# Computational Commutative Algebra

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## Lecture – 40
## Artin Rees Lemma

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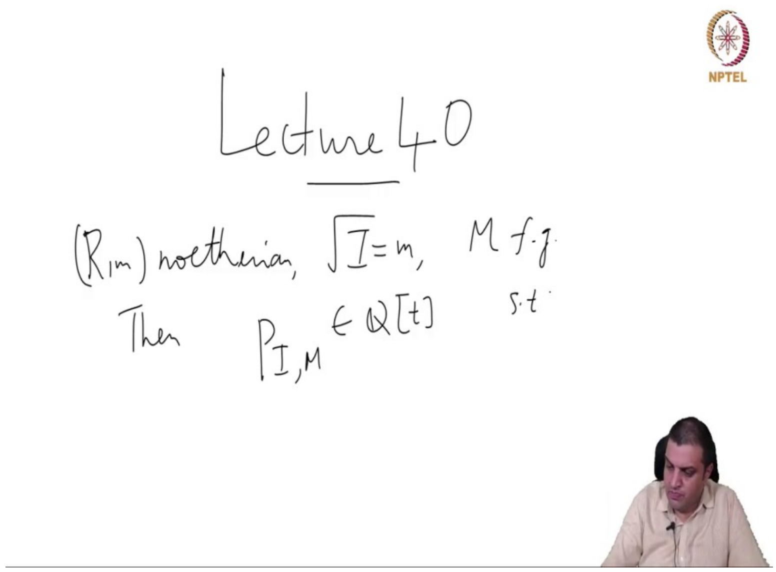


This is lecture 40. And so, let us just quickly recall what we have done. We have said that if R m Noetherian I an m-primary ideal and M finitely generated, then there is a Hilbert Samuel polynomial $P_{I,M}$ with rational coefficients.

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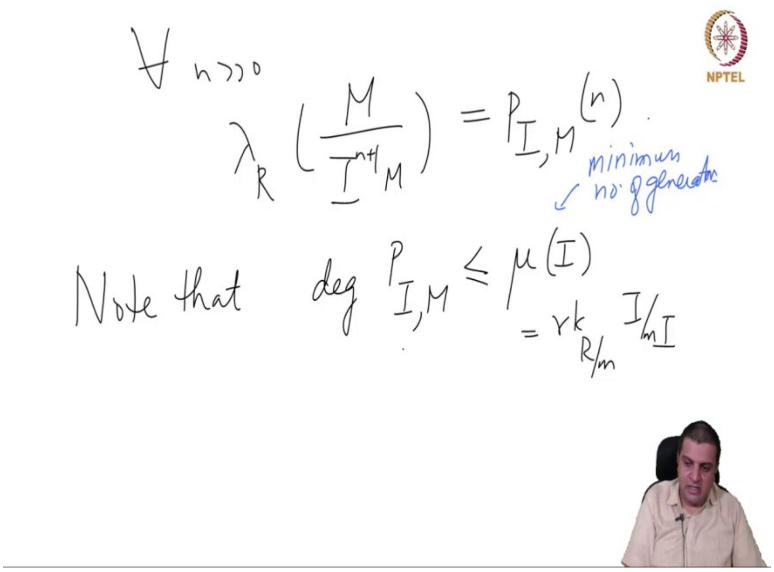


Such that for all sufficiently large n, $\lambda_R\left(\frac{M}{I^{n+1}M}\right)=P_{I,M}(t)$. This is called the Hilbert Samuel polynomial.

We now show that its degree does not depend on I. So, in other words the degree of a polynomial is an invariant of the module. So, before we do that note that degree of $deg(P_{I,M})\leq\mu(I)=rk_{\frac{R}{m}}\frac{I}{mI}$ So, this is a notation for I mean often used notation for minimum number of generators, which is well defined because by Nakayamas lemma.

But in any case it is some so, if you take any generating set, we can get a polynomial ring over $\frac{R}{I}$ mapping to this and. If you recall so, this $P_{I,M}$ was an addition of lengths which were the Hilbert polynomial for the associated graded ring. Associated graded ring was a finitely generated module over these many variables.

Therefore, the Hilbert polynomial of the associated graded ring had a degree at most one less than this number. And if you add them in successive degrees and successive degrees, then we will get a polynomial whose degree is bounded by above by $\mu_I$. So, this is just an observation.

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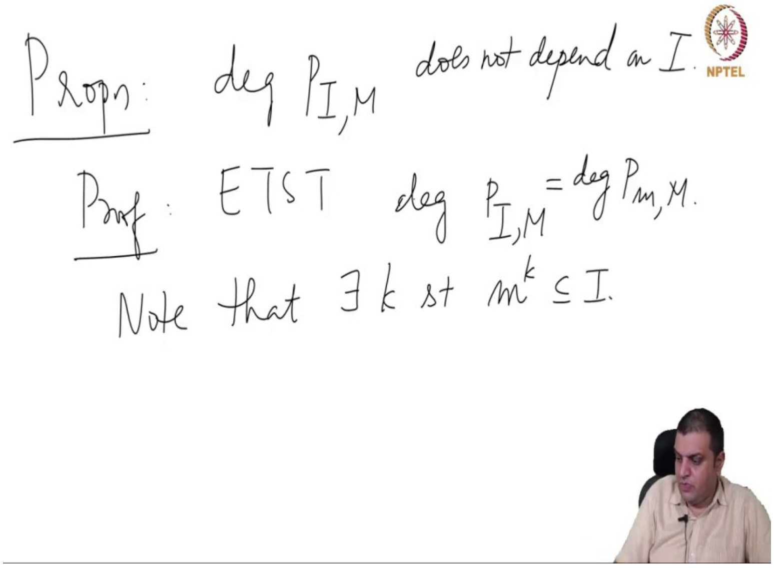


But what we need really is the following statement, which is that the $deg(P_{I,M})$ does not depend on I. In other words, if I and J are M primary ideals, then $P_{I,M}$ and $P_{J,M}$ will have a same degree not necessarily the leading same leading coefficient that is a different problem.

So, now, let us prove this ok. So, if you want to prove this well then it will also, we can fix some ideal here then prove that for every I it is the degree is equal to degree for the polynomial for that ideal ok. So, proof , enough to show that $deg(P_{I,M}) = deg(P_{m,M})$. If both are polynomials says they well defined, but so, note that there exist a k such that $m^k \subset I$ .

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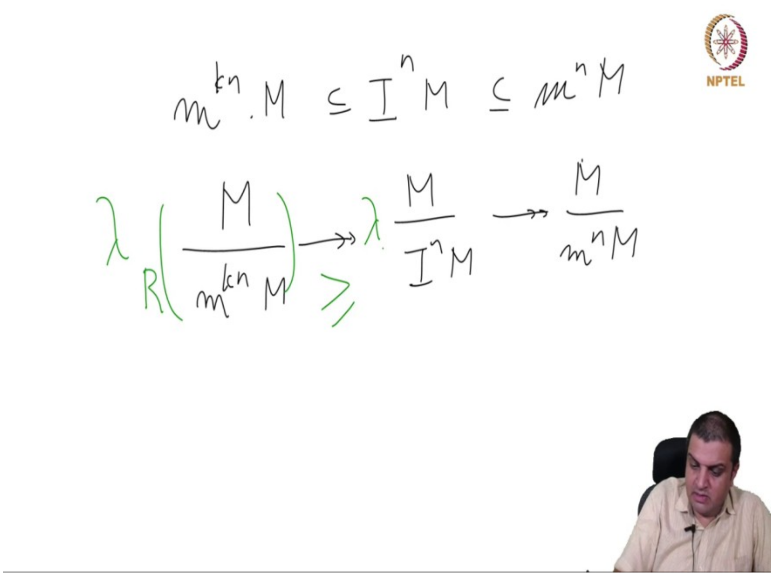


So, then we have containments $m^{kn}M \subset I^n M \subset m^n M$ . there will be a corresponding quotients $\frac{M}{m^{kn}} \to \frac{M}{I^n M} \to \frac{M}{m^n M}$ .

So, we can put lengths around it length over R, then $\lambda_R(\frac{M}{m^{kn}}) \geq \lambda_R(\frac{M}{I^n M}) \geq \lambda_R(\frac{M}{m^n M})$

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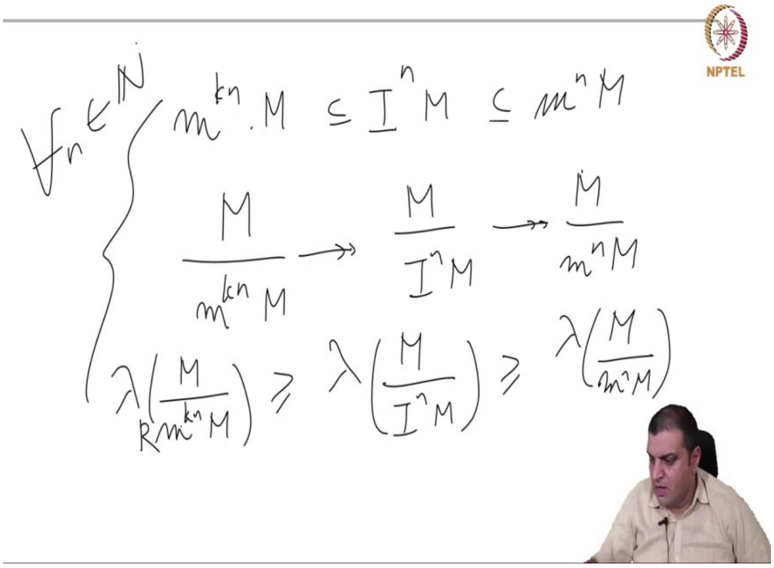

all of this statement is true for every n, but if you restrict ourselves just to sufficiently large n.

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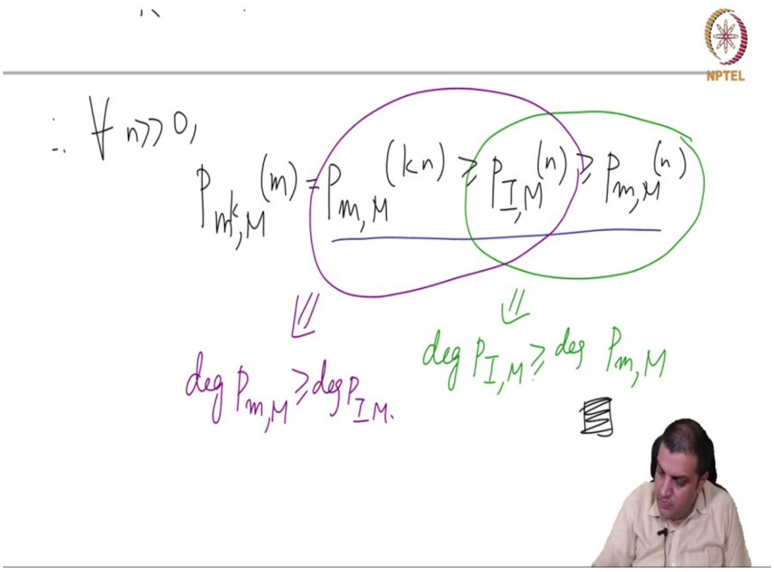

Then, therefore for all n>>0, this can be thought of in two different ways, this could be thought of as $P_{m',M}(n)=P_{m,M}(kn)\geq P_{I,M}(n)\geq P_{m,M}(n)$ all that matters is what is what the exponent there.

So let us look at these two inequalities. So, these are polynomials and their values are non negative and eventually. So, they should first of all have the same degree because if let us say this. This is the same polynomial evaluated at two different places.

If the $deg(P_{m,M})\geq deg(P_{I,M})$ , then eventually one would not get this inequality, . So, this one says that $deg(P_{m,M})\leq deg(P_{I,M})$ ; because these are polynomials which take non-negative values. So, the leading coefficients have to be non negative.

And if the leading coefficients are non negative and this ones degree is smaller than this that ones degree, 1 would not get this inequality for all sufficiently large n. So, it proves this one. Suppose this is a strict inequality. then see this k is fixed.

So, as you go further as n becomes larger and larger eventually the so, we are discussing if this is a high this is a strict inequality. Eventually the higher degree term of $deg(P_{I,M})$ , will take over any multiple of k. So, this one now implies that $deg(P_{m,M}) \geq deg(P_{I,M})$. Just think about polynomial of degree 5 and a polynomial of degree 6 both with positive leading coefficients.

And then irrespective of what constant $k_1$ puts. Eventually the polynomial of higher degree will take larger values than the polynomial of smaller degree. So, we using the same idea twice to conclude that this is equal. so, this is the proof of this proposition.

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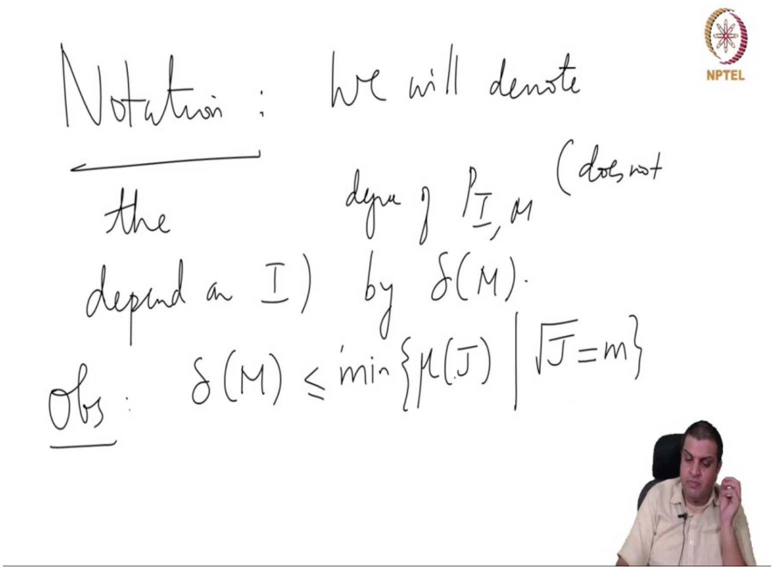


So we need this invariant so, we will notation . We need so, we will call this invariant, we will denote the common ddegree of $P_{I,M}$ ( does not depend on I) by $\delta(M)$ .

So , what we can observe at this point is that $\delta(M)$ is bounded so, observation $\delta(M) \leq min\{\mu(J) : \sqrt{J} = m\}$ . So, this is a bounded. So, these are non negative integers. So, this set I will have a minimum and choose a J with that minimum and then compute $P_{J,M}$ and then the number of variables that you would need here in the associated graded ring would be exactly $\mu(J)$ and therefore, this ones degree would be at most $\mu(J)$ .

So, this is what the observation that we need this is bounded. In fact, when we are going to study, this invariant we try to prove something stronger about $\delta(M)$ than this in than this property, but at least this is a useful observation now.

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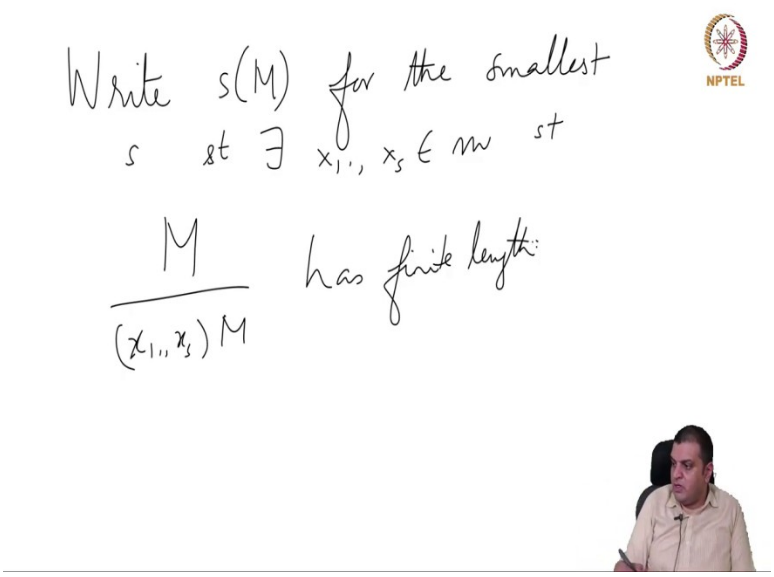


So, now, we want to define another invariant write s (M) for the smallest s such that there exist $x_{1,\dots},x_s\in m$ such that $\frac{M}{(x_{1,\dots},x_s)M}$ has finite length.

So, here we are not saying that we want to take an m-primary ideal, just take some elements with this property. Clearly if you take a elements that form a generating set for an m=primary ideal, this would be true. But, it is not necessary just any set with this property is enough.

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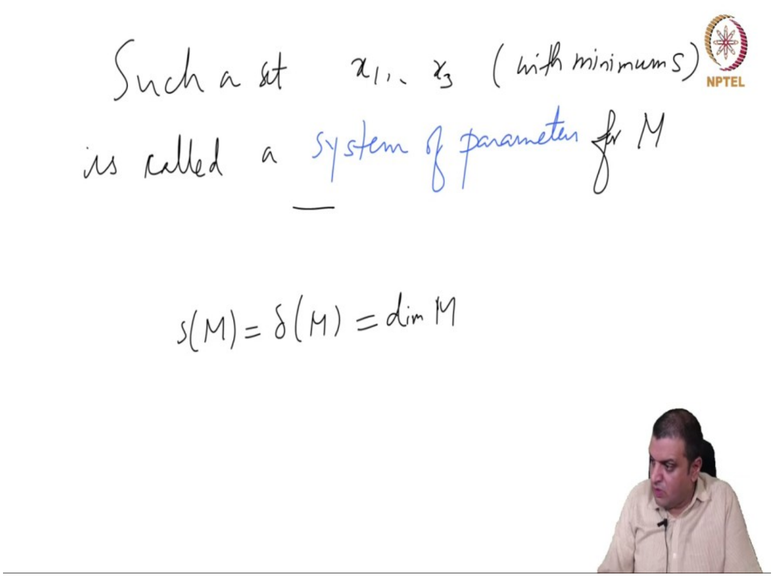

And such a set $x_{1,\ldots}, x_s$ ( with minimum s) is called a system of parameters for M.. And after we prove these theorems, we will come back and try to understand these things with examples what is going on etcetera. So, what we will we want to prove is the following, I will state the theorem again we just prelude.

So, what we want to prove is that the two invariants $s(M)=\delta(M)=dim(M)$ the Krull dimension of M, which I will define in a minute. So, first we want to understand $\delta(M)$ little better and not just $\delta(M)$ in a short exact sequence of modules how does $\delta$ behave so, that is we would like to understand that.

And that is typically proved by using a result called Artin-Rees lemma which has independent applications. So, we will first prove Artin-Rees lemma, then we will prove find a relation between delta or the Hilbert Samuel polynomials in a short exact sequence of modules.

We need that information to prove this theorem, we will define this later when we come to that thing. So, this is this is where we are going . So, now, now we discuss what is called Artin Rees lemma

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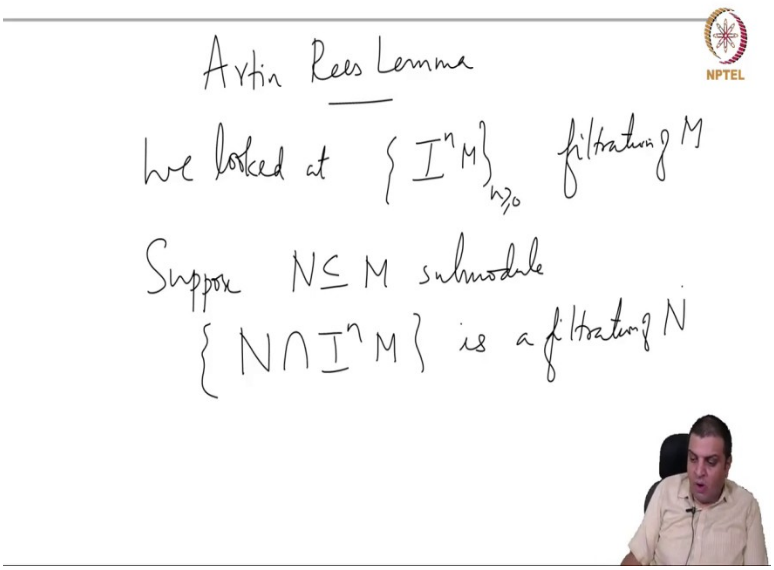


So, one of the things that we need to understand in this situation is the following. So, why do we need to work in slightly more generality than what we have done so far. We looked at a filtration given by $\{I^n M\}$ this is the filtration of M that is its a decreasing filtration and then, we took successive quotients. Then, we artificially constructed a new module called associated graded module.

Same thing for the ideal, we took powers of the ideal that is gives a filtration of R, and then we took successive quotients. And then we sort of patched them together put them together to give a new ring called associated graded ring. So, but suppose N is a submodule of M , if you take \{N intersection I^n M\} is a filtration of N.

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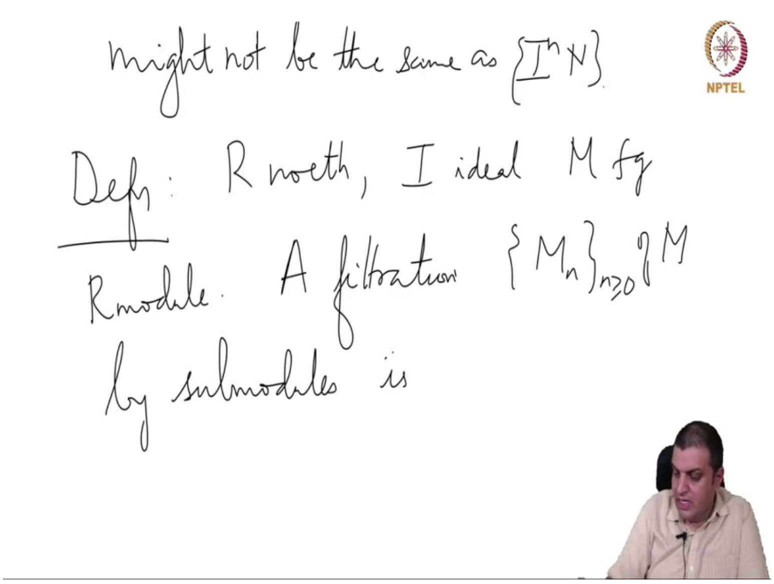


But this is not this may not be the same as and in general not, as the filtration given by $\{I^n M\}$ . w So, we need a way to take care of these filtrations also for that we make a definition . So, I will just define for Noetherian it is probably do it more generality, but any result we will prove it only for Noetherian rings.

So, R is Noetherian; I is an ideal, and M is a finitely generated R-module. Then a filtration $\{M_n\}_{n\geq 0}$ filtration of M by submodules is called an I-stable filtration.

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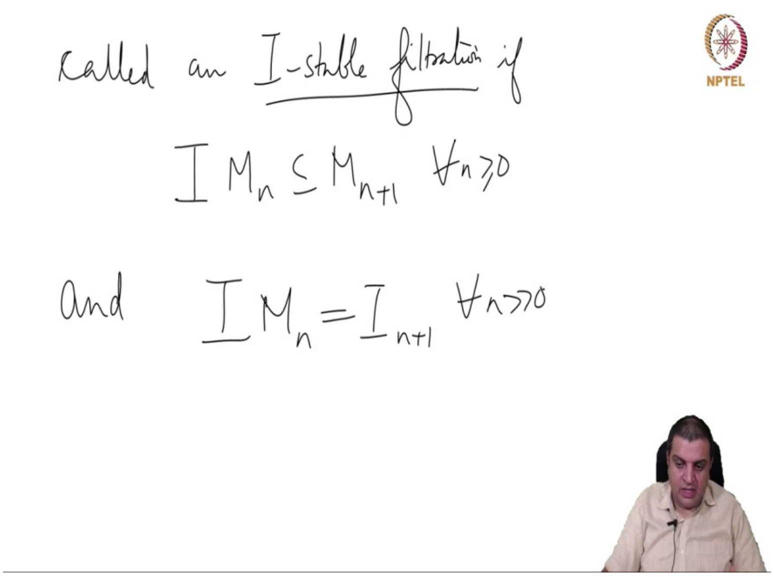


If first of all it is stable under multiplication by I, $IM_n \subset M_{n+1}$ for all n and $IM_n = M_{n+1}$ for all n>>0. So, this is what is called an I stable filtration. So, for example, the filtration given by powers of I is an I-stable filtration, although there are other things that we will we will worry about it . So, now, how will we identify an I-stable filtration? Ok. So, here iis proposition.

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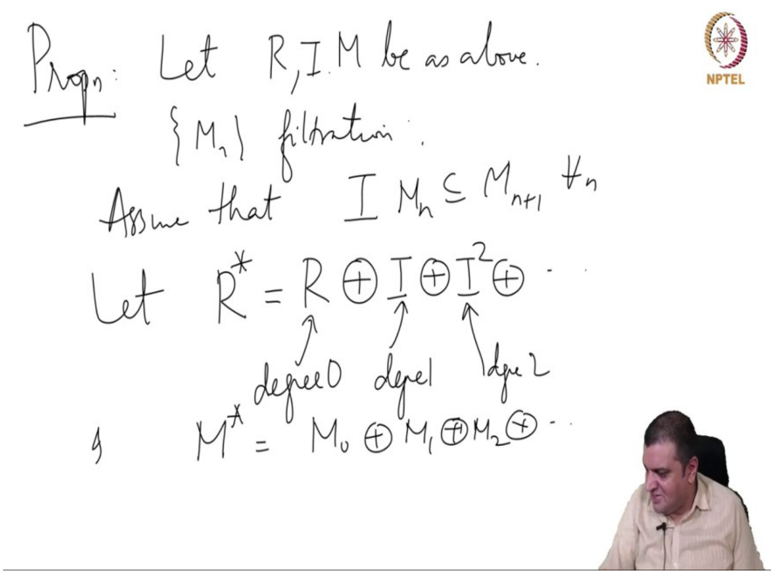

Let R, I and M be as above Noetherian finitely generated and $\{M_n\}_{n\geq 0}$ filtration . Then we ask so assume that, $I M_n \subset M_{n+1}$ for all n. And we ask when is this going to be an I-stable filtration, in other words when will the second condition also hold. So, we so, this will be expressed in terms of a new ring and a new module. Let $R^{star} = R \oplus I \oplus I^2 \oplus \ldots$ be the ring graded ring I is degree 1 R is degree 0, $I^2$ is degree 2 and so on and $M^{star} = M \oplus I M \oplus I^2 M \oplus \ldots$ same way this is in this degrees. So, you should check that $R^{star}$ is a graded R-algebra with this structure. And $M^{star}$ is a graded $R^{star}$ - module

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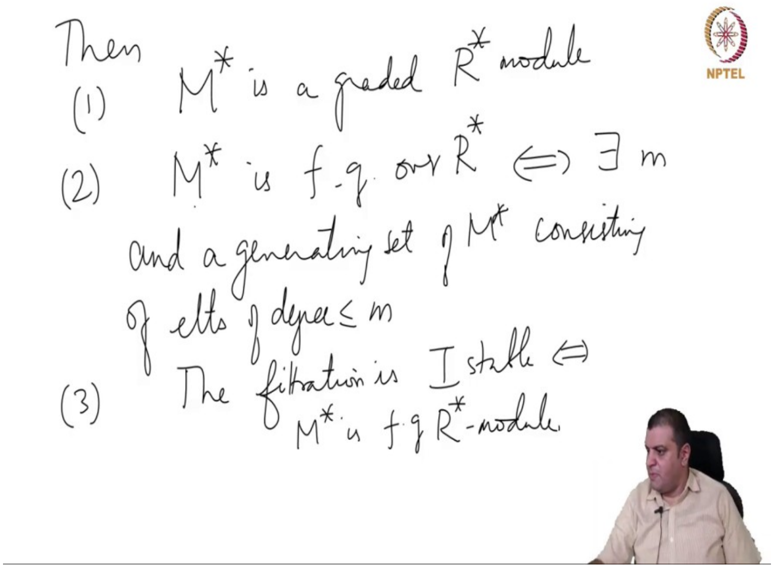


So, that I will leave as an so, let us so then let us write

1) $M^{star}$ is a graded $R^{star}$ - module. This is an $R^{star}$ is a graded R algebra. So, this you should check;

2) And $M^{star}$ is finitely generated $R^{star}$ - module if and only, if there exists some m and a generating set of $M^{star}$ consisting of elements of degree less than or equal to m .

3) the filtration is I-stable if and only if $M^{star}$ is finitely generated $R^{star}$ - module . So, the first please prove it $R^{star}$ is a Noetherian ring it is finitely generated R-algebra. So, it is Noetherian. So, now, let us try to understand what this means.

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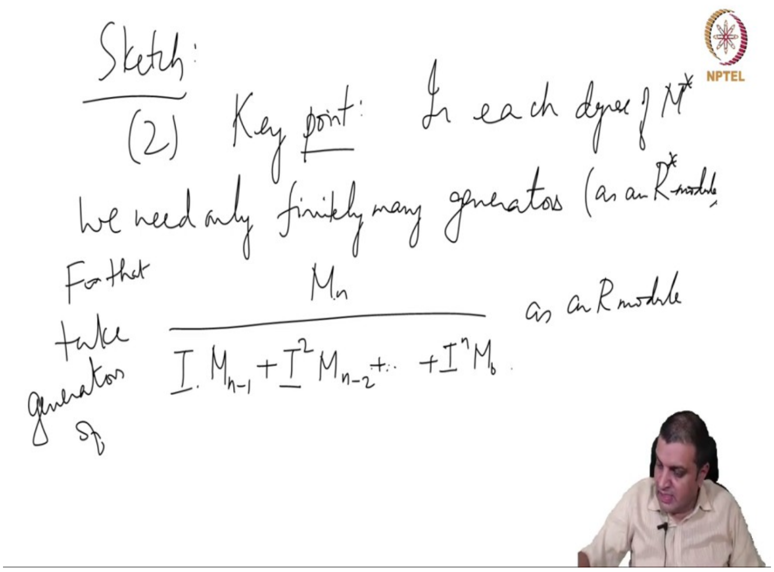


So, this is really a sketch this is of 2. What is a key point? . In each degree of $M^{star}$ ( $[M^{star}]_n = M_n$ ) we need only finitely many generators as an $R^{star}$ module . And what are those finitely many. So, what is in degree n? It is just $M_n$ .

Now, in $M_n$ how do we, why would we need generators in degree n? Well, generators in previous degrees in lower degrees are not enough to generate this module. So, how will we detect that? Well, we take this $\frac{M_n}{I M_{n-1} + I^2 M_{n-2} + \ldots + I^n M}$ and So, these things elements in these submodules have been taken care of by generators of previous degree, elements so, are the elements of this in fact, $I M_{n-1}$ is inside here so, we just need to worry about 1. But I am just writing this out to get some understanding what we are doing at this point

So, let us go over this thing again; when would we need a generator in degree n? So, in degree n is $M_n$ . In $M_n$ we would need a generator if some element of $M_n$ could not have been written by something of previous something in lower degrees times positive degree elements in the ring ok. So, how do we check that?.

So, this is in the immediate previous degree times degree 1 of the ring which is I, then 2 degrees in back in M and 2 degrees and degree 2 of R and so on. So, all of these are submodules of $M_n$ in fact, you only need to take 1, but let us this is for clarity. So, anything that is inside this submodule here has been taken care of by generators in previous degree.

So, you just need a generators for this. And for that we just take this module and take we need only finitely many generators as an R-module; so, for that take generators

$$\frac{M_n}{I M_{n-1} + I^2 M_{n-2} + \ldots + I^n M}$$ as an R module.

So if if M is finitely generated, then eventually larger degrees you do not need any generators, otherwise all the generators will leave in finitely many degrees. And in each degree there will be only finitely many generators.

So, let us go over this argument again all the generators will leave finitely many degrees, but in each degree you only need finitely many generator; so, altogether there are only finitely many generators. So, this is the key point in proving 2 and using 2 one can prove 3 immediately which is that which is just rewriting what we said here. So, 2 is the key point for proving 3.

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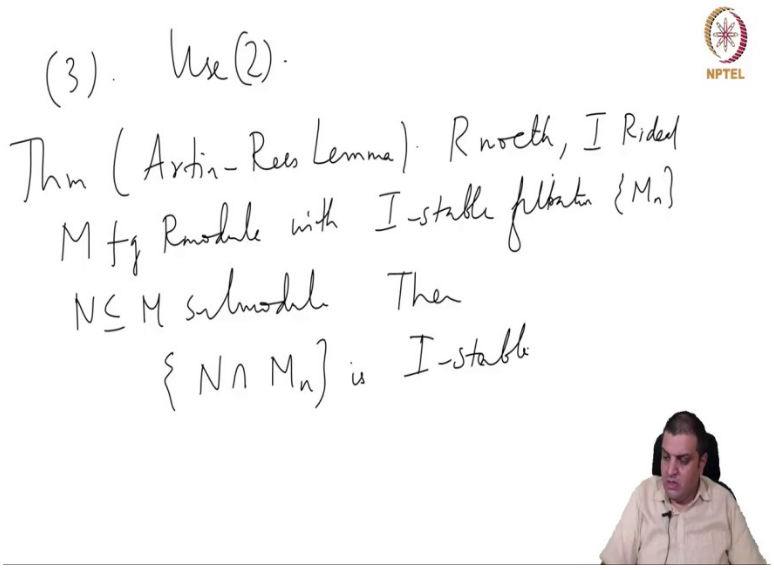


So, 3) use 2 ok. So, this gives us a characterization of I-stable filtrations. So, now, we want to prove what is called Artin Rees lemma. So, in the next lecture we will use Artin Rees lemma and compare the degrees of the polynomials in a short exact sequence. Again R

Noetherian, I R-ideal, M finitely generated R-module with filtration with I stable filtration \ $\{M_n\}$ , $N \subset M$ submodule, then the filtration given by $\{N \cap M_n\}$ is I-stable.So, given any I stable filtration we need to restrict it to intersected with N we would get a filtration and it is I stable.

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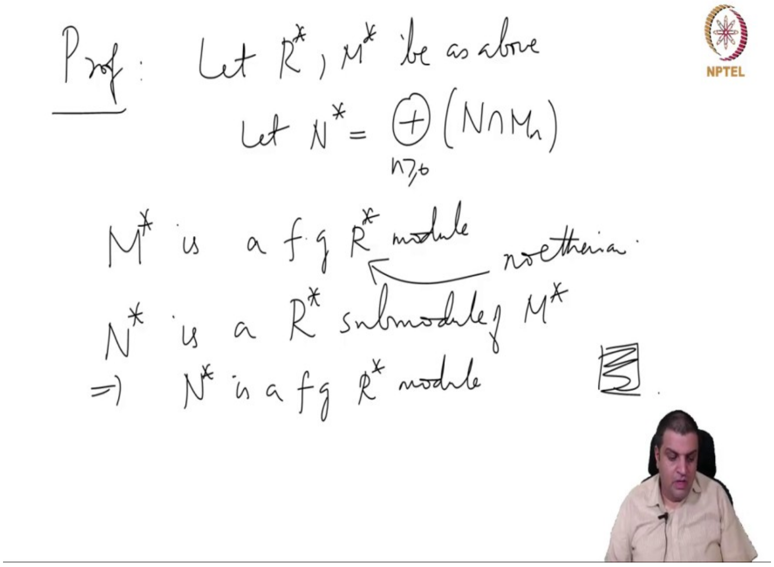


So, let $R^{star}, M^{star}$ be so, be as above let $N^{star} = \underset{n \geq 0}{\oplus}(N \cap M_n)$. So, N does not have any given filtration it is induced filtration from N. then $M^{star}$ is finitely generated $R^{star}$ module, because it is I-stable filtration this is a Noetherian.

$N^{star}$ is a $R^{star}$ -submodule of M, again not very difficult to check. Which means that $N^{star}$ is a finitely generated R* -module which is what we needed to check.

So, we will stop here and in the next lecture, we will prove that the degree of the polynomial is how it varies in a short exact sequence; short exact sequence of modules. And then we will prove the theorem that the three numbers are the same, what we are going to define dimension the Krull dimension of a module the invariant $\delta$ and the invariant s( M).

So, these three are the same we will prove it in a lecture 2. And, after we prove this we will prove what is called Krull principle ideal theorem, and a slight, I mean extension of that to not principle ideals. And, that would be the result with which we will go back to the polynomial case and prove that the polynomial ring over a field has dimension equal to the number of variables. So, that is where we are going.