Computational Commutative Algebra
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Lecture – 42
Dimension of noetherian local rings – Part 1

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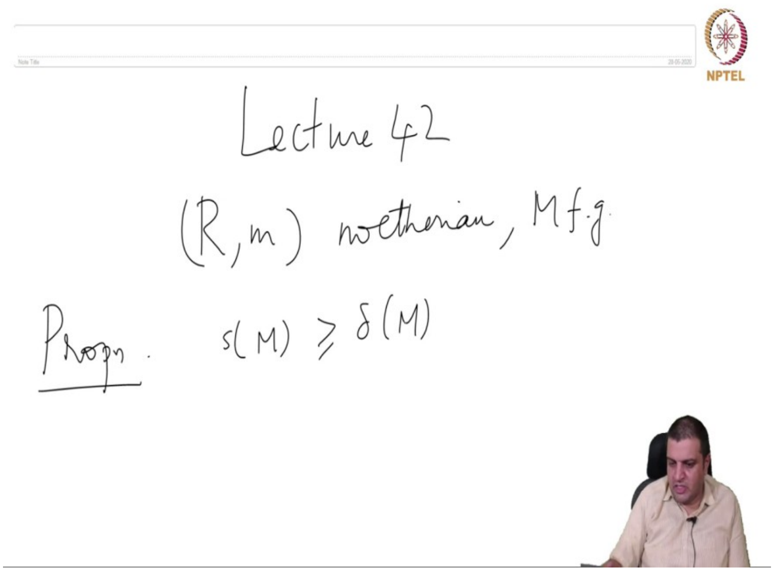

So, in this lecture we begin by looking at systems of parameters and then we will prove the Krull's theorem about relating three things dimension of a module $\delta(M)$ which is degree of a Hilbert-Samuel polynomial and $s(M)$ which is the length of a system of parameters. So, proposition. So, throughout again in this whole lecture (R, m) is noetherian and M is finitely generated R module and we will not explicitly state this all the time except maybe stating when we state the theorem.

So, another proposition $s(M)$ which is the minimum number s such that there exist some $x_1, \ldots, x_s$ so that $\frac{M}{(x_1, .., x_s)M}$ has finite length. So, called the system of parameters of for M. So $s(M) \geq \delta(M)$ which is degree of a Hilbert-Samuel polynomial for any ideal m-primary ideal I irrespective of the ideal.

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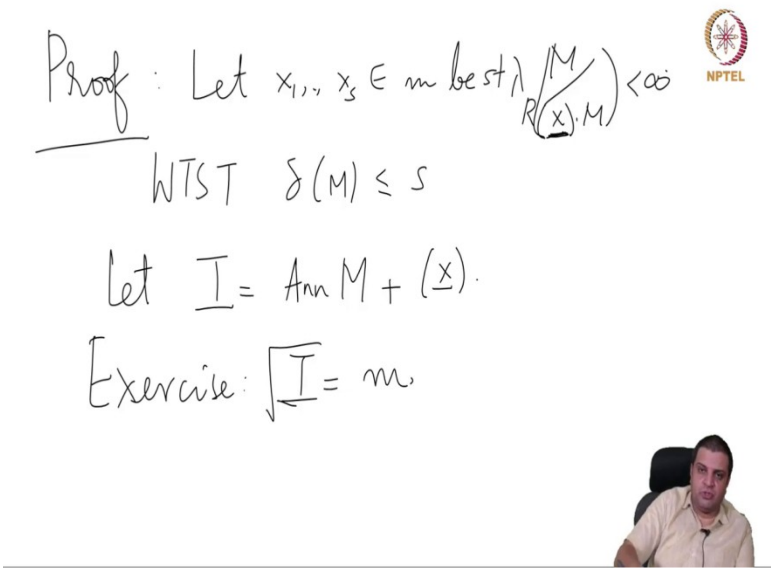

So, we can determine $\delta(M)$ by looking at $P_{I,M}$. So, we will try to use something that is convenient for us. So, let $x_1,\ldots,x_s \in m$ be such that $\frac{M}{(\underline{x})M}$. So, I will just put underline to denote all the x's together times M , the length of this module is finite.

So, it is enough to show that $\delta(M) \leq s$. If this is the case then the minimum of all such possible also will be greater than or equal to this if this is true for every s. So, this is what we want to show. So, we would like to choose some convenient I construct with respect to a generating set for that I and then conclude this.

So, let $I = Ann\,M + (\underline{x})$ and it will be an exercise to check that this I is an m-primary. In fact, a sequence like this will have this property if and only if annihilator M plus that ideal generated by that sequence is m-primary.

So, exercise $\sqrt{I} = m$. So, we can we can try to determine $\delta(M)$ by looking at $P_{I,M}$ for this I. So, how would we do this we first have to write the associated graded ring as a quotient of a polynomial ring.

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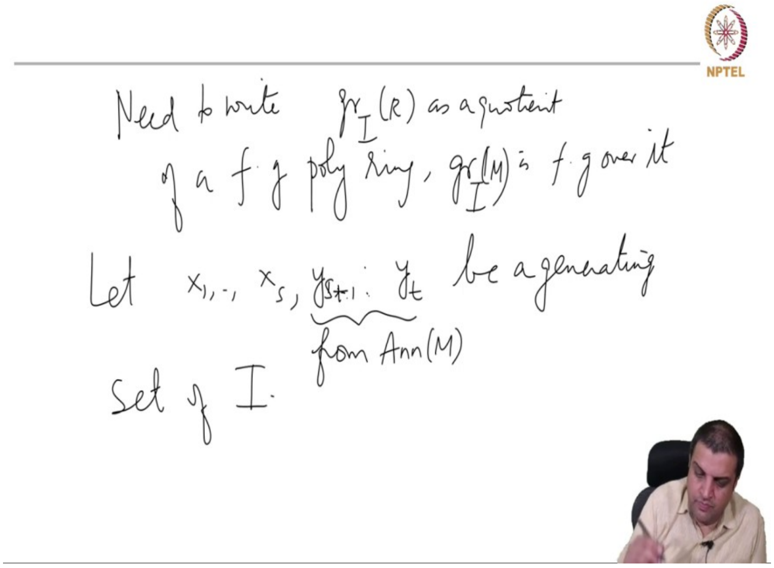

Associated ring for I. So, I need to write the associated graded ring of R as a quotient of a finitely generated polynomial ring and then M is finitely generated over it. I mean that is sorry, not the associated graded ring associated graded module of M is finitely generated over it and then we will use the property about the Hilbert function of such a module.

So, in order to write this we have to make a choice a choice of generators of M. So, we will choose the following. So, let $x_1, \ldots, x_s, y_{s+1}, \ldots, y_t$ and these y's from the annihilator of M.

So, take the x's and generators for this need not be a minimal generating set for I, but that is just need to take some generating set be a generating set of I it need not be minimal, but that is fine.

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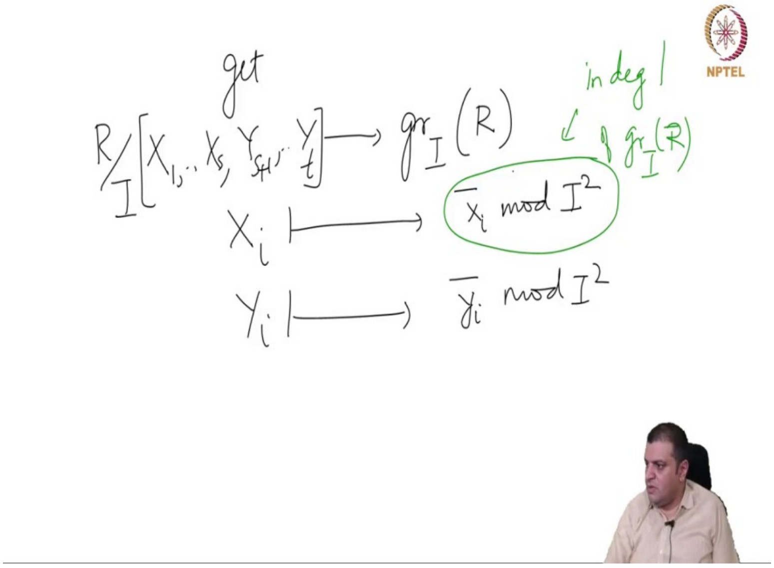

So, now we can get the associated graded ring of R as a quotient of $\frac{R}{I}[X_1,\ldots,X_s,Y_1,\ldots,Y_t]$, these are variables. What is the map? The map is the variable X_i goes to the element $\overline{x_i}\,mod\,I^2$. So, this is the degree one part of.

Remember this is in degree 1 of the associated graded ring and Y_i similarly, goes to $\overline{y_i}\,mod\,I^2$. So, we get a polynomial ring and the associated graded module is a module over this. So, therefore, $gr_I(M)$ is annihilated by $Y_{s+1},\ldots,Y_t$.

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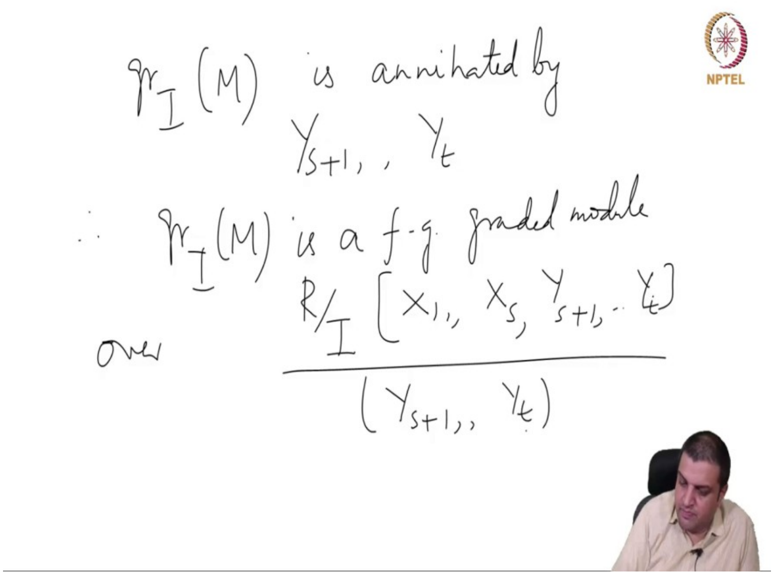

Because if we take an element in this it is the residue class of some element of the module on Y_i acts as $\overline{y_i}$ and what is the multiplication $\overline{y_i}$ times some $\overline{m}$ where m is an element inside here M is just $\overline{y_i m}$, but $y_i m=0$. So, these elements are annihilated by this.

Therefore, $gr_I(M)$ is a finitely generated graded module over the over the quotient ring. So, this polynomial ring modulo the Y's because they anyway kill over that is

$$\frac{\frac{R}{I}[X_1,\dots,X_s,Y_{s+1},\dots,Y_t]}{(Y_{s+1},\dots,Y_t)}.$$

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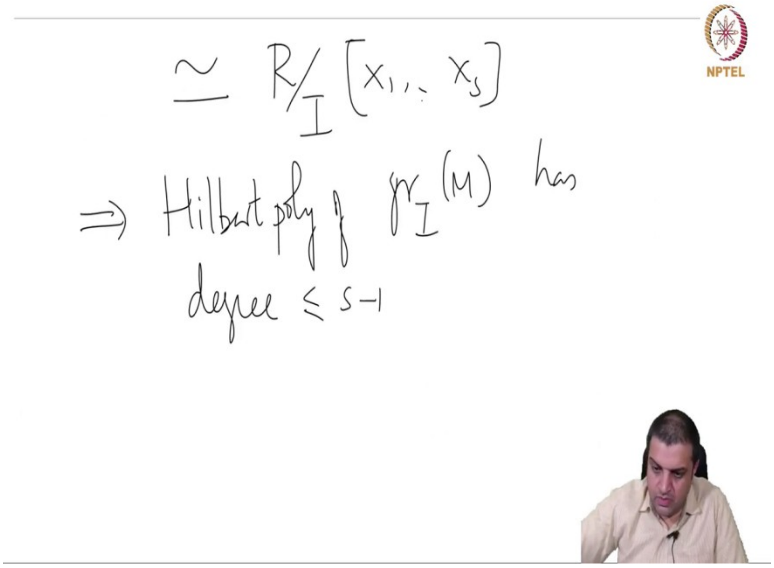

But, this is also a polynomial ring recall that this is also $\frac{R}{I}[X_1,\dots,X_s]$. So, associated graded ring is a finitely generated module over this ring that is the observation. Of course, we cannot change this to X's. Here we are using I it is just we only need the generators of the generators corresponding to the system of parameters and not for the an annihilator that is the observation that we want to make .

Then this one says that the Hilbert polynomial of the associated graded ring of M has degree at most $s-1$.

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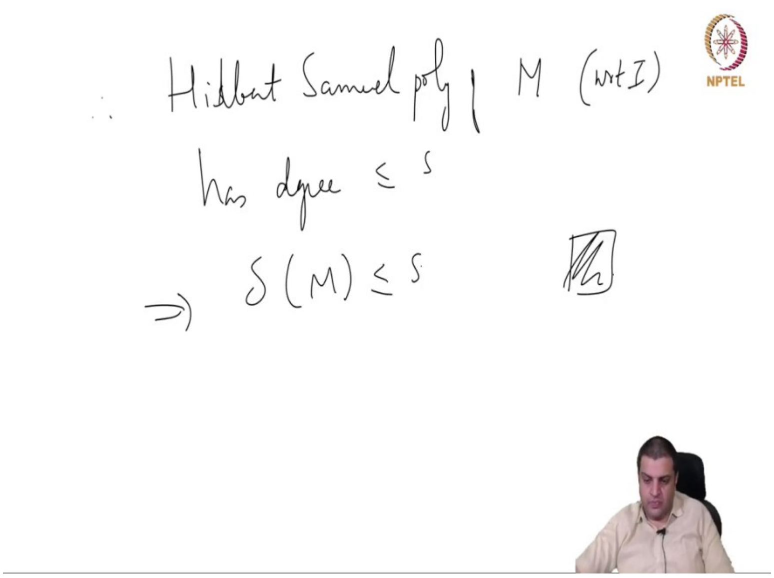

Therefore, the Hilbert-Samuel polynomial of M with respect to I has degree at most s. This now implies that $\delta(M) \leq s$. So, in Krull's theorem this is one step, but we will we want to look at some quick examples in Macaulay before we prove the theorem Krull's theorem.

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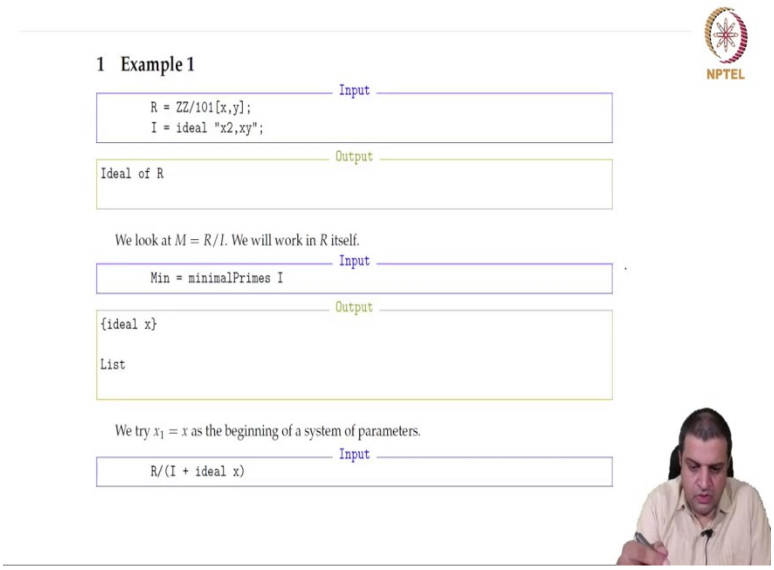

So, now let us look at these this example. So, the first example. So, we have seen variations I mean these ideals earlier. So, the in the input we take some field adjoint two variables and we look at the ideal generated by x^2, xy. What we wanted to consider is the the modulus the quotient ring, but it is a little difficult see some of these functions are not defined for quotient rings. They are defined for ideals and modules.

We have to each time we construct a quotient ring we have to tell Macaulay to convert it to a module etc these are just changing their types. So, instead of worrying about all of these things we will just worry about the minimal just work directly in R, it is equivalent anyway.

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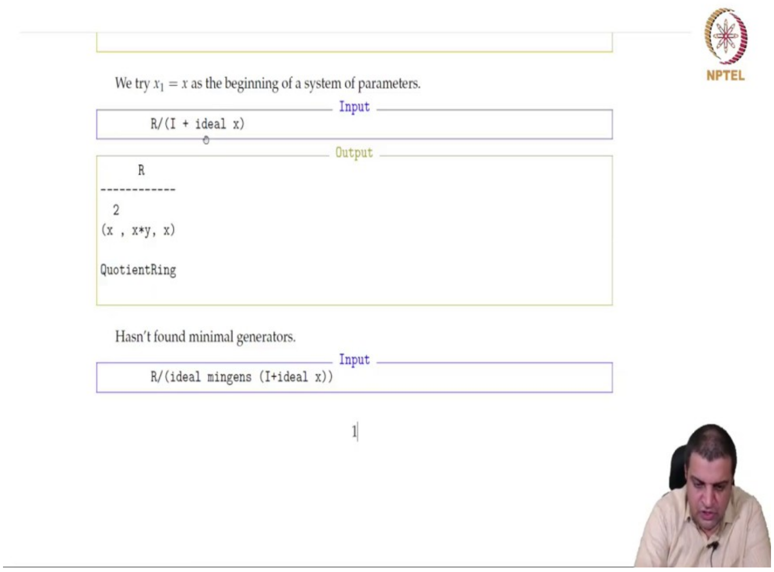

So, we ask for minimal primes of I, we have computed this earlier any prime must contain x because of x^2 and (x) itself is a prime ideal in R. So, it is a primary ideal in the quotient and therefore, it is the minimum prime. We are searching for a way to find systems of parameters or what would be a system of parameters for M. So, we have to do some $x_1,\ldots,x_s$. So, let us start with $x_1=x$.

So, we ask $\frac{R}{I+(x)}$. So, this is not it did not; it did not minimize the generators we just give that way. So, here I just ask. So, it has not found minimal generator. So, I ask to find the minimal generators of $I+(x)$ and then construct the ideal from that.

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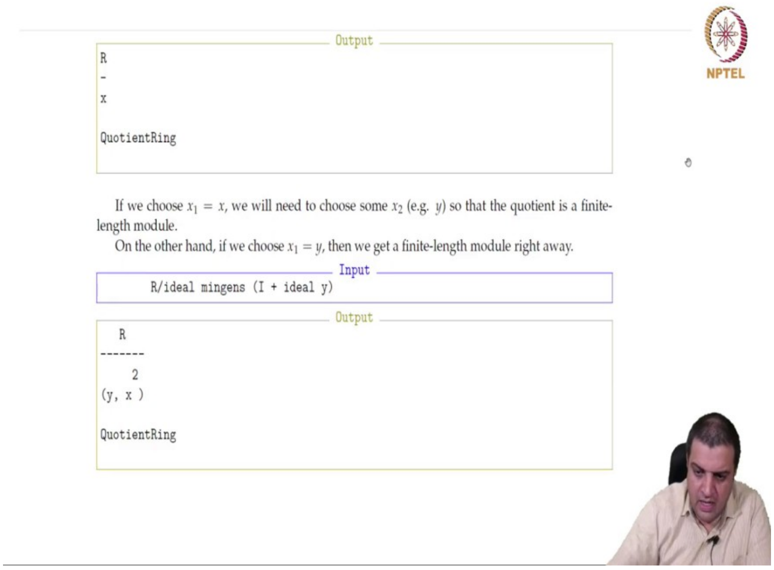

So, then if we ask that this is the output it says output is $\frac{R}{(x)}$ which is what we expected. So, if we choose x_1 equals x then we will need to choose some other x_2, for example, y because $\frac{R}{(x)}$ is isomorphic to $k[y]$. So, we need to use something so that the quotient is a finite length module. I mean $k[y]$ is not a finite length R modulo.

So, we have to choose two things if we started this way at least two things. In fact, only two things, but on the other hand if we choose x_1 to be y. So, let us check that thing. So, we ask the same thing I just add the ideal take min gens and then construct the ideal.

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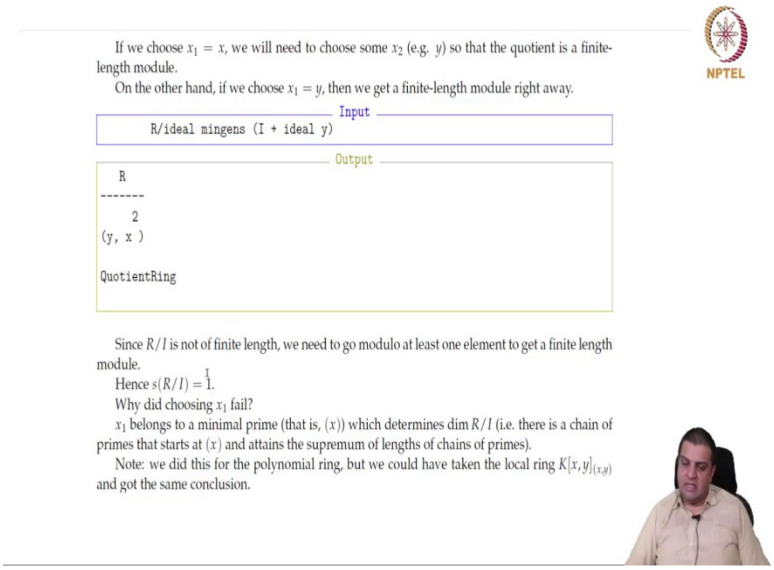

So, we would get (y, x^2). So, we got one element. So, therefore, the this is $s(M)$ in this case M being $\frac{R}{I}$ is at is at most one. But, on the other hand, $\frac{R}{I}$ itself is not a finite length because we know that if this has finite length then it would be a finite dimensional vector space, but in this case every power of y remains in the quotient.

So, it is not a finite length. So, we need to go modulo at least one element and in this case we have found one element. So, we are done. So, that says $s(R)=1$.

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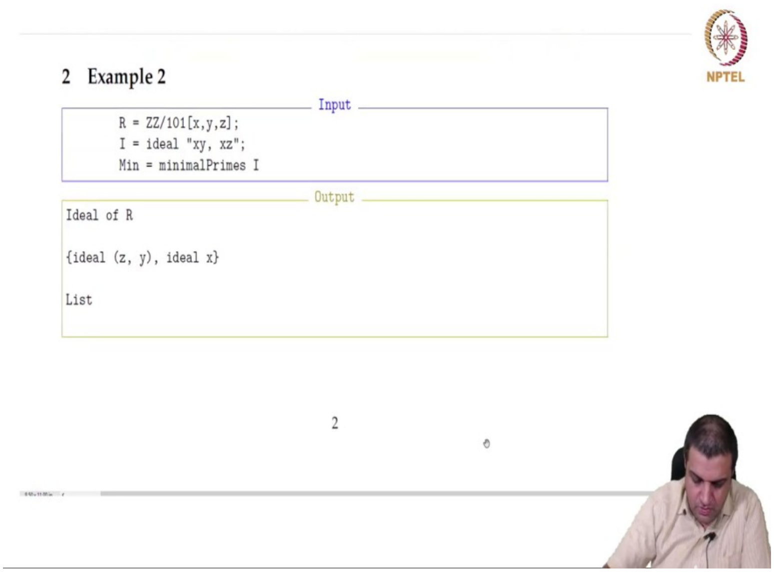

Now, let us ask why did choosing x_1 fail? So, the reason why it failed is that choice of x_1 belongs to a minimal prime which determines dimension of $\frac{R}{I}$ in other words from this minimal prime to a maximal ideal there is one of maximum length that is you start with (x) and then go to (x,y). So, there is one.

So, there is a chain of primes that starts at (x) and then attains the supreme of lengths of change of primes. So, therefore, that is the reason why this failed. So, one thing to note at this point is we did this for the polynomial ring, but we chose elements inside the maximal ideal (x,y) all throughout.

So, that if you just localized at the maximal ideal (x,y) we would have gotten the same conclusion and there is this; there is this going back and forth between local rings like this or and graded rings which we will explore after we prove these results for. And, that is crucial to build some intuition about various things. So, we will have couple of lectures on doing that thing and this and all the more.

So, because macaulay2 is or any computational algebra system is very I mean is more adept at handling a graded situation than handling an arbitrary ring. So, therefore, we will do this translation between graded case and local case on and off and we will go over it in detail later.

So, but in this particular case it is easy to see that if you have done the same calculations over the local ring we would have got the same result. Now we joined three variables and take the ideal (xy,xz) , this it is minimal primes are (y,z) and (x).

So, the first generator says x must be in it or y must be in it. Second one says x must be in it or z must be in it; therefore, if x is not there in a minimal prime in any prime then y,z must be there. So, that is the only minimal prime.

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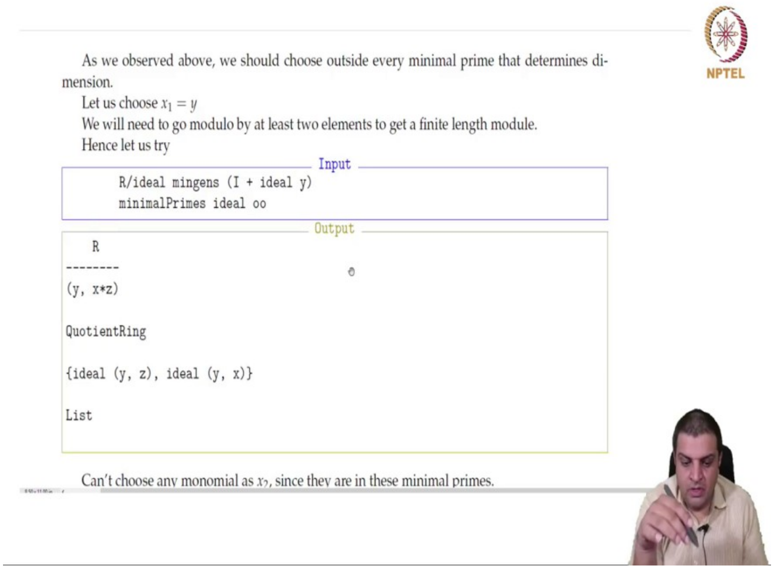

So, we already observed that there is no point picking elements inside a minimal prime that determines dimension. So, here there are two minimal primes; here there are two minimal primes here there are two minimal primes one from this minimal prime (x) there is a chain of length at least two which is (x, y) and (x, y, z).

Well, here we can probably so, any prime ideal containing yz must correspond to a prime ideal in $k[x]$ when $k[x]$ is PID. So, we can only have 0 or maximal ideals. So, any chain from here to a maximal ideal we will have length at most 1, this and the maximal ideal as opposed to this there is one at least 2. So, we should probably avoid this and we should pick one from there. So, that is what we are saying here.

So, we have to avoid x. So, we could pick something in the other prime. So, we could pick for example, y and then if we add that we will get (y, xz) then we ask for it is minimal primes. So, this is considered as a quotient ring of R and if you ask ideal of a quotient ring it would give the ideal for which it was quotient by. So, it just showed up it just said what that ideal is which is just this ideal (y, xz)minimal primes over that.

So, now, we see that we cannot pick any monomial because every monomial in R must be in these two ideals right. If it is divisible by x it will be inside here; if it is not divisible by x it has to be divisible by y or z, I mean a monomial different from 1 therefore, it must be inside here.

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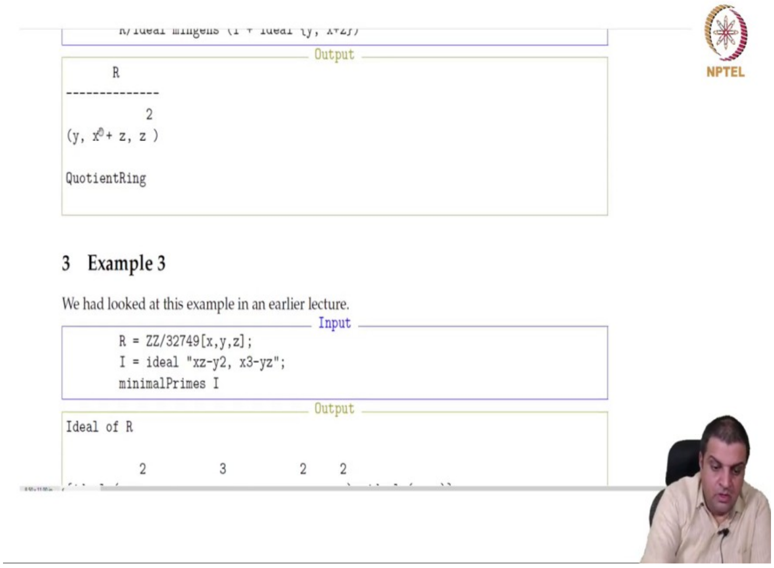

So, we cannot choose a monomial as x_2. So, these are every monomial is in this ideal. So, we could try $x+z$, then we ask R mod ideal mingens I plus the ideal generated by y and $x+z$ and that gives $(y, x+z, z^2)$ square it rearranges some term. And, this we can see leading term of this is y, leading term of this is x and this is the monomial. So, its own leading term.

So, x, y and z are leading term, xy and the power of z are leading terms and this is finite length. So, we have picked a system of parameters y and $x+z$. So, this is third example we will discuss after we prove the theorem and I think.

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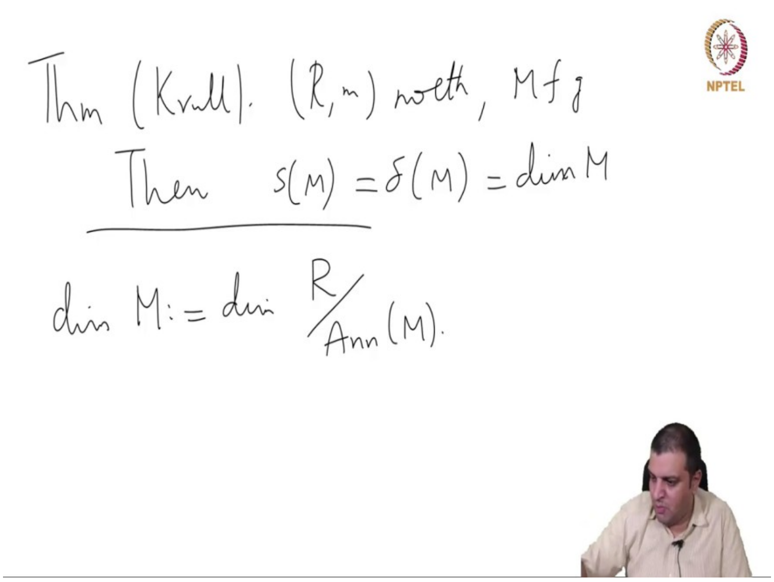

So, let us state the theorem. This is Krull (R,m) noetherian M finitely generated then $s(M)=\delta(M)=dim\,M$, $s(M)$ which is the length of a system of parameters on M is equal to $\delta(M)$ which is a degree of Hilbert Samuel polynomial $P_{I,M}$ for any m-primary ideal I equals the Krull dimension of M and what is dim M? $dim\,M=dim\,\frac{R}{Ann(M)}$

So, this is we have said what dimension of quotient rings are chain of primes in that quotient. So, this is the theorem. Just a remark just to warm up for the proof.

See in the examples there is macaulay that we saw we said we had to take an element when we are constructing a system of parameter, we have to take an element that avoids primes that are that determine dimension so that each time you go modulo you know the beginning of a system of parameters in appropriate choice will also ensure that dimension goes down.

Hopefully, I mean one can use this to prove the theorem except there is one catch that we so, we could use this probably to prove by induction on dimension except one catch which is that we do not know yet whether dimension is finite. It is only after we prove the theorem or the first part of the proof of the theorem that we will know that, only then we can induct on dimension, but that is we will need to use it. So, that is one.

So, the idea of going modulo the beginning of a system of parameters progressively has the effect of cutting down dimension. The only point is that we have to ensure that dimension comes down to 0, when we are stuck when we cannot find any more system of parameters that is. So, that we will need to prove I mean we will need to prove an equivalent statement. So, now, let us prove the theorem.

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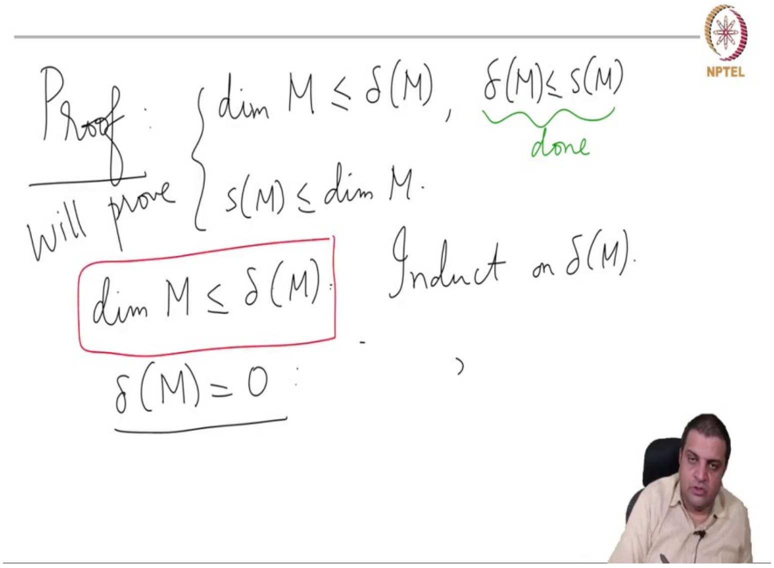

So, we will prove three statements which is one which we have already proved. We will prove that $dim\, M \le \delta(M)$ but the degree is independent less than or equal to. We already proved that this is less than or equal to $s(M)$ this we already proved and then finally, we will prove that $s(M) \le dim\, M$.

So, the three inequalities if we prove we will finish. So, we will prove these things. We will prove these two, this already done. So, let us prove this let us prove this and this. So, we know that this is a finite number because it is a degree of a polynomial. We know that $s(M)$ is a finite number because maximal ideal is finitely generated and definitely system of parameters cannot have length greater than or equal to a generating set of maximal ideal.

So, these two quantities are finite $\delta(M)$ and $s(M)$, this will ensure that dim is also finite and then we will be able to do induction using them. So, this is induct on δ what does that mean. So, when would $\delta(M)=0$. So, this function n going to may be .

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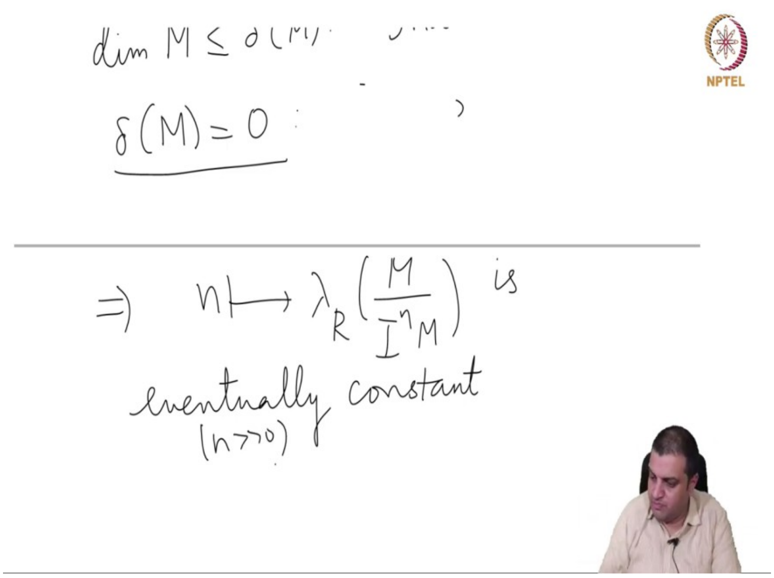

So, $\delta(M)=0$ means that this implies that the function which takes n to the length of $\frac{M}{I^n M}$ is eventually constant that this is just because a polynomial has degree 0. So, this is an eventually mean for all n much larger than 0, it is constant.

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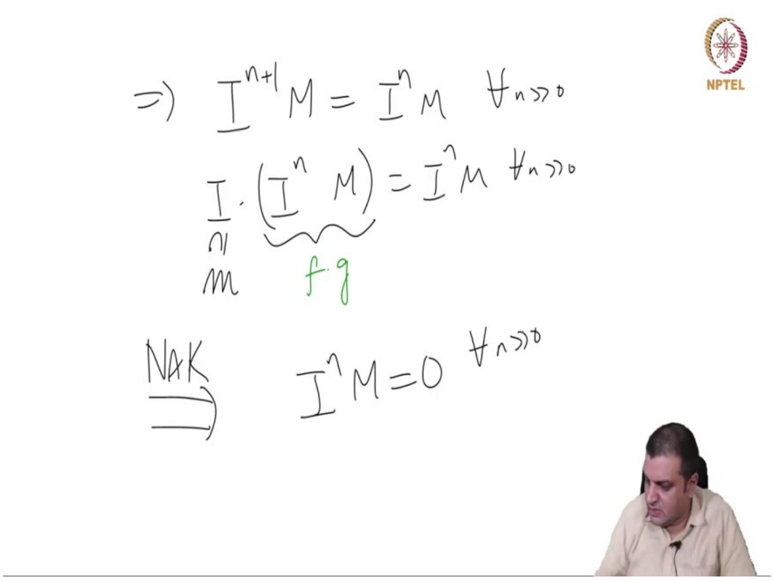

But, that would imply that $I^{n+1}M = I^n M$ for all n sufficiently large. But, this is just $I\left(I^n M\right)=I^n M$. So, this is inside the maximal ideal; some proper ideal over and this is finitely generated. So, this is inside the maximal ideal this is finitely generated therefore, by Nakayama lemma $I^n M = 0$ for all sufficiently large n.

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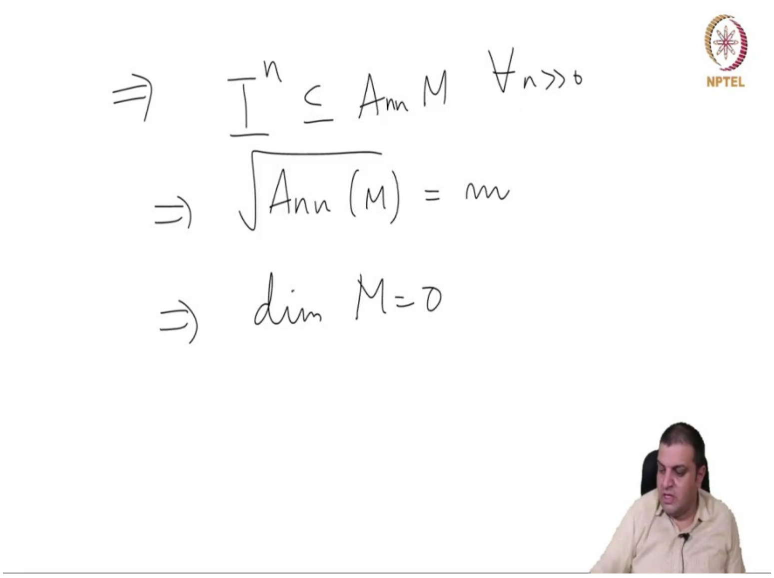

In other words, $I^n \subseteq Ann\, M$ for all sufficiently large n or in other words, the annihilator of M is m-primary and M is finitely generated. So, M is artinian and this implies that dimension of M is 0. So, the inequality is preserved both are 0 if this is 0 then this is also 0 remember we are trying to prove this. So, if the right hand side is 0 left hand side is also 0, that is the first that is the base case.

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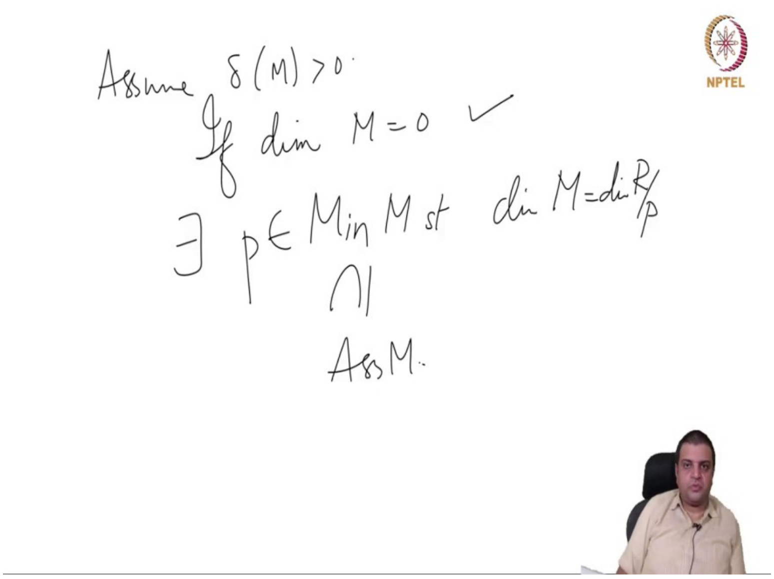

So, now assume that $\delta(M) > 0$. If $dim\, M = 0$, then there is nothing to prove. So, if it is not or maybe M ring. There exists so, notice that there exists $p \in Min\, M$ such that $dim\, M = dim\, \frac{R}{p}$. That is because dimension of a module is the dimension of $\frac{R}{Ann\, M}$ and minimal prime over M means minimal prime at which M_p is nonzero.

So, therefore, those are exactly this they will be minimal prime in R mod minimum prime over the annihilator of M and this is. So, we have this, but minimal primes are associated.

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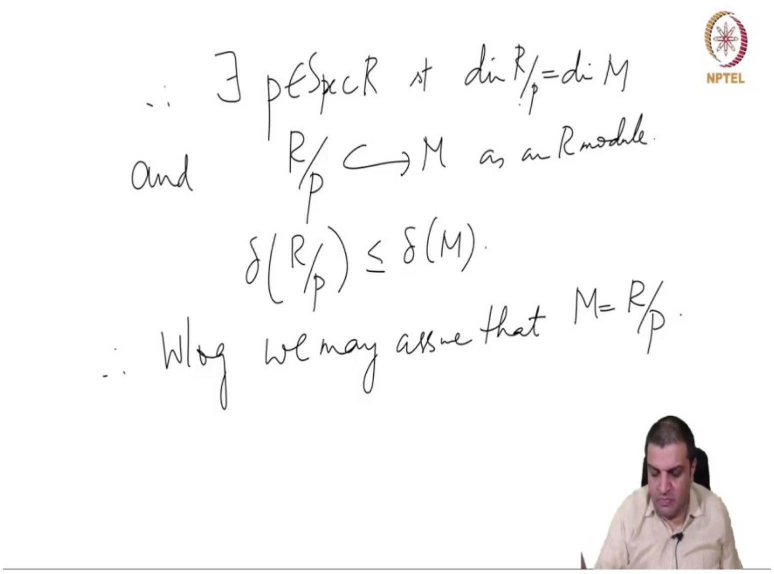

Hence there exists $p \in Spec\, R$ such that $dim\, \frac{R}{p} = dim\, M$ and $\frac{R}{p}$ injects into M as an R module. So, we proved earlier that delta of a sub module is less than or equal to of a module and dimension of this is equal to dimension of that. So, if you prove the inequality for $\frac{R}{p}$, then we are done. Therefore, without loss of generality we may assume that $M = \frac{R}{p}$ we will see why.

This is helpful just to recap we did we want to replace by some nice module. So, we look at a minimal prime that determines dimension so that on the lower side there is no a change. This is we have this equality and on the right side there is a desirable inequality. So, if we prove for this it will also hold for M.

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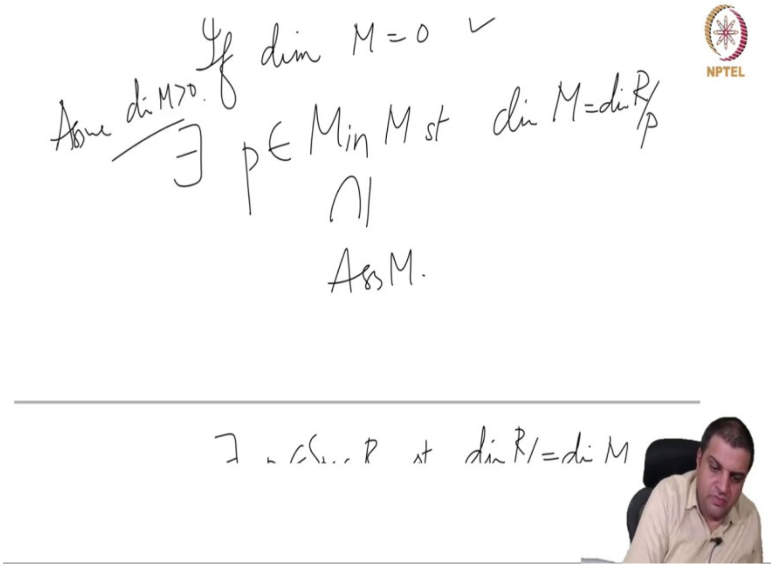

So, so much of the argument does not use that dimension of M is nonzero. So, this is ok. So, assume dimension of M is positive if it is 0, there is nothing to prove.

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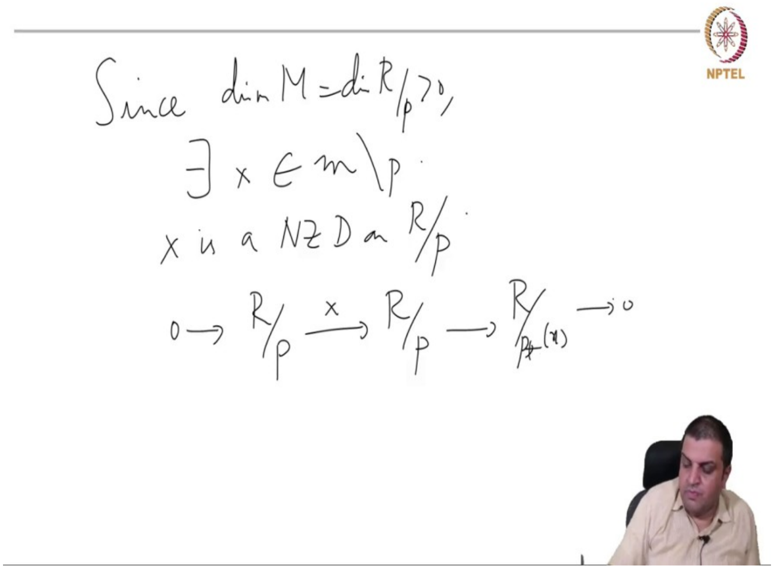

Since dimension of M which is dimension of $\frac{R}{p}$ is positive there exists some $x \in m$¿ such that there exists x is a nonzero divisor on $\frac{R}{p}$. Because if anything kills if x kills some $\overline{y}$ inside this ring then $xy \in p$, but then $x \notin p$. So, y must be inside p. So, therefore, $\bar{y} = 0$ it is a nonzero divisor on this. So, now we have $0 \to \frac{R}{p} \to \frac{R}{p} \to \frac{R}{p+(x)} \to 0$.

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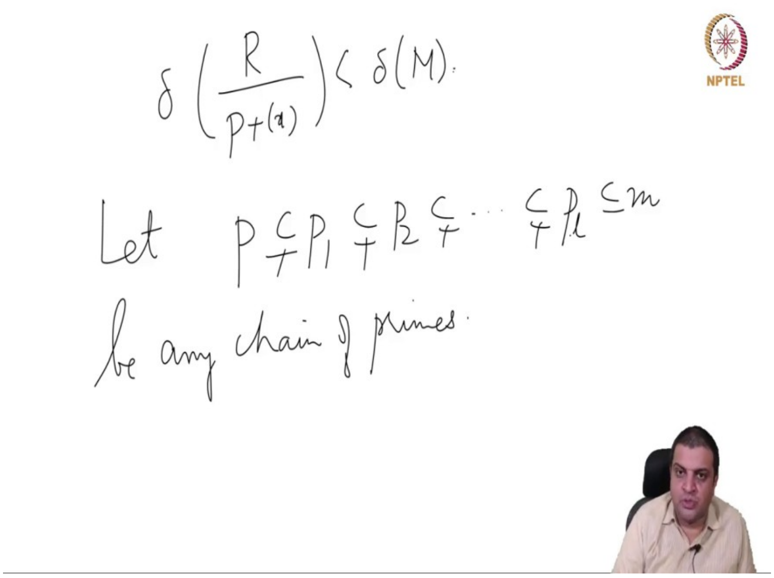

So, we saw that $\delta\left(\frac{R}{P+(x)}\right) < \delta(M)$. So we would like to argue that sorry, we have to be little bit careful about choosing this x.

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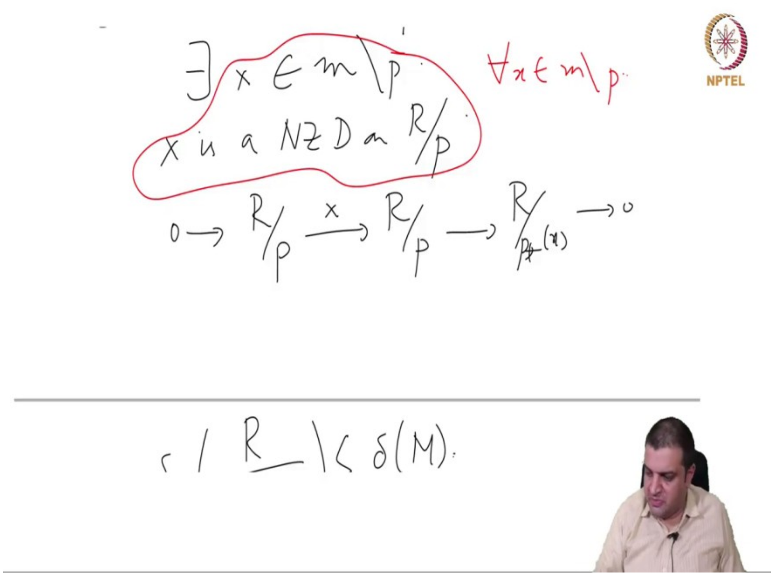

So, this is true for every p. So, let me just make an make an emphasis of this that actually this is true for not there exist, but for all $x \in m$¿. So, just an observation. So, I mean we have to use the induction hypothesis; this is how we will use it. Now, we will have to choose x carefully to get the desirable inequality in dimension.

So, let $P \subseteq P_1 \subseteq \ldots \subseteq P_l \subseteq m$ be any chain of primes. If we show that l is less than $\delta(M)$, then the supremum of all such l will also be less than or equal to $\delta(M)$.

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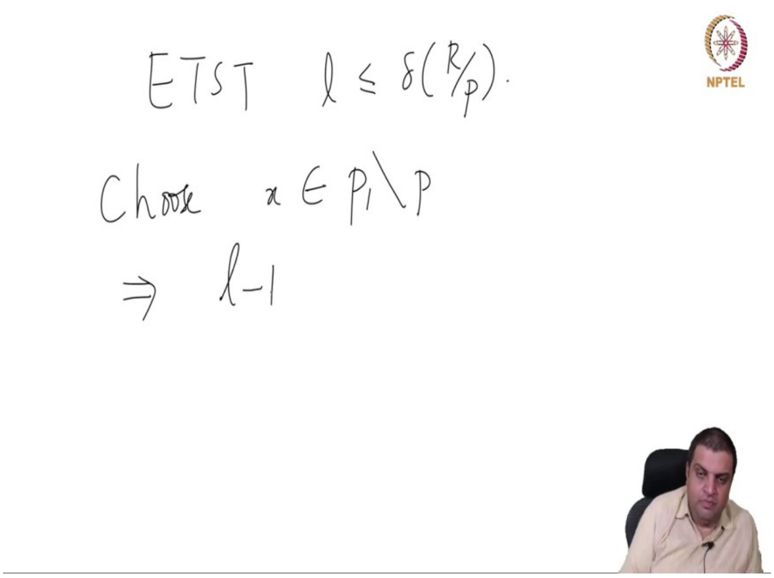

So, it is enough to show that $l \leq \delta\left(\frac{R}{P}\right)$ because the $dim\frac{R}{P}$ is a supremum of all such l if for any chain the length is less than or equal to $\delta\left(\frac{R}{P}\right)$, then it would imply that for the dimension also, its supremum also. So, now, choose $x \in P_1$¿. So, this is the choice of that we needed.

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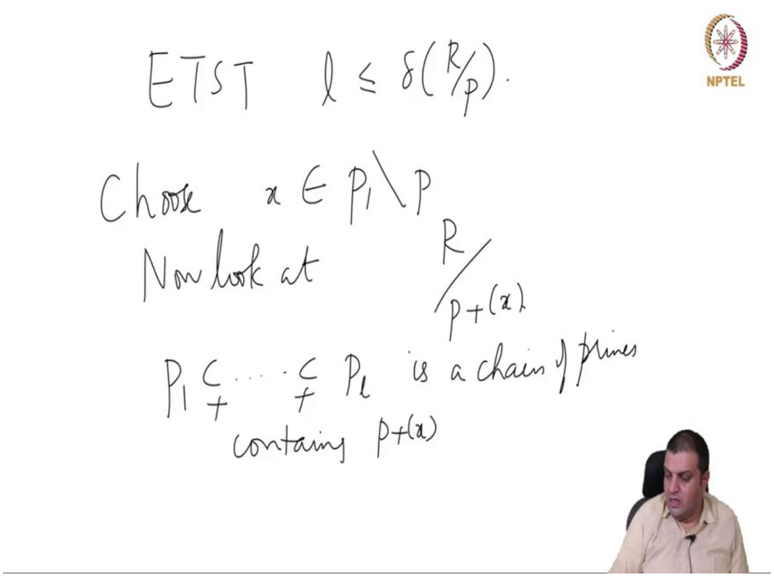

So, now look at $\frac{R}{P+(x)}$. $P_1 \subseteq \ldots \subseteq P_l$ is a chain containing $P+(x)$.

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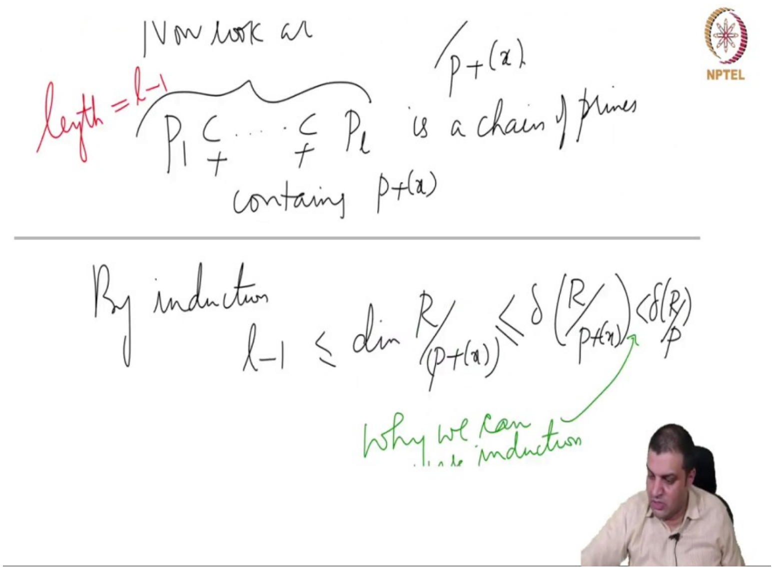

Therefore, by induction, $l-1$ which is the length of this is remember this is length is equal to $l-1$. It starts from P_1 to P_l. So, there is only $l-1$. So, $l-1 \leq dim \frac{R}{P+(x)} \leq \delta\left(\frac{R}{P+(x)}\right) < \delta\left(\frac{R}{P}\right)$. So, this is the why we can use induction.

So, we are working for a module with a smaller δ. So, for that dimension is less than or equal to δ and this $l-1$ is less than or equal to dimension. So, now, therefore, $l \leq \delta$, but then we have to worry about supremum of such l, once you add 1 on both sides throughout one would lose which is ok.

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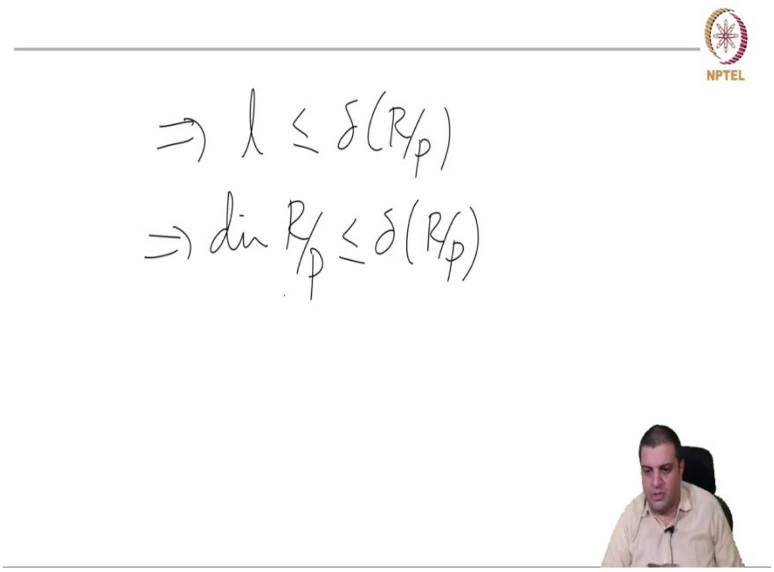

So, this implies that $l \leq \delta\left(\frac{R}{P}\right)$. So, this is true for every l therefore, supremum also is true this is true for every chain of primes. So, therefore, $dim\frac{R}{P} \leq \delta\left(\frac{R}{P}\right)$ and we are already observed that if we prove for primes like this then we have also proved for all modules.

So, this proves one of the inequalities. So, let us just briefly go back and see what was. So, what was the strategy? The strategy was to prove that dimension is less than δ; and δ is less than S which has already been established and now we need to establish this which we will do in the next lecture.