Computational Commutative Algebra
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Lecture – 43
Dimension of noetherian local rings - Part 2

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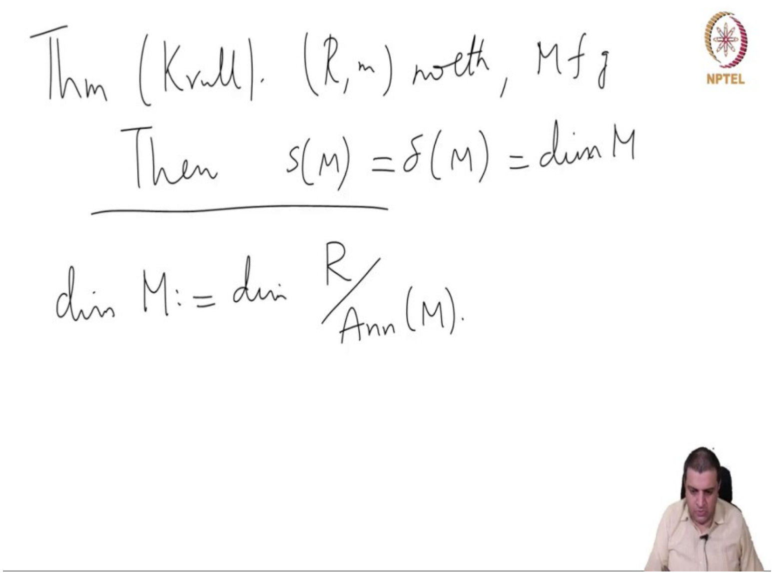

This is lecture 43 and we are continuing the proof of Krull's theorem. So, this was from the last lecture (R,m) noetherian local, M is finitely generated; then $S(M)$ which is the length of a system of parameters equals $\delta(M)$, which is the degree of Hilbert Samuel polynomial equals dimension M, where whole dimension is defined as this.

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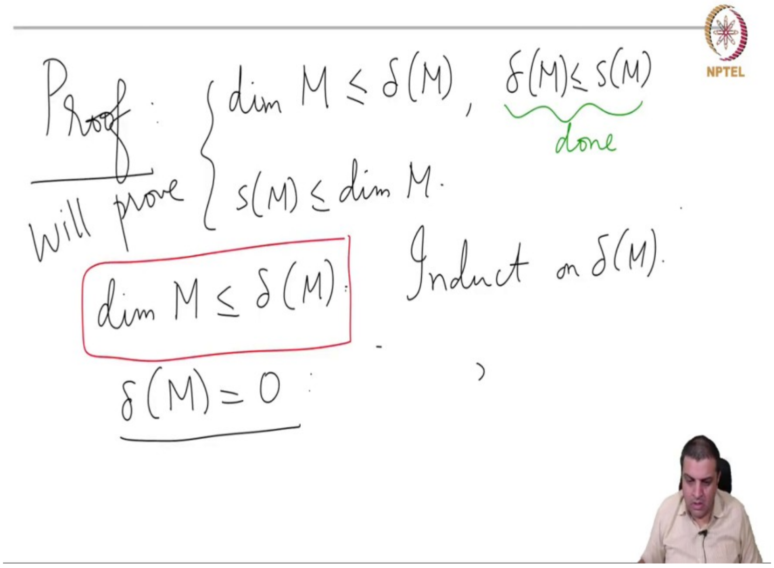

And in the proof, we were going to show the three inequalities in going in a circle. The first one we had done in the previous lecture; this one we established in the last lecture, and now we will prove this final part.

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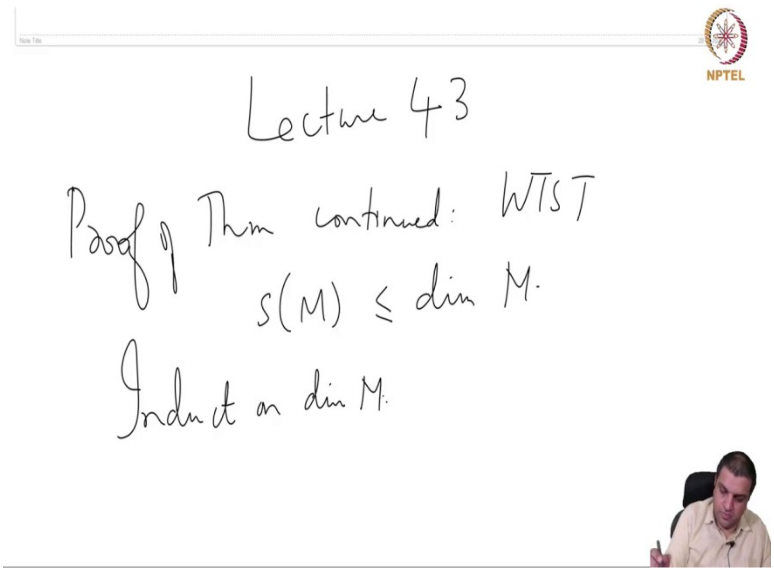

So, this is proof of theorem continued. So, we want to show that, $S(M) \leq dim\, M$. So, now because we know it is a finite number, we can induct and prove this statement. So, induct on dimension M. If dimension of M is 0, then it says that M has finite length.

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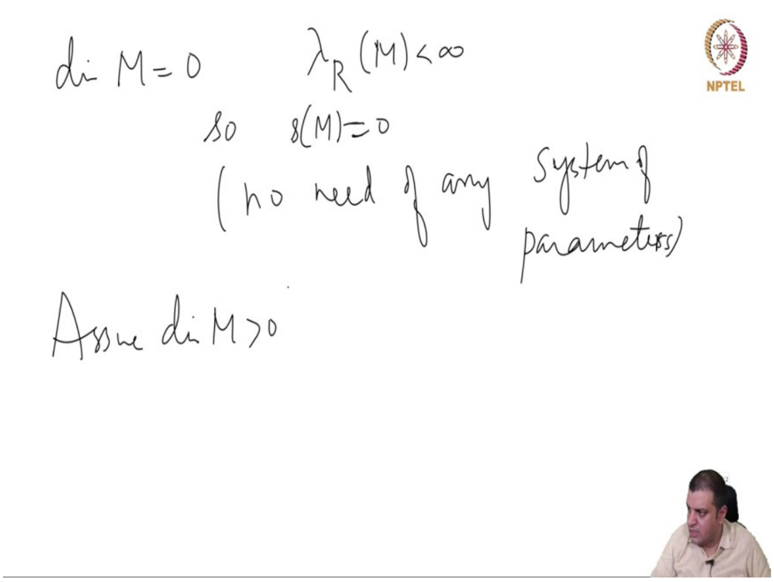

Dimension of M is 0 means that, annihilator of M is M primary and therefore, length of this is finite. So, in other words $S(M)=0$; in other words we do not need any x's to make it, so that the quotient is finite length.

So, no need of any system of parameters, it is already finite length parameters. So, now, let us assume dimension is positive and assume that it is true for all modules of smaller dimension.

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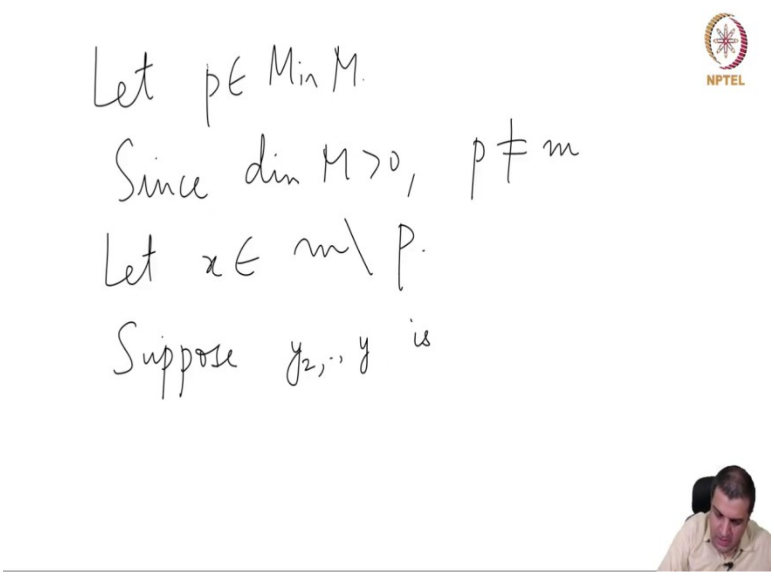

So, now, let us take let $P \in Min\, M$. Then since $dim\, M > 0$, $P \neq m$. So, we can pick an $x \in m$¿;

we need to pick such x, so when you take $\frac{M}{xM}$, we get some non-zero module.

Because if you take just any x outside P, it could be an invertible element and if you would do $\frac{M}{xM}$, where x is an invertible element, we would just get 0. So, the reason to choose is we want $\frac{M}{xM}$ to be non-zero and we can actually do some induction. Now, let us look at.

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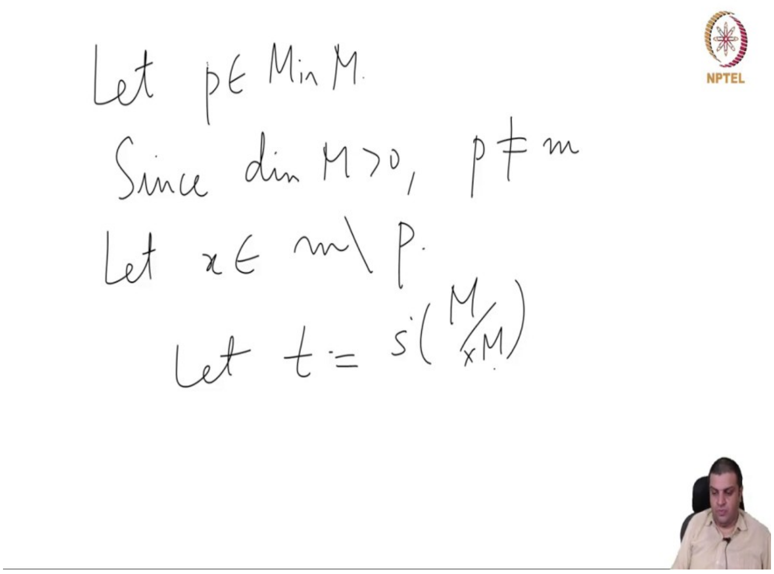

So, let, $t = S\left(\frac{M}{xM}\right)$; what is that say?.

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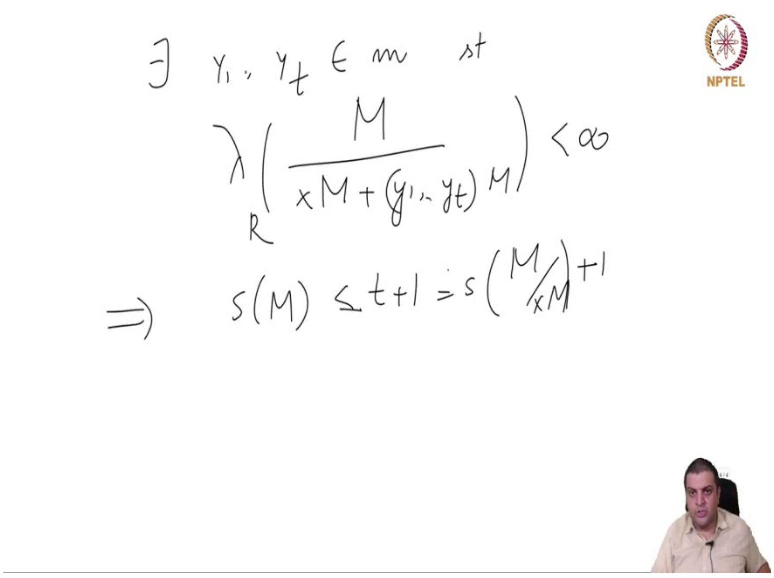

It says that, there exist some $y_1,\ldots,y_t \in m$, such that $\lambda_R\left(\frac{M}{xM+(y_1,\ldots,y_t)M}\right)<\infty$. So, this now means that, $S(M)$ cannot be longer than this, which is $t+1$, this is $S\left(\frac{M}{xM}\right)+1$. So, but what about this number by? Now, we need to ensure that, but if you kill M.

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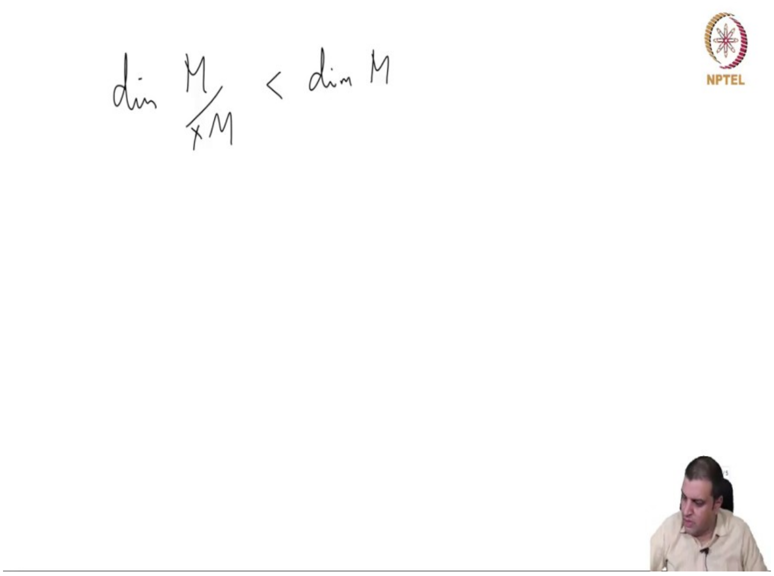

And, what about $dim\frac{M}{xM}$? On the other hand $dim\frac{M}{xM} < dim\,M$ because if you choose a prime ideal, this avoids a prime ideal below ok, sorry I got prime ideal, sorry.

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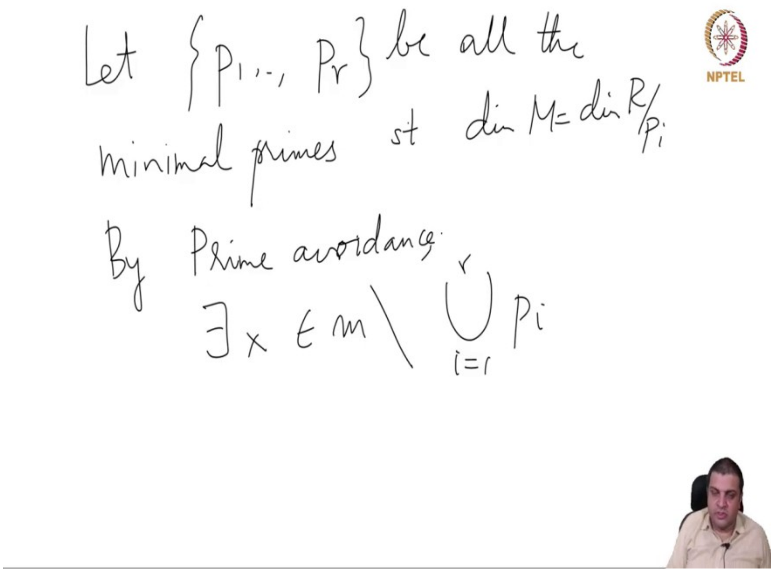

Let us choose an x properly. So, let $\{P_1, \ldots, P_r\}$ be all the minimal primes, such that

$dim\,M = dim\frac{R}{P_i}$.

Remember the dimension of a module is the maximum of dimension of $\frac{R}{P}$ as P runs over the minimal primes; among them some of them are equality, and some of them might be smaller. So, just pick the ones that have equality, do not care about anything else.

This is how we worked out in the previous example; we wanted to pick the start of a system of parameters by avoiding primes that determine dimension. So, I had not picked that thing here at this stage, we need to pick that way. By prime avoidance, $\exists x \in m \setminus \bigcup_{i=1}^{r} P_i$. This is what prime avoidance said.

If the ideal is not in a collection of finitely many prime ideals in any one of them, then it is not there in the union. And there are strengthening of this; we do not need all of them to be prime, which you must, I mean which you should have done in the exercises. So, there exist a name with this property.

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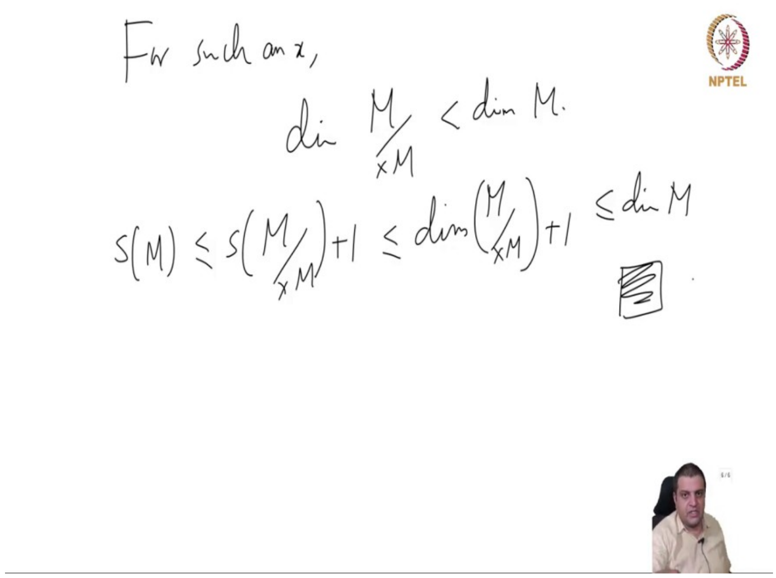

Now, for such an x, $dim\frac{M}{xM}<dim\,M$. That is because if you need a prime ideal which contains x and the annihilator of this, it cannot be one of these, because none of these contain x. So, therefore, any chain in which goes from annihilator of this module, can be lengthened by at least by one to get a chain inside, a chain containing annihilator of this.

So, this dimension is strictly less than. So, now, therefore, induction applies for $\frac{M}{xM}$ and so, now, let us put these things together; $S(M)\leq S\left(\frac{M}{xM}\right)+1\leq dim\frac{M}{xM}+1\leq dim\,M$. By induction this is less than or equal to $dim\frac{M}{xM}+1$, and because of this condition, this is less than or equal to dim M, which is what we wanted to prove.

So, this proves the theorem and now we want to draw some corollaries of this theorem. One of them is called. So, remember we have to do all these things to understand the dimension theory of noetherian rings; we got motivated into this, because we wanted to understand what is the dimension of polynomial ring over a finitely many variables over a field.

We believe that it should be, I mean it looks like it should be the number of variables; but we need proper understanding of dimension to prove that statement. And this is a much more, I mean much more general works in local rings etc; but we will need to use these things to prove that, and you know needs a little bit more steps to finish that.

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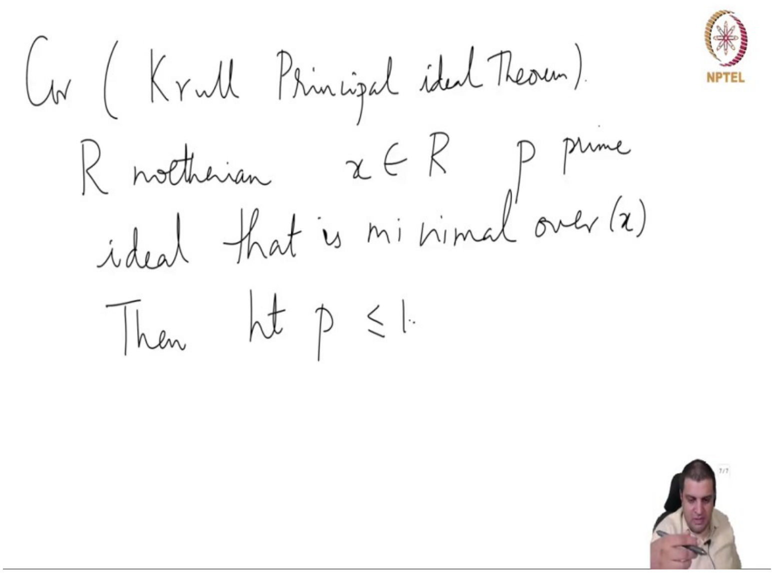

So, now, let us draw an important corollary of this called Krull's principle ideal theorem. So, R noetherian, $x \in R$; P prime ideal prime ideal that is minimal over (x). Then $ht(P) \leq 1$. I mean one, maybe this is a believable statement; but it is not, I mean at the generality of noetherian rings, this needs a proof.

So, the way we will prove it is; we will just derive it as an immediate corollary of the dimension theorem that we proved earlier. There are other proofs; one can prove it more directly without doing that much work, but some work has to be done, this is not a very simple observation, it is not an immediate observation from other things. And in fact, it is false for non noetherian rings. So, one has to use something, I mean some something about noetherian rings in the proof.

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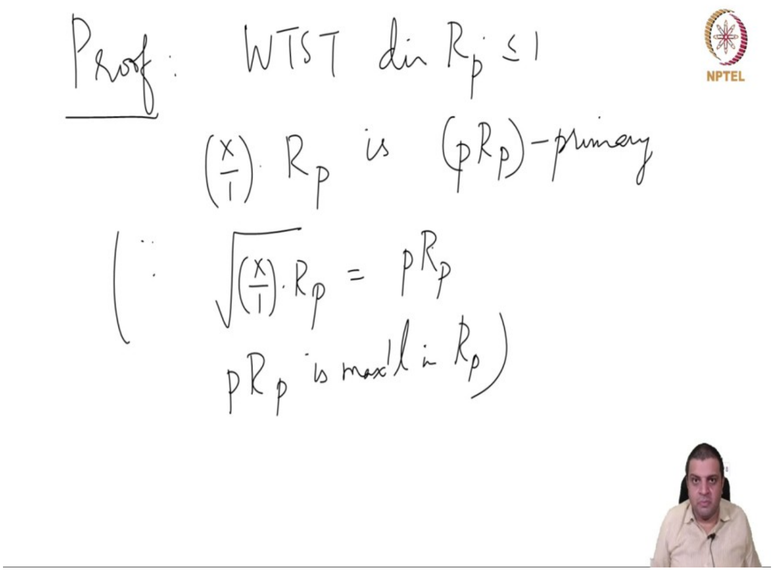

So, now let us proof is, I mean after we have done that theorem; this proof is very straightforward. So, we wanted to show that $dim\, R_P \leq 1$. Notice that, $\left(\frac{x}{1}\right) R_P$ is $P\, R_P$- primary.

Because there is nothing between; that is because the radical of this is $P\, R_P$. In that local ring, there is nothing between this ideal; there is no prime ideal between this ideal and that maximal ideal. So, this is $\sqrt{\left(\frac{x}{1}\right) R_P} = P\, R_P$; there are no other prime ideals, between these two ideals this is minimal and use the fact that $P\, R_P$ is maximal in R_P. So it is $P\, R_P$ primary.

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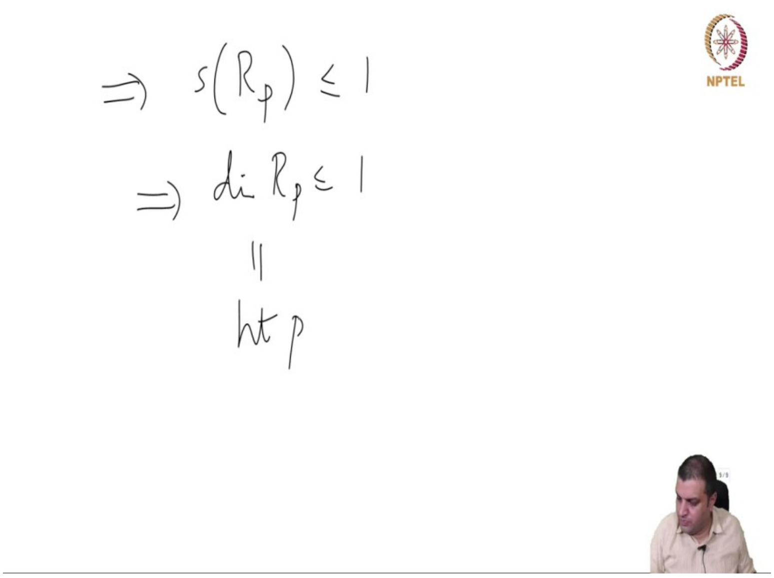

So, this now implies that $S(R_P)\leq 1$ but we know that S is same as dimension. Now, this is equal to height. So, as I mentioned there are other proofs of this theorem; one such proof without going to the Krull's dimension theorem is given in Heisenberg's Commutative Algebra book.

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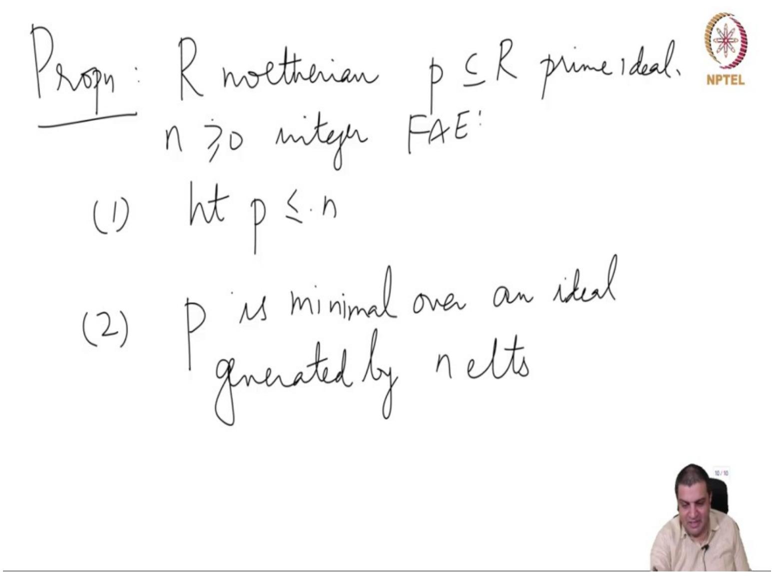

So, another purpose, which is also building up from this; we need some property of prime ideals, which we will need to use in polynomial rings. So, R noetherian, P is a prime ideal, n some positive integer, then the following are equivalent; One, $ht\,P \leq n$ and two, P is minimal over an ideal generated by n elements. What is called principal ideal theorem is just when n equals 1. We need this version to be used.

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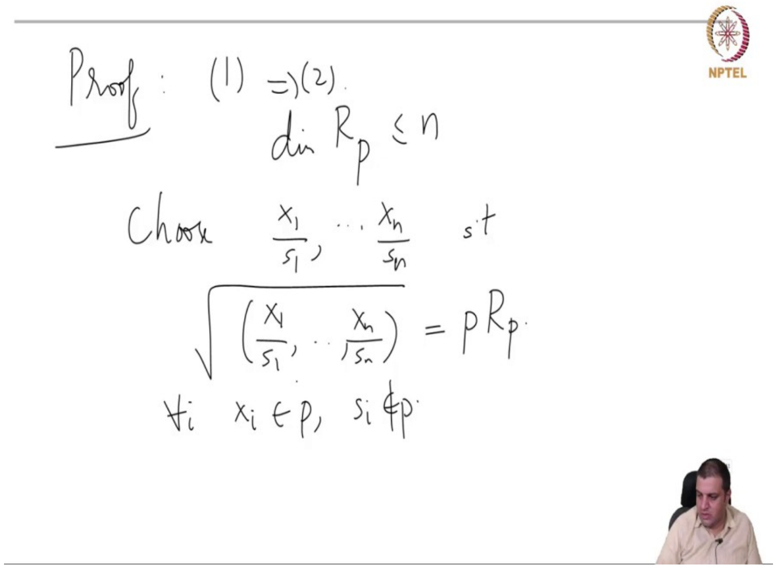

So, proof. So, 1 implies 2. 1 says that $ht\,P \leq n$. So, this means that, $dim\,R_P \leq n$; choose $\frac{x_1}{s_1}, \ldots, \frac{x_n}{s_n}$ such that the ideal generated by them, radical of this is $P R_P$, where $x_i \in P$ and $s_i \notin P$; there could be units for this purpose. So, this is for all. So, in the localization, there is such a ideal; this is again using the theorem.

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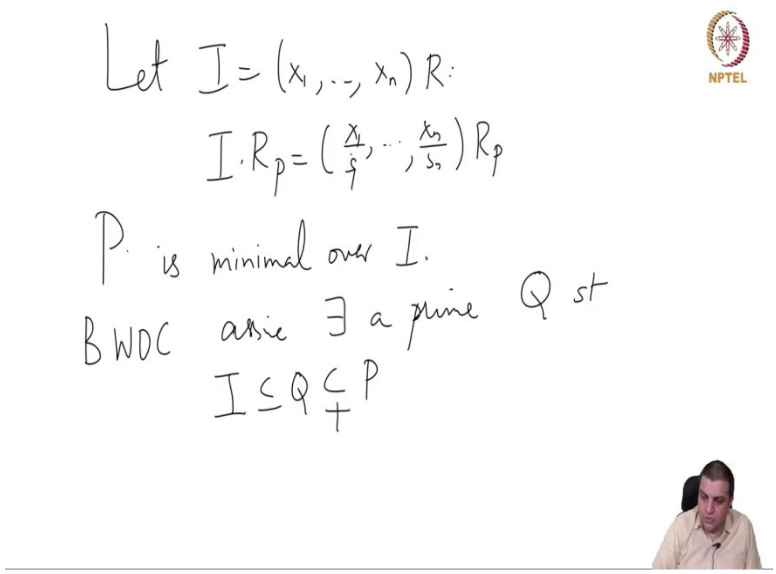

So, now, let $I = (x_1, \ldots, x_n)$ in R. Then of course, $IR_P = \left(\frac{x_1}{s_1}, \ldots, \frac{x_n}{s_n}\right)R_P$; because these are anyway units in this ring, so whether they are inside or not it is immaterial.

Now, what is the claim is that P is minimal over I. By way of contradiction, assume there exist a prime Q such that $I \subseteq Q \subsetneq P$. So, P is not a minimal over I.

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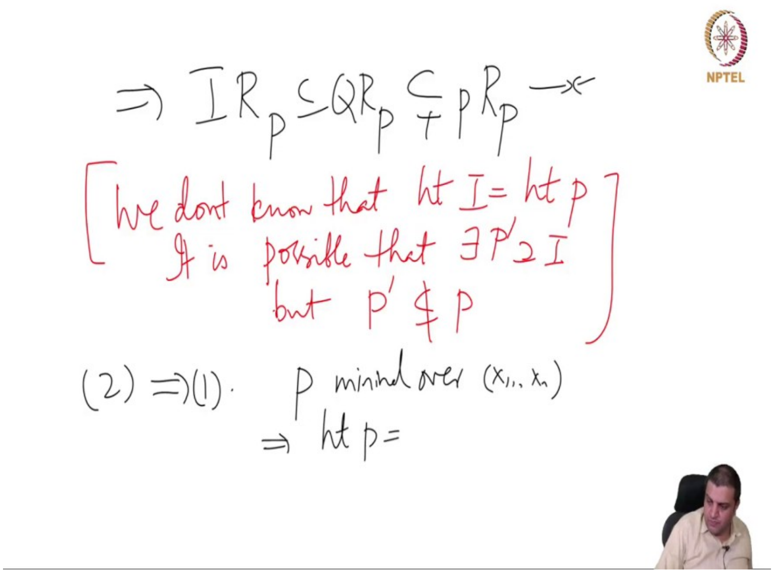

Then this one would imply that $IR_P \subseteq QR_P \subsetneq PR_P$ and that is a contradiction. So, this proves that P must be minimal. So, take any system of parameters in the local ring.

So, there will be fractions; just look at the numerators and look at the ideal generated by that and that is enough. That would generate, it would be minimal over P, just, maybe I will just mark put as a note, we do not know that $ht\, I = ht\, P$. Remember we are not saying height of P is n; it is just less than or equal to n. It is possible that there is some $P' \supseteq I$, but $P' \not\subseteq P$. So, once you localize P' became the whole ring, that will one would not see P' in the local ring at R_P; but there might be some p' of smaller height.

So, there is, ok. So, we are not saying anything about height of I. This is that we just proved 2 implies 1, this is just. So, P minimal over $(x_1, \ldots, x_n)$ implies that, height of P which is equal to dimension of R_P, which is equal to which is less than or equal to n.

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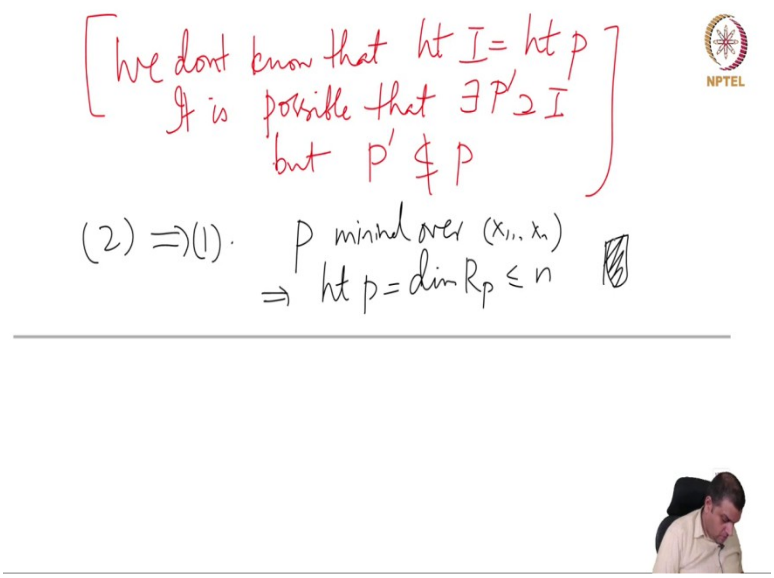

Because again the same reason, these things form a system of; I mean these things form something m-primary, something primary to the maximal ideal is in this local ring and that should that cannot be greater than n then. So, this is the end of this.

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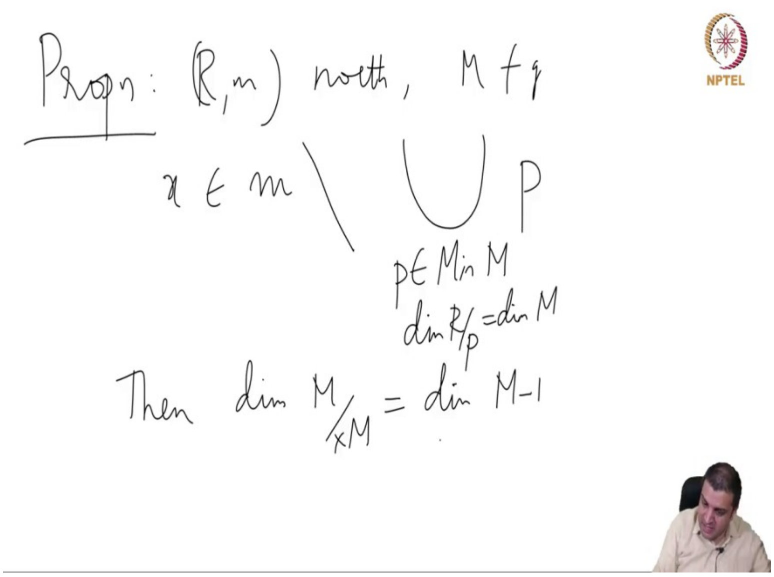

So, one more proposition. So, this is the choice of x that we made in the proof. So the same notation as a theorem (R,m) noetherian and M finitely generated.

And $x \in m\{P$ ¿ where $P \in Min\,M$ and $dim\,\frac{R}{P} = dim\,M$. So, an x with which you can actually cut dimension; so we just observed that, then $dim\,\frac{M}{xM} = dim\,M -$ ¿1.

So, in the proof of the theorem we used that dimension of $\frac{M}{xM}$ is strictly less than dimension of M and that is how we built the induction; but it is in fact equal to $dim\,M - 1$. That is because, if you go back to the idea behind the proof of theorem; we have to pick such an x, if you want to start constructing a system of parameters we have to pick such an x.

So, this is what we want. So, we will use the observation that we already made which is that, this is less than or equal to. So, we will just prove the other inequality.

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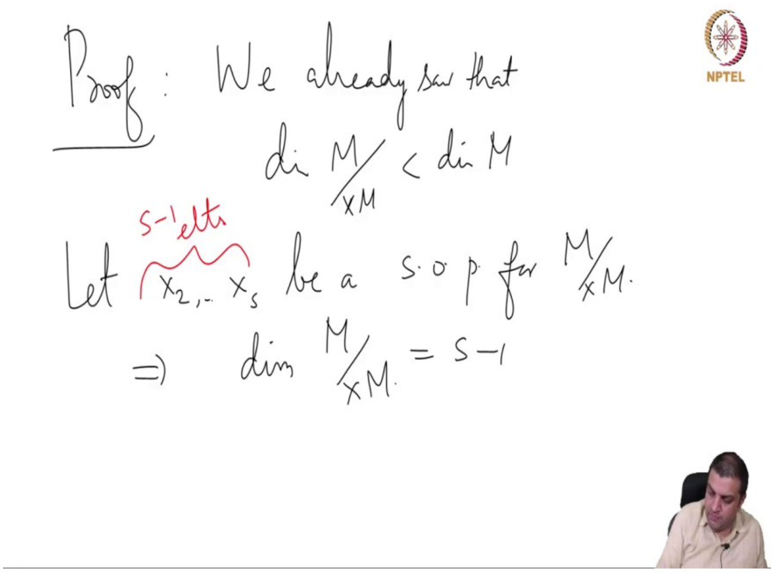

So, proof, we already saw that $dim\,\frac{M}{xM} < dim\,M$ that is we used in the proof of the main theorem. So, we just have to prove that dimension of M is less than or equal to this plus 1.

So, let $x_2, \ldots, x_s$ be a system of parameters for $\frac{M}{xM}$. So, this now means that, $dim\,\frac{M}{xM} = s - 1$. So, this is where we are using the theorem; there are $s - 1$elements.

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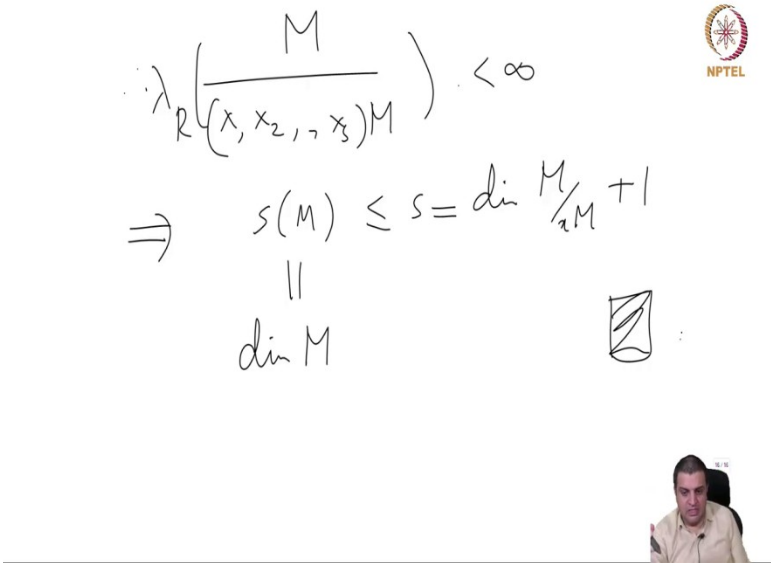

Now, if you choose therefore, $\lambda_R\left(\frac{M}{(x, x_2, \ldots, x_s)}\right) < \infty$ which implies that $S(M) \leq s$. But this is by the theorem $S(M) = dim\, M$ and $s = dim\frac{M}{xM} + 1$ which is what we wanted to prove dimension is less than this and we already proved the other inequality. So, this proves that.

So, this is a statement that is not; I mean that is sort of in, that is not counter intuitive. In the sense that, if you choose some element, you know in fact most elements, then when you go modulo that the dimension will go down. So, you can think of it as, in geometrically one can think of it as; if you have some $Z(I)$, then if you impose one more condition and if that condition is sort of independent, then the size of $Z(mod\, I)$ will come down.

Remember dimension of M or the is; I mean it is a subset of spec and that is one can analogously think of it as $Z(I)$ for some ideal. So, as you as we kill more and more equations, the solution set will come down.

And if you choose; at each stage if you choose a general enough equation, the solution set as, solution sets dimension whatever it means comes down one step and one at a time. So, that is what this theorem is this proposition is saying; but although it is not counter intuitive, it still needed some amount of work to prove that statement.

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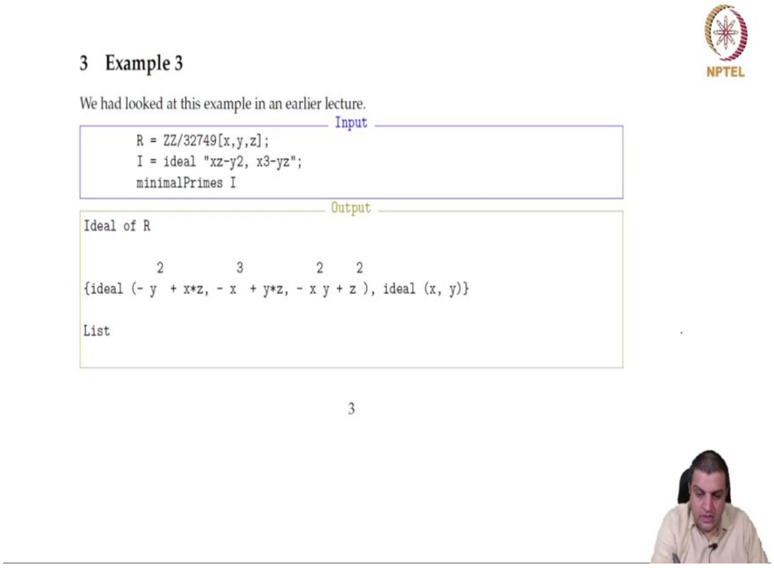

So, this time I chose a different field, some a very big field with 32749; this is a prime number, this is the largest prime number that I call it to handles or maybe sorry, I do not know what the latest version the largest prime number is bigger, but you know for some time this used to be the largest prime number. So, this is a large enough field and I will explain why it is necessary to choose a large enough field, polynomial ring in two variables.

And here is an equation that we saw we did our saturation example by hand, by hand meaning with the help of macaulay with this thing; $(xz-y^2, x^3-yz)$. And we know already that we had identified these minimal primes. But if you go back few lectures ago. But one part was easy that we observed, (x, y) is minimal prime which contains this.

And we also already observe that anything that, in that lecture we already observed that anything that is not this prime minimum must contain x and y. So, these two are only minimal primes and we also proved that this is the kernel of a map from $k[x, y, z] \rightarrow k[t^3, t^4, t^5]$. So, this is also a prime. So, this is, but anyway we just; in this case I just asked macaulay to calculate this.

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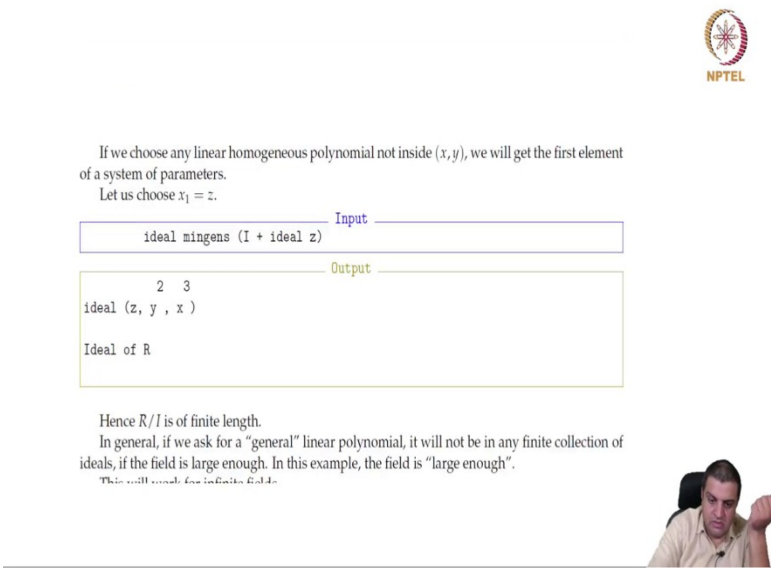

Let us, now, we would like to find, we would like to find a system of parameters; but the point of this example is, it is actually not at all difficult to find. If you choose any linear homogeneous polynomial not inside (x, y); then we will get the first element in the system of parameters, because of course we cannot choose anything inside (x, y).

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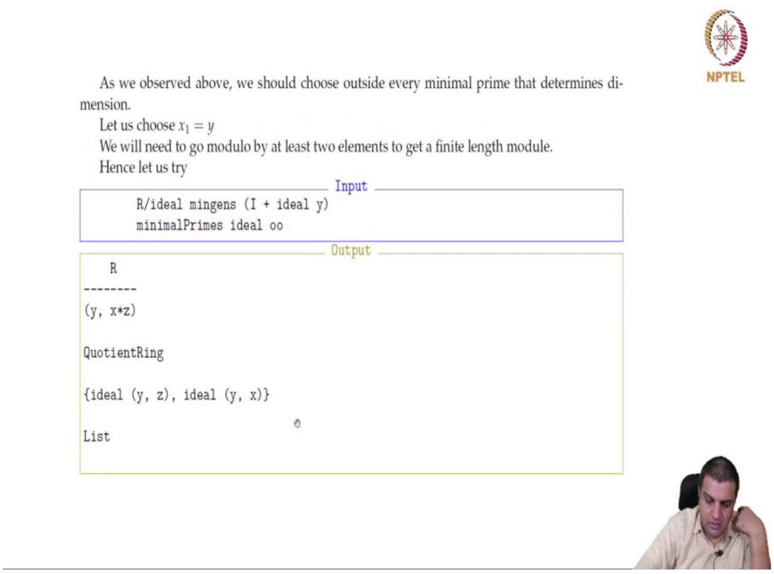

And I chose linear because; see in this polynomial, this prime does not contain any linear, linear homogeneous polynomials. In homogeneous, we will have to check a little bit; I mean it will not contain, but we have to check it with a little bit more difficulty. Linear homogeneous just by looking at these terms it is ok; this is already inside the square of the ideal (x, y, z), and hence it cannot contain anything which is linear, which needs to involve x, y and z, linear terms x, y and z, right.

So, for example, we could choose z; let us kill z and then ask for the ideal generated by the minimal primes, we just get (z, y^2, x^3). Let us check here. So, if you sorry, here if you kill z, we get a y^2; because z will get an x^3 and then there is a z. So, that explains this thing. So, $\frac{R}{I+z}$ has finite length.

So, therefore, we just need. So, the dimension of this ring; again when we say dimension of this ring, we mean dimension of this ring localized at the homogeneous maximal ideal (x, y, z). We do not need to worry about homogeneity in this problem; but at the maximal ideal generated by the variables.

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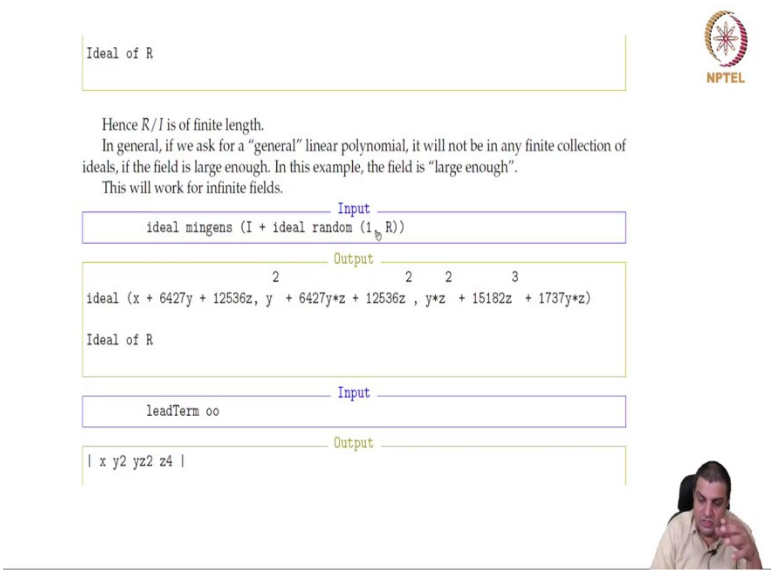

So, in general if you ask for a general linear polynomial in this sort of situations, it will not be in any finite collection of ideals. So, this goes back to prime avoidance theorem and this is was an exercise; if the field is large enough. So, typically large enough means infinite; I mean for sure it will work in infinite fields. In this particular case, this 32749 was also large enough characteristic that; so we can just look at this thing here. I will take a random element of degree 1 in R. So, I got $x+6427y+12536z$.

So, the point is that random element; I mean most likely, of course we cannot we ask for random element, macaulay generates it through some random number generator, there are no true random number generators, these are all. So, it may. So, end up that the random element does not involve z and then we know, it would not work.

But if you do random; if you run this command enough number of times; then most of those times you will get an element which involves z. And in this particular example as soon as this linear polynomial involves z that would be enough as a system of parameters. So, that is the thing.

So, this is a typical thing that one does in macaulay that, if you just work; if you do this, if you want to identify a system of parameters or something or if you want to find a general polynomial, we would just run the random command enough times. And whatever is true for general, will hopefully be true most of the times in this running of random command.

And just because it is not immediate from here that this ideal is; just by looking at this is not immediate that this ideal is m-primary, a primary to the ideal generated by the variables. So, we just ask for the lead terms in sum Grobner base thing and we get (x, y^2, yz^2, z^4). So, this is finite length.

So, in the next lecture, we will use these ideas; we will use not the example, but the Krull principal ideal theorem to and the characterization of height in terms of being minimal over some number of elements. We will use that to prove that for a noetherian ring R, dimension of $R[X]$ is $1+dim R$.