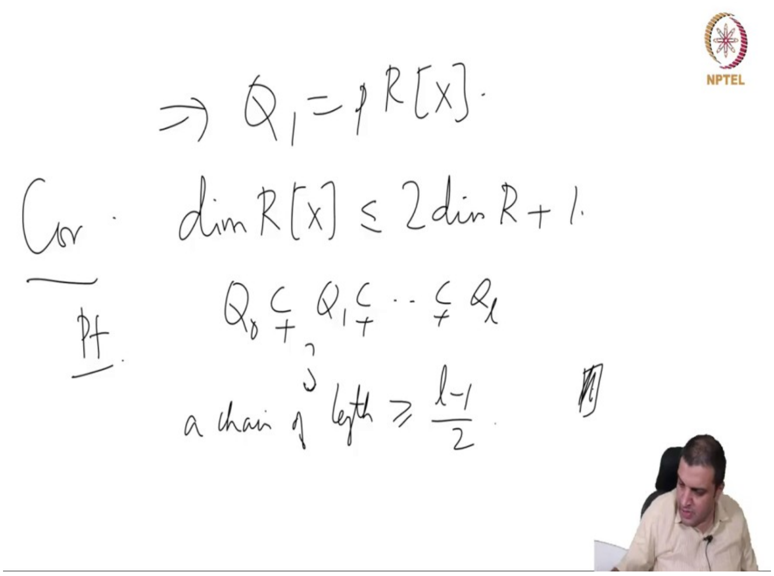

But, then when we walk backwards this just says that $Q_1 = PR[X]$. So, this is the proof and as an immediate corollary we can get the following which is still not good enough for us. That is because if you take any ok.

So, proof if we take $Q_0 \subsetneq Q_1 \subsetneq \ldots \subsetneq Q_l$. So, this will give a chain if you contract them chain of length greater than or equal to $\frac{l-1}{2}$ because three things in this together cannot contract to something I mean if three distinct things in a chain cannot contract to the same P below.

The previous one says only two of them can and that is what we get this the statement. So, this is in the generality of arbitrary commutative rings this is all that one can say. So, now, we go back to this theorem if R is noetherian then dimension of $R[X]$ is one more than dimension of R and as an immediate thing we see this corollary.

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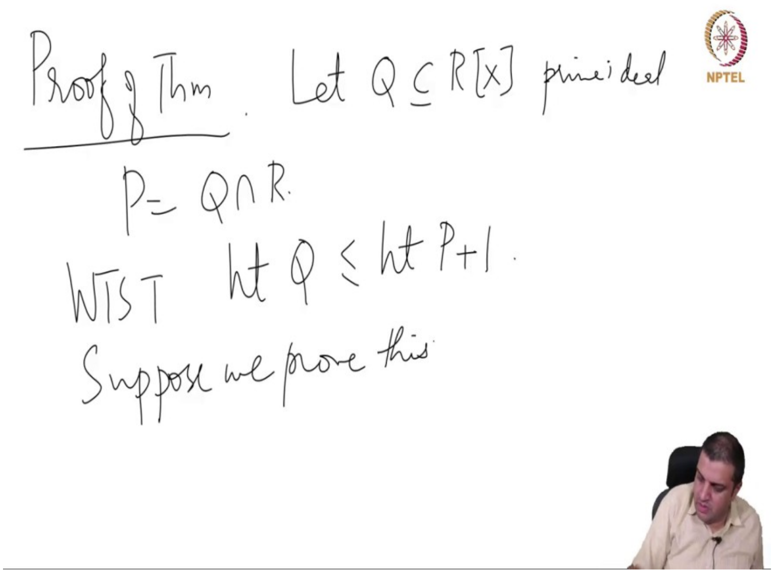

So, proof of now we can just prove the theorem. Proof of theorem. So, we take a prime $Q \subseteq R[X]$ and P is the contraction of that prime ideal into R. So, what we want to show is that $ht\, Q \leq ht\, P + 1$. If we show this then well if we take any chain, then the supremum length will be less than the supremum of these such things plus 1.

But then supremum of such things is dimension of R.

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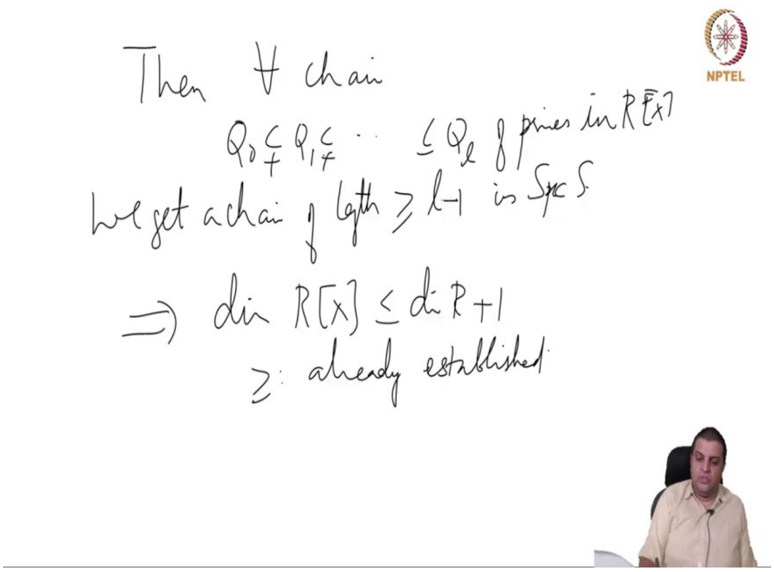

So, suppose we prove this, then for every chain $Q_0 \subsetneq Q_1 \subsetneq \ldots \subsetneq Q_l$ of primes in $R[X]$. We get a chain of length greater than or equal to $l-1$ in spec S. So, therefore, we take supremum of such l, then here we will get supremum of such $l-1$. So, from this we would get that $dim\, R[X] \leq dim\, R+1$. The other inequality has already been established.

So, therefore, we want to show this. So, we are discussing something about height of some two primes Q and P and to determine this we need to worry only about primes inside P. So, we could just localize inside in R we could just invert everything outside P and do the same thing in $R[X]$ just invert everything outside P.

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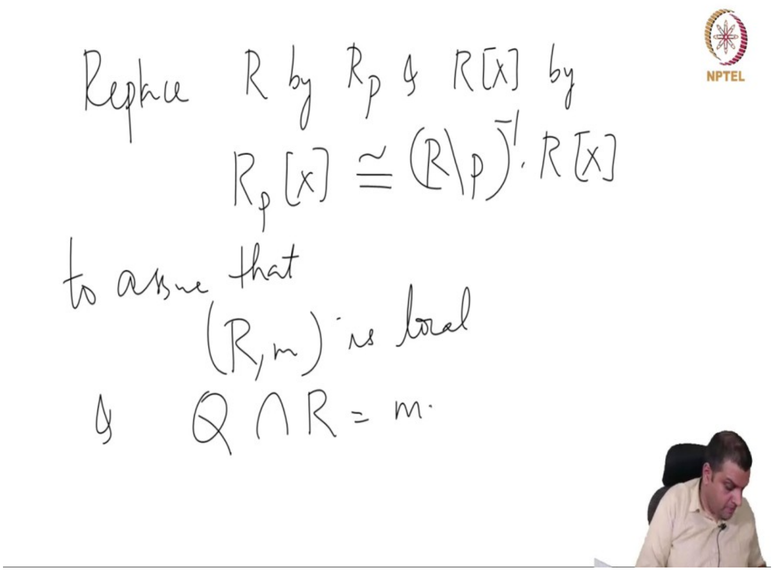

So, replace R by R_P and $R[X]$ by $R_P[X]$ which is also the same thing as $(R ¿)^{-1} R[X]$ R (Refer Time: 13:06). To assume that (R, m) is a local ring and $Q \cap R = m$. So, this is the reduction that we have done.

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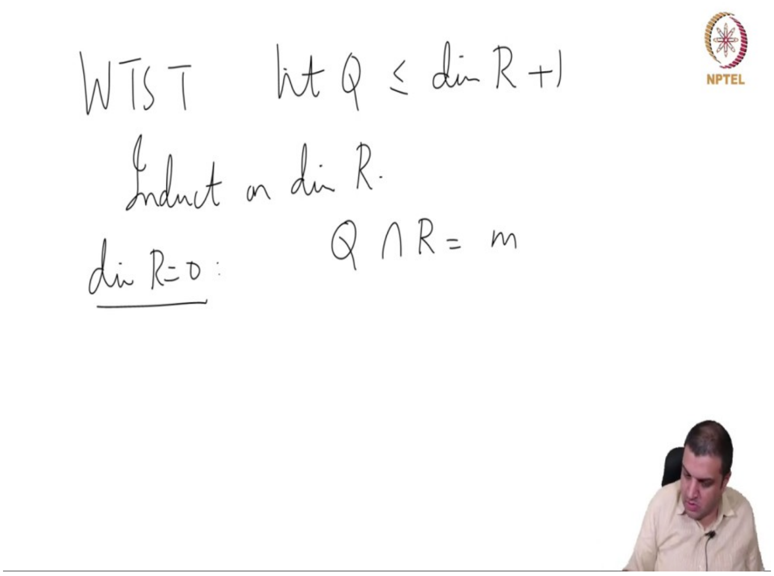

And, we wanted to show that $ht\, Q \leq dim\, R + 1$ that is how we just translated this thing. So, now, we induct on dimension of R. Dimension of R is 0, then $Q \cap R = m$. What about the height of?

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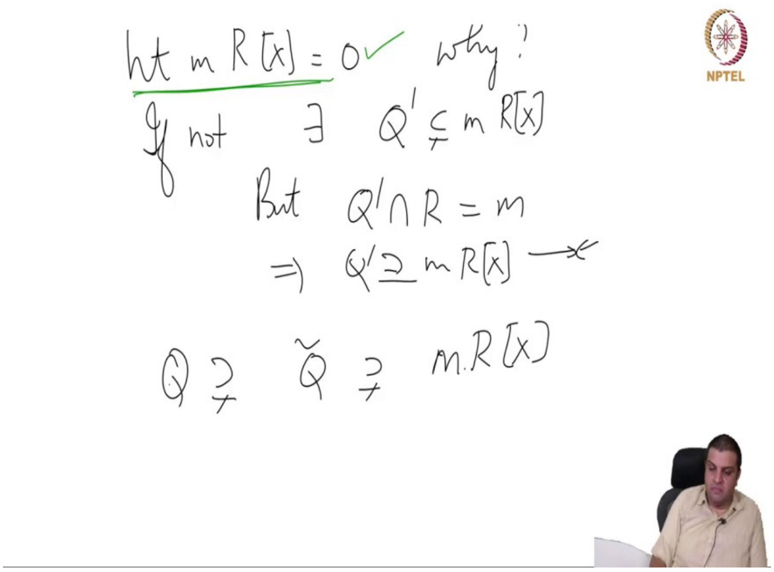

So, now let us look at. So, now, we want to argue that height of $mR[X]$ is 0. Why? If not there exists some $Q^{'} \subsetneq mR[X]$ right this is not a minimal prime inside $R[X]$, it is a prime. What we have to claim here it is not a minimal prime inside $R[X]$, but what can this contract to?

Well, it has to contract to a prime ideal, but there is only one in this assumption there is only one prime ideal in R. So, this must be equal to m, but this contradicts.

So, this say that $Q^{'} \supseteq mR[X]$ and this is the contradiction because we just said that it is proper subset of $R[X]$. I mean any ideal contains the extension of its contraction. So, this is just ok. So, therefore, this that proves the statement that has been I mean because of this that has been established.

So, now, let us look at height of Q. So, Q of course contains $mR[X]$. Suppose there is a Q tilde here, but then all three of them will contract to $mR[X]$.

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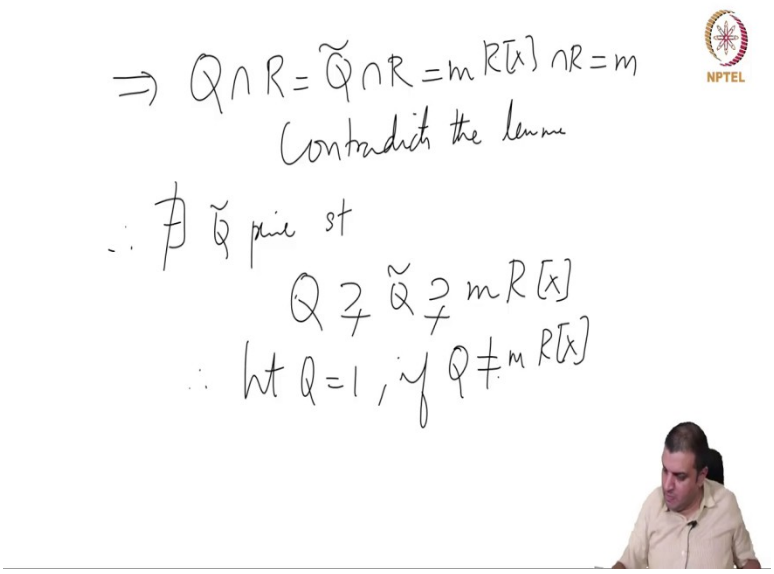

This will imply that $Q \cap R = \widetilde{Q} \cap R = mR[X] \cap R = m$; three distinct things I mean in a chain contracting to the same prime ideal contradicts the lemma. Therefore, there does not exist any $\widetilde{Q}$. Sorry, by this I did not write it, but by this I meant prime such that Q contracts to such that Q properly contains $\widetilde{Q}$ and properly contains $mR[X]$.

And, now one can conclude therefore, that any minimal prime of any prime here has to contract has to map to $mR[X]$, you cannot have three things in a row. So, any prime ideal therefore, it will be $mR[X]$ and then Q that is all therefore, $ht\, Q = 1$. Then somewhere we have assumed that Q is not $mR[X]$; let us check that yeah if.

So, if Q is $mR[X]$ then it is size 0 that we already checked. So, this proves the base case dimension is 0.

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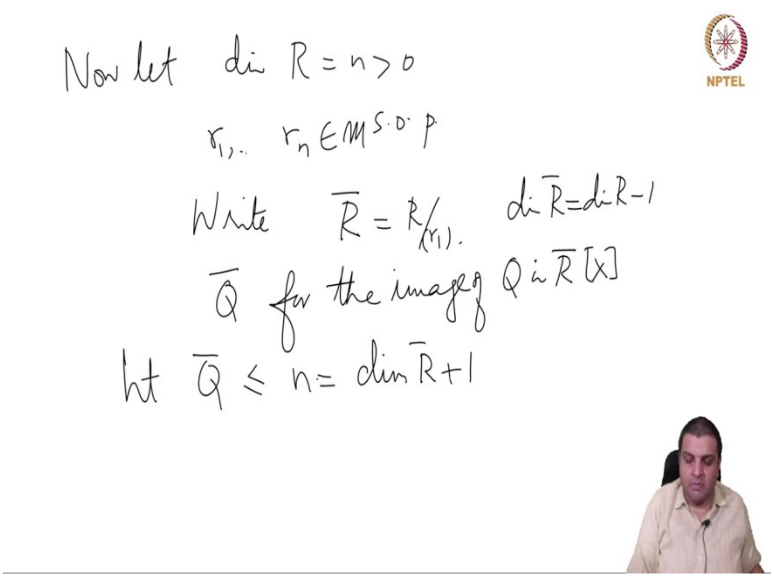

So, now let us assume that dimension of R is positive. So, now, let $\dim R = n > 0$, take a system of parameters. So, I mean and this is just one could have done using principal ideal theorem, but it is equivalent saying that by of that dimension there is a system of parameters.

Let us write sorry not X just to avoid confusion let us call them $r_1, r_2, \ldots, r_n$ system of parameters inside m, then write $\overline{R} = \frac{R}{r_1}$. We know that dimension of $\overline{R}$ is $\dim R - 1$ and write $\overline{Q}$ for the image of Q in $\overline{R}[X]$. Q contains r_1 because Q contains m which contains r_1.

So, therefore, $ht\,\overline{Q} \leq n = \dim \overline{R} + 1$. This is the induction statement height of Q is less than or equal to $\dim R + 1$ which is what we wanted to prove. So, we can assume this for $\overline{R}$ and $\overline{Q}$, and $ht\,\overline{Q} = \dim \overline{R} + 1$ which implies that $\overline{Q}$ is minimal.

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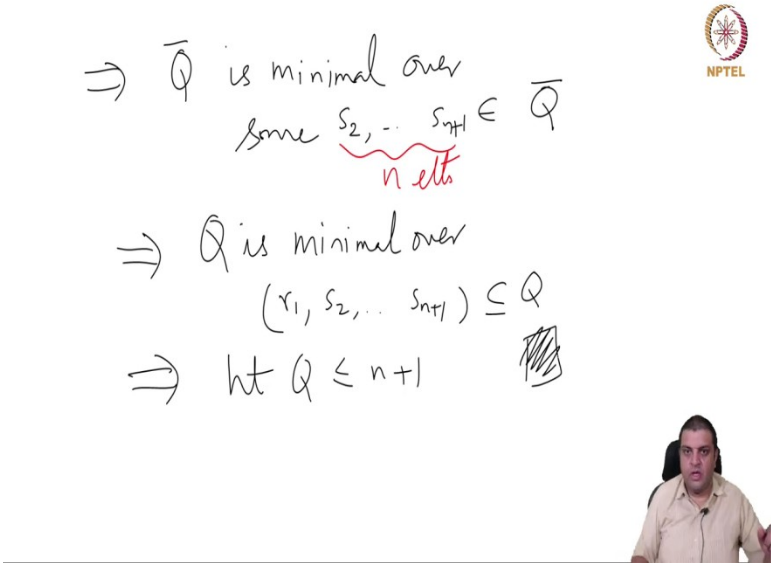

So, this is where we are using Krull's theorems minimal over some $s_2, \ldots, s_{n+1} \in \overline{Q}$. It is a prime ideal of height at most n. So, it is minimal. So, these are $n+1$ elements, these are n elements.

So, now, this implies now that Q is minimal over $r, s_2, \ldots, s_{n+1}$. This is inside Q. This ideal is inside Q because r_1 is in m. So, when we kill that we killed something inside Q. So, this now implies that $ht\, Q \leq n+1$ which is what we wanted to know.

So, now we have established the theorem which we mentioned at the beginning and hence also the corollary. So, let us quickly look at one more statement about finite type algebra it is over fields. So, now, at least we can go back to noether normalization.

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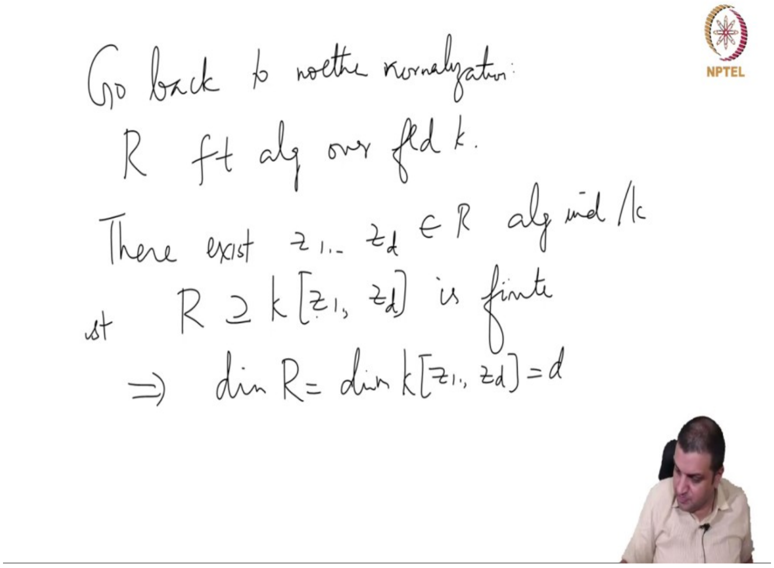

So, if we go back to noether normalization so, we have R finite type algebra over a field k. Then there exists $z_1,\ldots,z_d \in R$ z 1, algebraically independent over k such that $R \supseteq k[z_1,\ldots,z_d]$ is finite. Since they are algebraically independent it is a d-dimensional polynomial ring.

Now, that we have established that there are d of them then their Krull dimension is d, R over this is finite. So, this is what we had proved. So, and from this we are also by looking at integral extensions, we had known that dimension of R equals dimension of this polynomial ring. So, this was the d that came in the proof.

So, dimension of R is some integer and as many as whatever that integer d is that is the that is exactly the large that is the largest number of algebraically independent elements that one can find inside R and then at that point it is finite.

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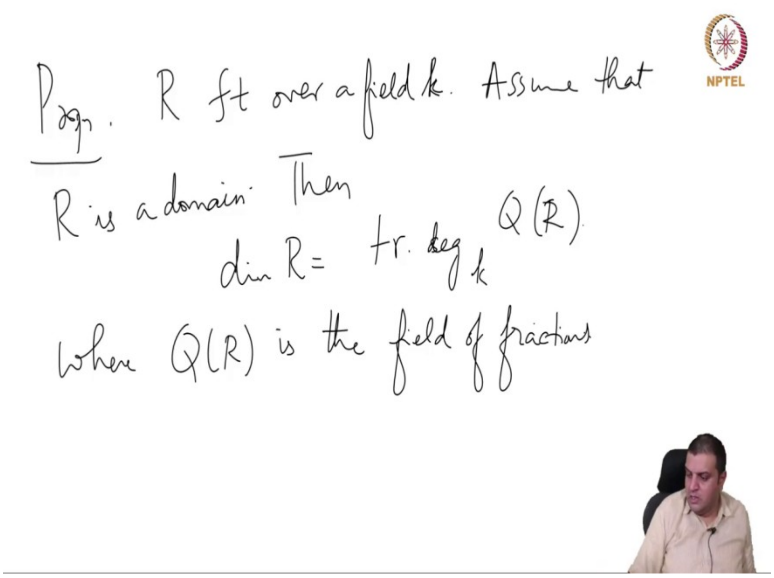

So, in this context we can prove the following proposition. Suppose that R is finite type over a field k meaning finitely generated as an algebra over a field k assume that R is a domain, then dimension of R is the transcendence degree of the field of fractions of R. So, over little k this is where $Q(R)$ is the field of fractions. To talk about the field of fractions one has to be in a domain. So, the proof is.

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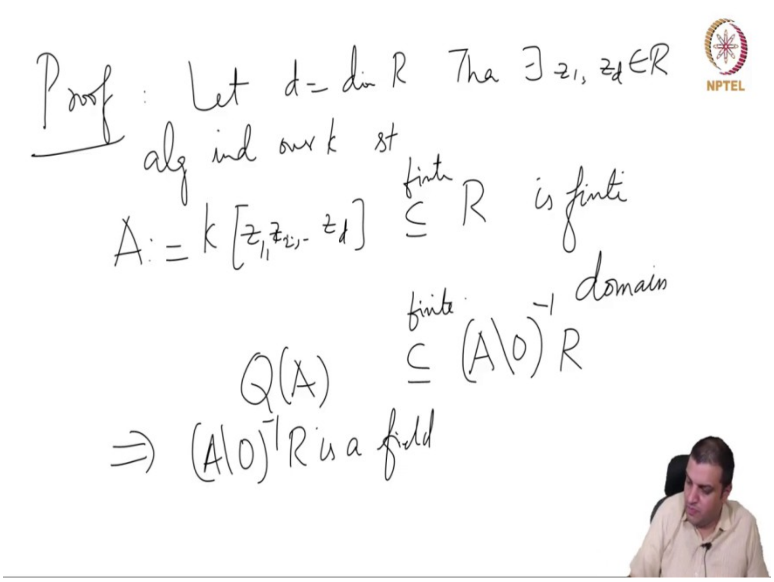

Let $d = dim R$, then there exists $z_1, \ldots, z_d \in R$ algebraically independent over k such that the polynomial ring generated by this algebraically independent elements sitting inside R is finite.

So, now let us take the field of fractions here. So, let us call this ring A. Let us take the field of fractions A inside here which means we inverted every nonzero polynomial in this. Well, this thing to here whatever we get so let us just for now let us just write $(A ¿)^{-1} R$. So, to go from A to $Q(A)$, we just invert all nonzero elements of A.

But, the property of being finite or the property of being integral is preserved under localization. So, this is finite. So, this is also finite. This is a domain and we already saw that if you have a domain containing a field over which it is finite, then this is also. So, this implies that is a field. This is a domain that is R was a domain and we just inverted some nonzero elements of R and so, this is a field.

And, then one can check that it has to be the fraction field because R to this field there must be a map and it must be same as a fraction field. It is first of all smaller than the fraction field because we did not invert in theory we did not invert all the nonzero elements of R we only inverted those of A, but it must have inverted everything because this is already a field. So, check that.

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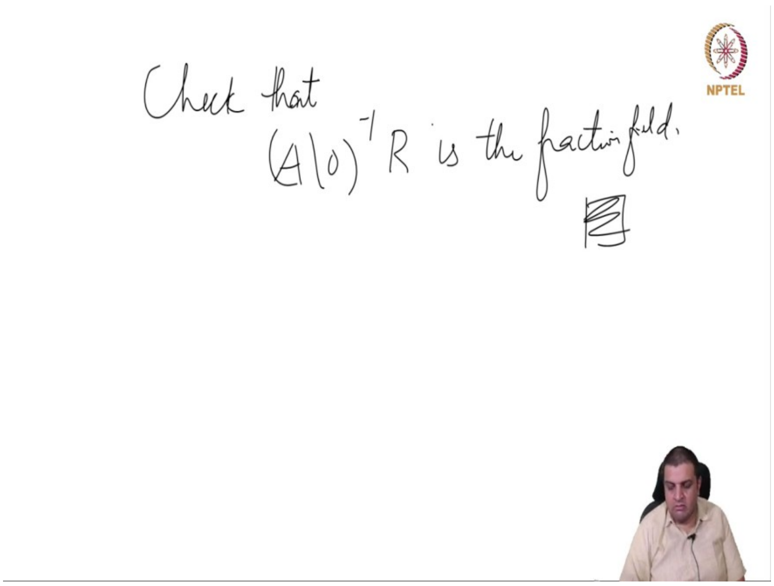

Check that if you take if you invert all nonzero elements inside A this is the fraction field. So, this is the end of this proof.

So, in the next few lectures we will concentrate on homogeneous ideals over polynomial rings over fields because they are more amenable to computations there are lot of things to learn from that and then we will maybe a lecture also more about generalities about polynomial rings over fields a property called a catenary which we will do without proofs.

Because there is more theorems to be proved, to prove that statement and we will not attempt that, at least we will become familiar with the notion even without a proof. And, then in the final 8 to 10 lectures will be on homological algebra what else can we use these computational techniques to what else can we learn about these things.