Computational Commutative Algebra
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Lecture – 41
Degree of Hilbert-Samuel Polynomials

Welcome, this is lecture 41. In this we look at this Hilbert Samuel Polynomial and we look at its degree and prove some elementary results about it, which we will need in the proof of the main theorem.

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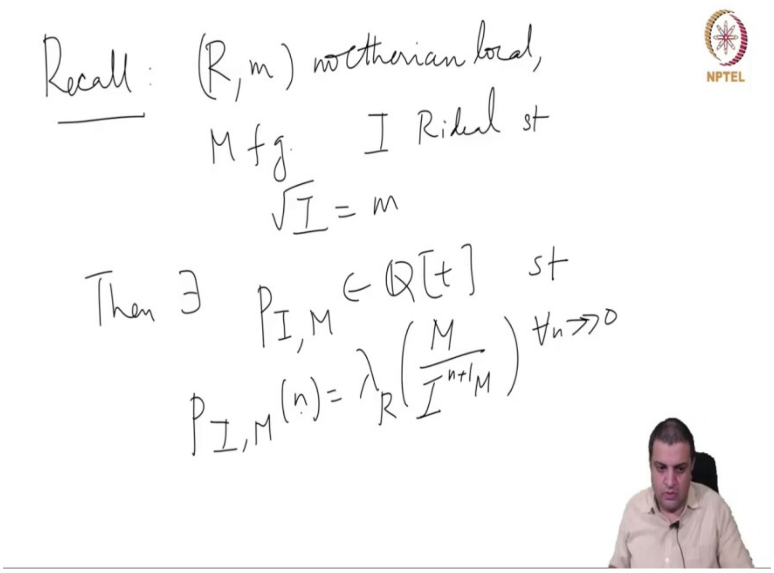

So, recall (R, m) noetherian local M finitely generated, I an m-primary ideal. Let me remind you m-primary is equivalent to saying that $\sqrt{I}=m$ because it is a maximal ideal. Then there exists a polynomial with rational coefficients such that, the polynomial gives the value of the length of this module for all n greater than sufficiently large and because of this property this module has finite length.

And the way we proved this was to look at the associated graded module for M over the associated graded ring of R. In fact, the polynomial ring which surjects onto the associated graded ring and there we proved that the individual graded pieces the lengths vary by a linear combination of those binomial looking polynomials. And this is a sum of those things and therefore, this will also have that property, but we do not need that information for now.

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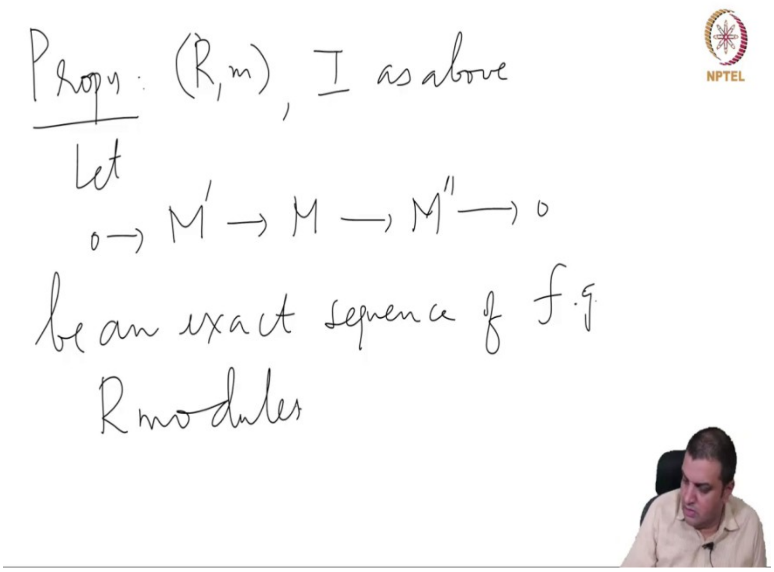

So, here is a proposition that we would need for later, (R,m), I as above. Let $0 \to M' \to M \to M'' \to 0$ be an exact sequence of finitely generated R modules.

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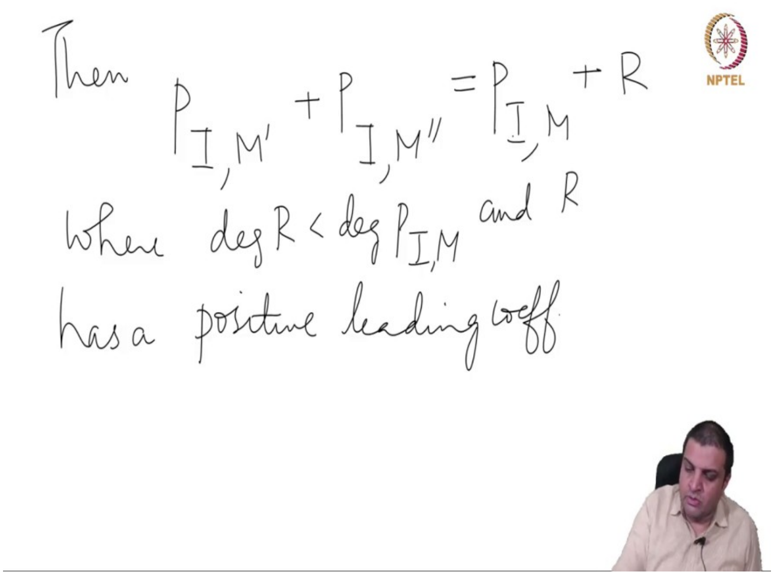

Then we have 3 polynomials, the Hilbert Samuel polynomial for each one of them. $P_{I,M'}+P_{I,M''}=P_{I,M}+R$. This is the length of I this module I mean the second polynomial gives the length of this module mod times the power of modulo appropriate power of the ideal one would.

So, that would be a potential candidate for the corresponding object for this, but they so this is but it is not exactly this. There is some extra term which is also going to be a polynomial, because these two are polynomials where $deg\,R < deg\,P_{I,M}$ and R has a positive leading coefficient. So, this is the proposition; so how it behaves in a short exact sequence.

So, now, proof is an application of the Artin-Rees lemma. So, this is not exactly equal there will be some remainder term, but the remainder term. So, what does this mean? Remainder term grows I mean is a polynomial of course, because the difference of a polynomial, it grows as n grows. That is what the positive leading coefficient means. And it grows much slower than the that than $\frac{M}{I^{n+1}M}$ because the degree is smaller, but ok.

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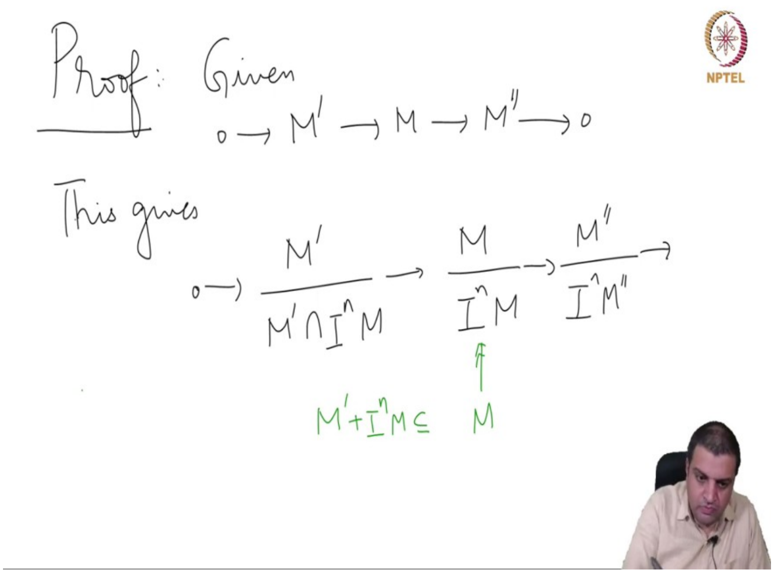

So, let us prove this. So, proof is just an I mean an application of Artin-Rees lemma after we sets something up. So, we are given $0 \to M^{'} \to M \to M^{''} \to 0$. So, this gives

$$0 \to \frac{M^{'}}{M^{'} \cap I^n M} \to \frac{M}{I^n M} \to \frac{M^{''}}{I^n M^{''}} \to 0.$$

So, let us just discuss this just briefly. So, we wanted to go modulo I^n times these modules respectively.

So, if you additionally kill that $I^n M$ here, it will also kill $I^n M^{''}$ because the image of $I^n M$ inside here is $I^n M^{''}$ and we get a surjective map. So, that explains the surjective map.

So, now, let us look at what happens what is the kernel of this. So, what if you think about M, if you think about M mapping to this, so let us think about M mapping to this and that mapping to this. So, in this composite what goes to 0. Well, first $M^{'}$ goes to 0 and then $I^n M$ goes to 0. So, the kernel of this map M to this is exactly $M^{'}+I^n M$.

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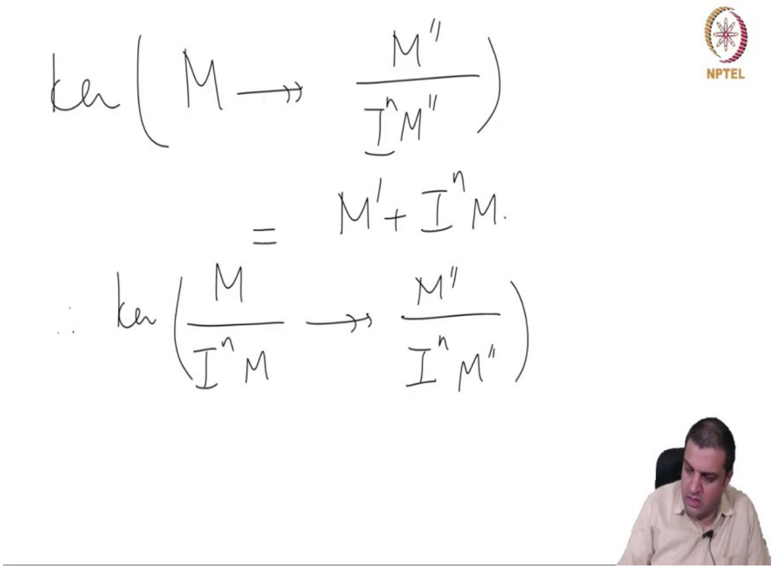

So, therefore, the kernel for this map would be. So, if you if you look at this map; what I just said is if you look at this map, kernel is $M^{'}+I^n M$ therefore. Therefore, kernel of $\frac{M}{I^n M}$ surjecting on to $\frac{M^{''}}{I^n M^{''}}$.

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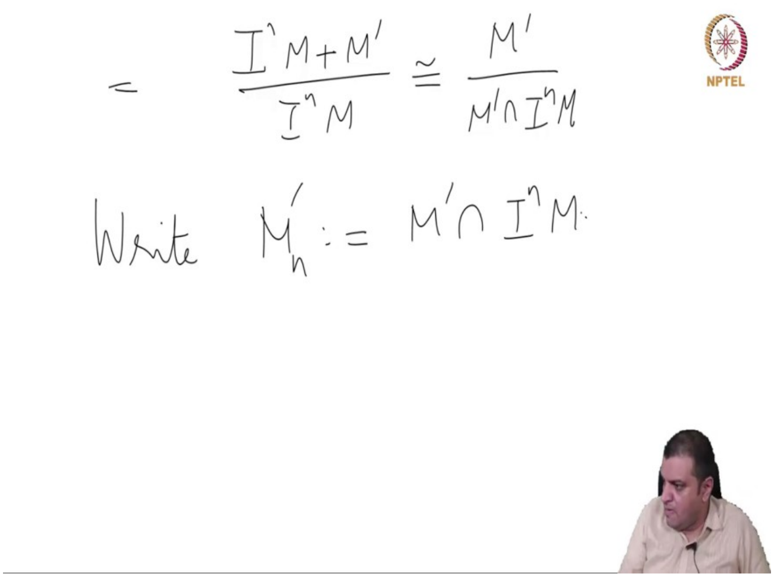

This is just $\frac{I^n M + M^{'}}{I^n M}$, which is same thing as $\frac{M^{'}}{M^{'} \cap I^n M}$. So, that is what we wrote here. So, that gives us this term so it is this short exact sequence that we wanted to consider.

So we are given a filtration of M by powers of I and here also by powers of I. Here we get a different filtration. So, this is the context in which we will need to apply Artin-Rees lemma. Write $M_n^{'} := M^{'} \cap I^n M$. So, what does this now tell us? It tells us that length of this, minus the length of this equals this length.

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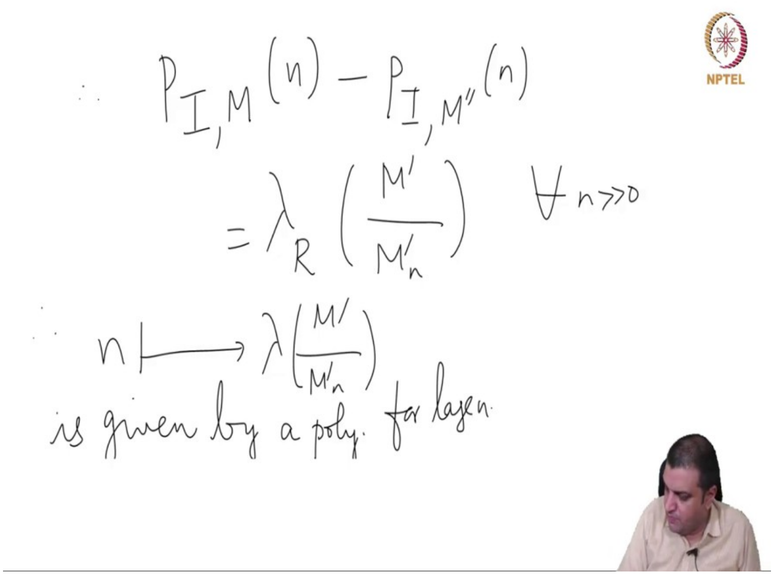

So, therefore, we get $P_{I,M}(n) - P_{I,M''}(n) = \lambda_R\left(\frac{M'}{M''_n}\right)$ for all sufficiently large n. So, remember these gives the length only for sufficiently large n. So, the however, this is the difference of the lengths not say difference of the polynomials.

So, the conclusion is that, this function. So, this function n mapping to this length. This function is given by a polynomial therefore, is given by a polynomial for large n. So, exactly I mean sort of like a Hilbert function itself; so, now, by Artin-Rees lemma.

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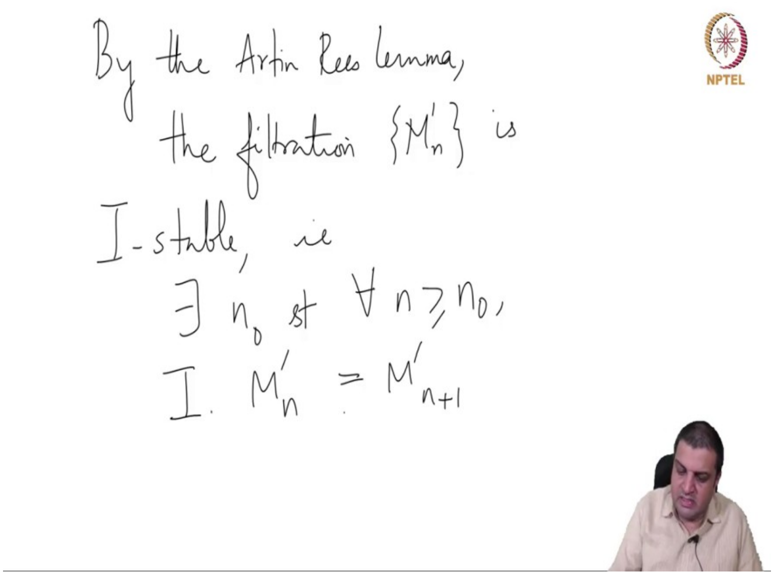

So, by the Artin-Rees lemma, what does Artin? So, Artin-Rees lemma says that if you look at the filtration this is I. So, Artin-Rees the filtration given by $\{M'_n\}$ of M is I stable that is there exists some n_0 such that, for all $n \geq n_0$, $I M'_n = M'_{n+1}$.

So, this in other words that module the module $M^{¿}$ that we constructed for this M', over the ring $R^{¿}$ it is finitely generated. So, this is equivalent to that statement.

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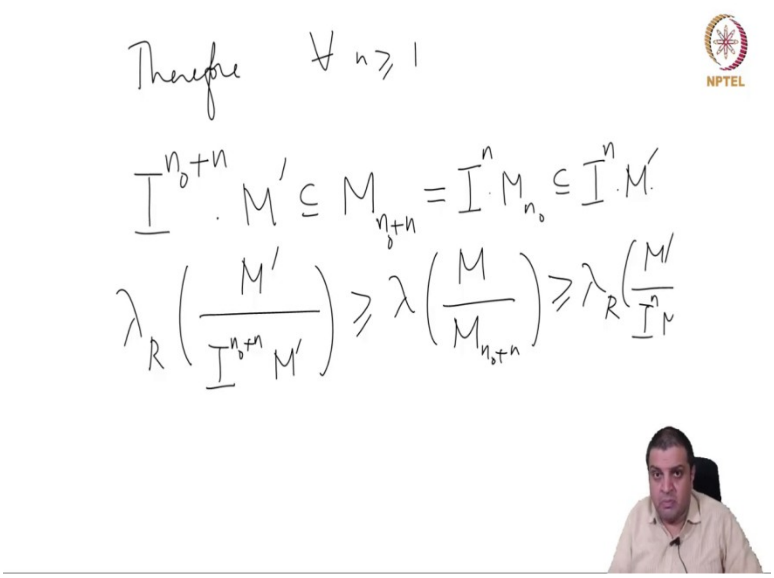

If we take I to some large power which is bigger than n_0, $I^{n+n_0} M' \subseteq M_{n_0+n}$. That is a property of I stable filtration that first of all. There is this containment for every n and for sufficiently large n, I times nth piece is equal to (n+1)th piece. So, that is Artin-Rees lemma. So, this is inside here, but repeatedly applying this.

So therefore, for all $n \geq 1$, if you take this and repeatedly applying. So, there is a slight change in notation here, please sorry please bear that in mind here it is all $n \geq n_0$ but here I am taking that as a difference from n_0. So, this is same thing as $I^n M_{n_0}$ that just repeatedly applying this result and this is inside $I^n M'$.

So we get this relation. Now let us take lengths, $\lambda_R\left(\frac{M'}{I^{n_0+n} M'}\right) \geq \lambda_R\left(\frac{M}{M_{n_0+n}}\right) \geq \lambda_R\left(\frac{M'}{I^n M'}\right)$. So, we are killing a smaller sub module. So, we know that these lengths for sufficiently large n are given by polynomials.

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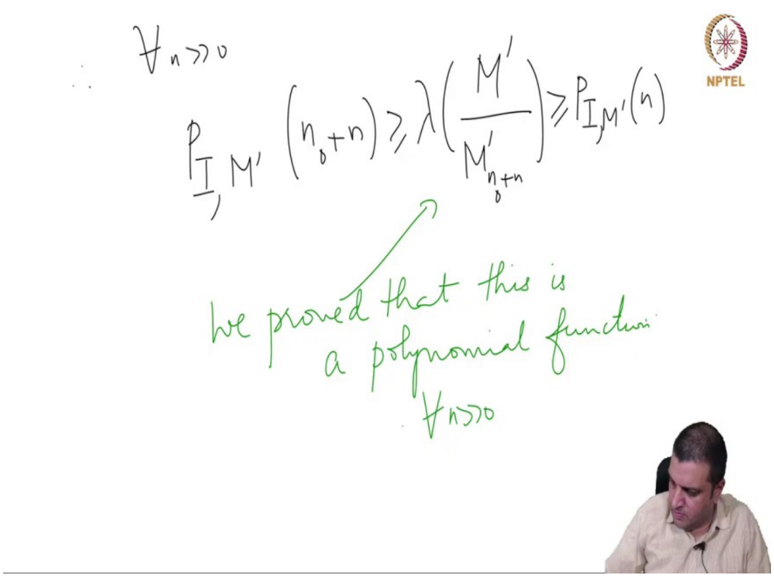

Therefore, for all n sufficiently large, $P_{I,M^{'}}(n_0+n) \geq \lambda\left(\frac{M^{'}}{M^{'}_{n_0+n}}\right) \geq P_{I,M^{'}}(n)$. Since we are arguing for all sufficiently large n between these two terms there is an offset of 1, but that is ok.

This is for all sufficiently large n. So, now we already proved that this is a polynomial function $n \rightarrow \lambda_R\left(\frac{M^{'}}{M^{'}_n}\right)$. This says that their polynomial function with the same degree and same reading coefficient. So, this is a polynomial function we prove that this is a polynomial.

That is because for large n that, but that is enough the asymptotic behavior will be determined by its reading coefficient and degree. So, they all have to be the same in these things. So, which now proves that yeah so now what can what do we know so this.

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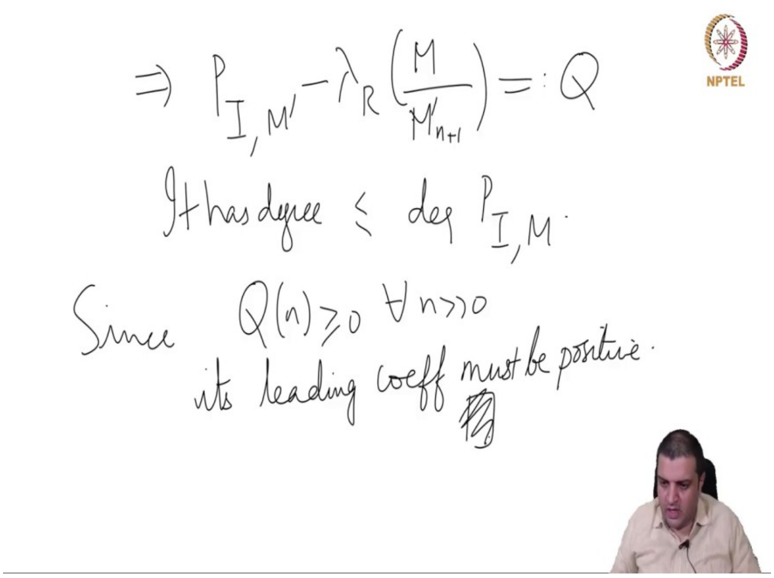

So $P_{I,M'} - \lambda_R\left(\frac{M'}{M'_n}\right) = Q$. So, define this. So, this is the R that we had, the difference of these two polynomials is the Q that we would define. This defined to be the function the polynomial R. So, remember, maybe I should just use some Q here.

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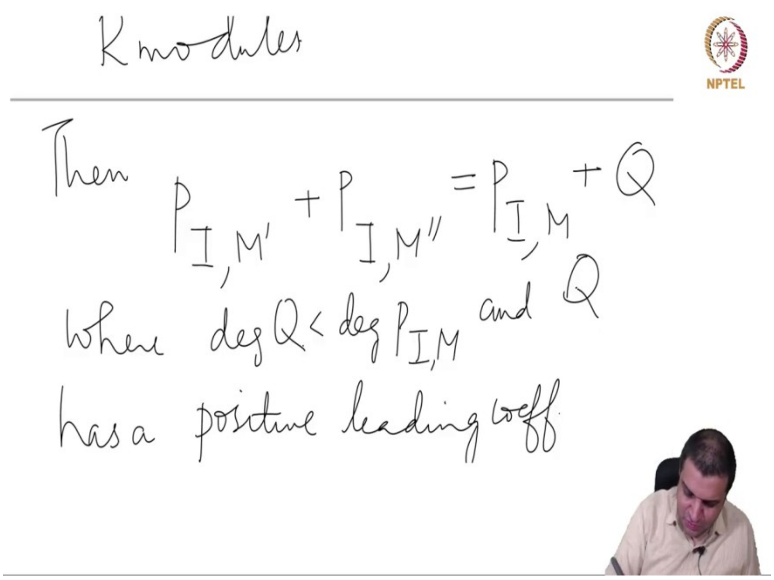

Let us not ever used afterwards. So, this is the polynomial Q that we asserted exists. So, this is a polynomial it has degree less than or equal to the degree of $P_{I,M}$ and moreover its leading coefficient has to be positive because as n goes to infinity this difference is positive.

Since $Q(n)$ is non-negative for all sufficiently large n, that is from this relation so it is the difference of these two that we called Q. It is leading coefficient must be positive. So, this is the end of this proof. So, we won't need the full detail of this proof to for us to use. But we do need the fact, we do need to use it in the following way corollary.

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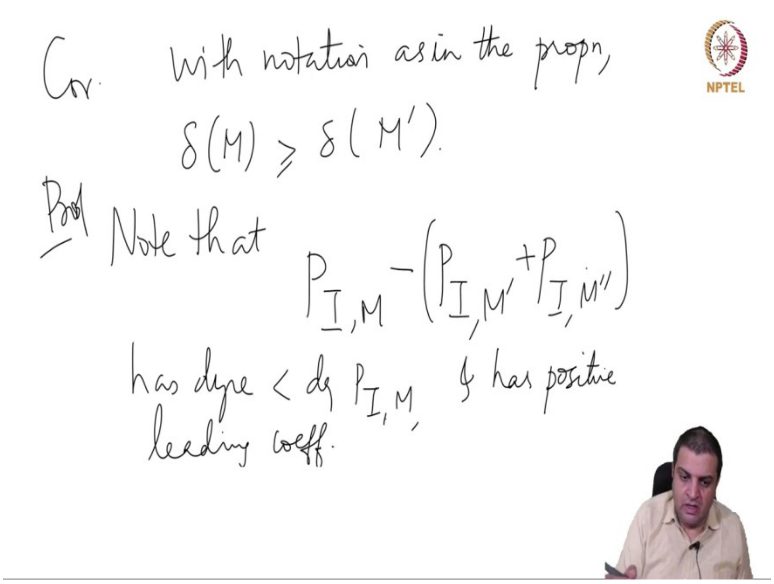

With notation as in the proposition $\delta(M) \geq \delta(M')$; and this is greater than or equal to $\delta(M'')$ it is sort of easier to see because of the surjectivity. I mean this length is always less than or equal to this length.

So, this degree of this polynomial for M'' will always be less than or equal to the degree of $P_{I,M}$. It is the other one that required some this argument, is yeah of this is; why is that the case? So note that proof, $P_{I,M} - (P_{I,M'} + P_{I,M''})$ has degree less than degree of $P_{I,M}$ and has positive leading coefficient.

In other words, if this sum here let us write like this if this sum here had a degree which is bigger than degree of this, then that thing the degree of the difference would be the degree of this.

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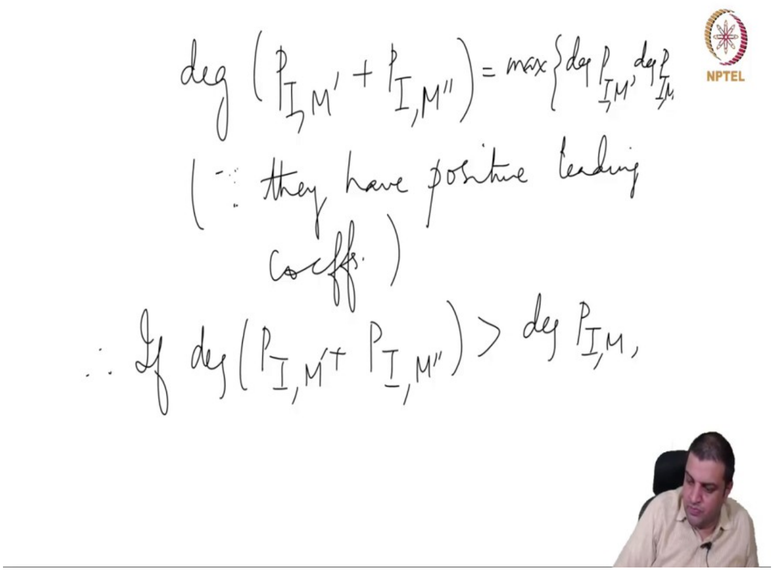

So, let us just write this if $deg\left(P_{I,M'}+P_{I,M''}\right)=max\{deg\,P_{I,M'}, deg\,P_{I,M}\}$. Let us just quickly observe this part degree of the sum is equal to the maximum of their individual degrees that is because they both have positive leading coefficients. The leading terms can never cancel each other when you take the sum.

So, using this yeah so this side and the difference has smaller degree. So, therefore, the leading term of this and leading term of that if the if this thing had strictly had positive degree strictly bigger than the degree of the of P I M then the difference will have degree equal to the degree of this ok. So, therefore, if $deg\left(P_{I,M'}+P_{I,M''}\right)>deg\,P_{I,M}$.

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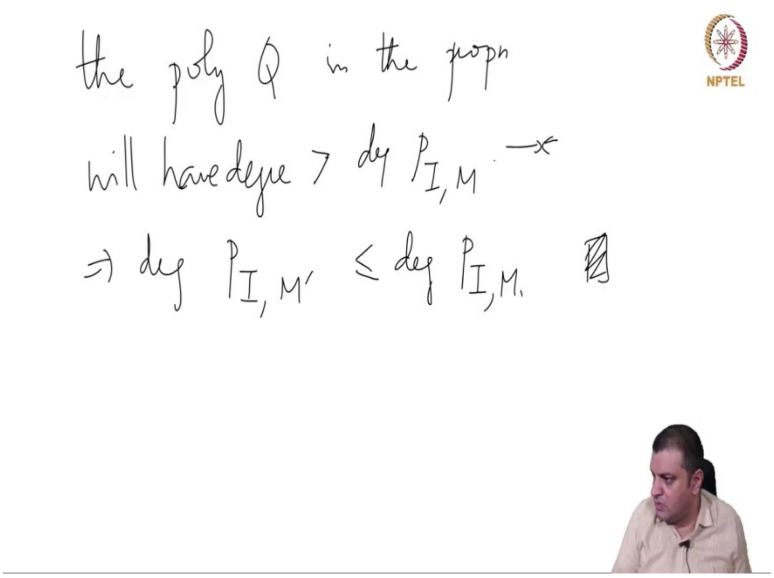

The polynomial Q, in the proposition will have degree greater than degree of $P_{I,M}$, which is a contradiction. So, in other words $deg\,P_{I,M'} \leq deg\,P_{I,M}$, because the other one is anyway smaller the max of this is less than or equal to this.

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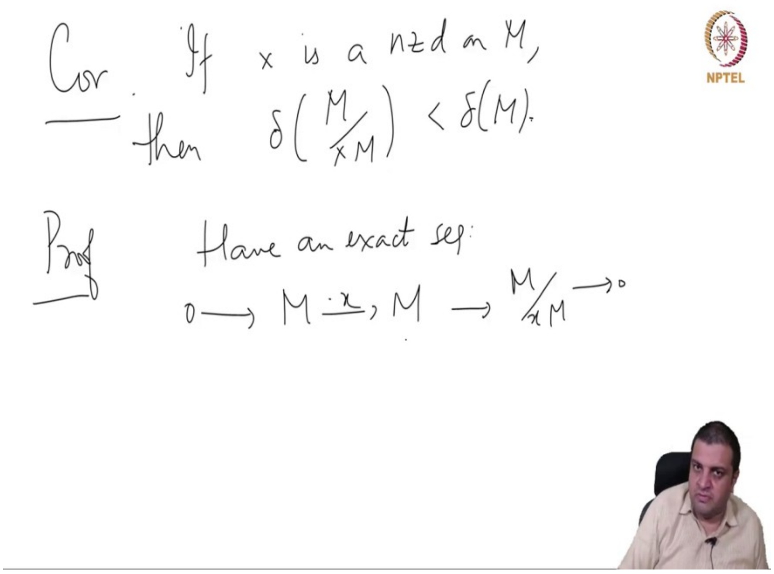

Another corollary is if x is a nonzero divisor same notation as in the theorem, nonzero divisor on M then $\delta\left(\frac{M}{xM}\right) < \delta(M)$.

And why is this? This is, because so consider proof we have an exact sequence $0 \rightarrow M \rightarrow M \rightarrow \frac{M}{xM} \rightarrow 0$ and this is injective here because x is a nonzero divisor. So, then we apply the proposition to this it says; the sum of these two is this plus some difference which is positive leading coefficient and smaller degree.

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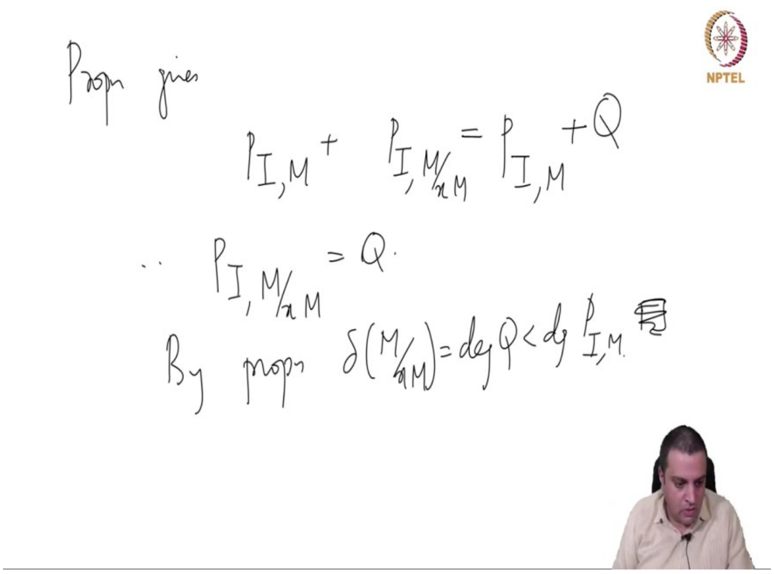

So, proposition gives. So, the outer two which is $P_{I,M} + P_{I,\frac{M}{xM}} = P_{I,M} + Q$. Therefore, $P_{I,\frac{M}{xM}} = Q$ and by proposition $\delta\left(\frac{M}{xM}\right) = deg\, Q < deg\, P_{I,M}$ which is what we wanted to prove.

So, if you have a nonzero divisor and you go modulo that then delta strictly decreases. So, we need one more observation about the relation one relation among the number of the length of a system of parameters and delta. So, which sorry slightly I mean so we need to see the proof.

So, we will do that first thing in the next lecture. And then we will prove we will look at a little bit more about systems of parameters in Macaulay 2 and then we will prove the theorem.