Computational Commutative Algebra
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Lecture – 04
Noetherian Rings

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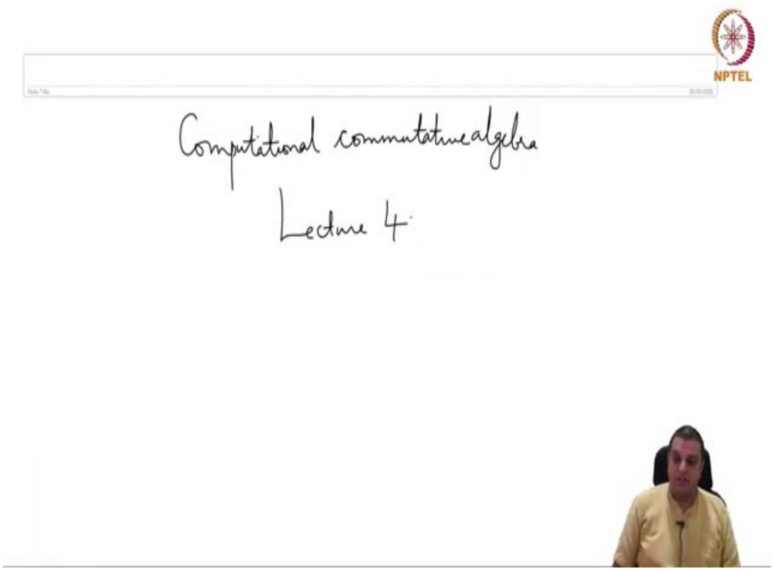

Welcome to the 4th lecture on Computational Commutative Algebra.

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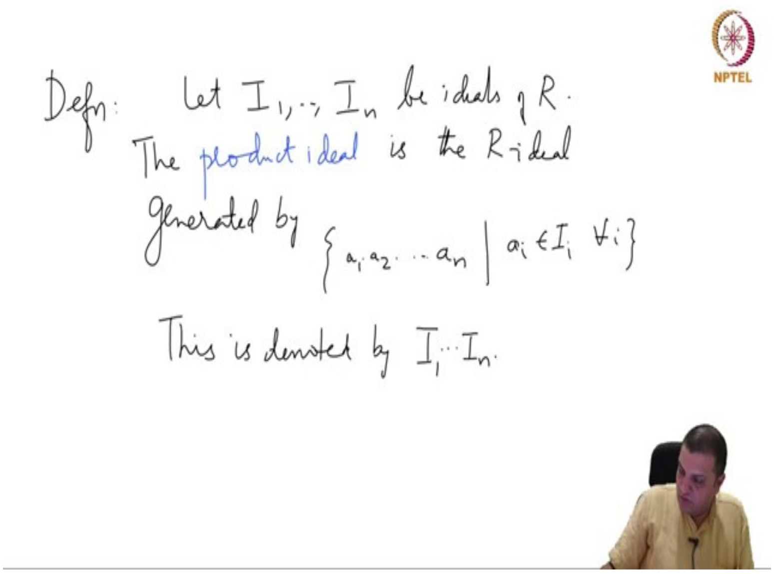

First we lo at products of ideals. So, we are continuing our discussion of operations on ideals. Let $I_1, I_2, \ldots, I_n$ be ideals of R. The product ideal is the R ideal generated by the set $\{a_1 a_2 \ldots a_n \vee a_i \in I_i, \forall i\}$.

So, from each one of these ideals in each one of ideals in this family take one element a_1 from I_1, a_2 from I_2 and so on; take the product and do this for all possible choices. This gives us only a set, not an ideal and lo at the ideal generated by that thing. And, this is denoted the product ideal is denoted by the products.

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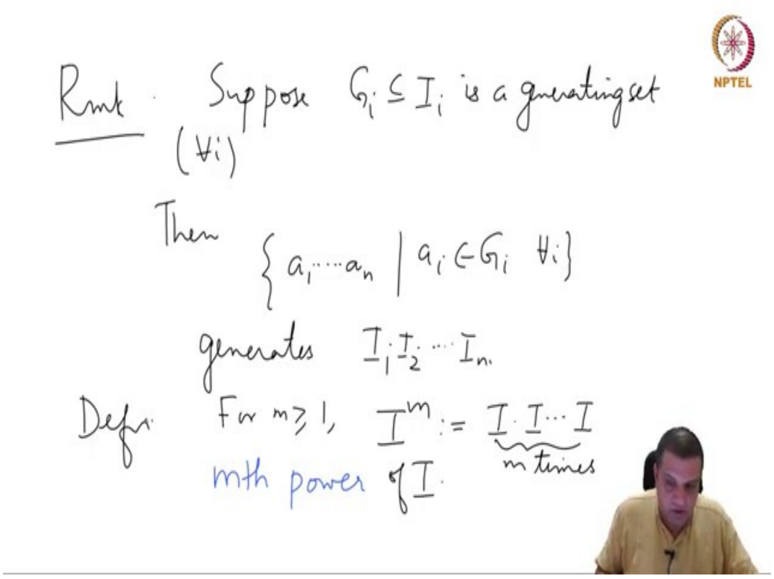

So, just one remark; suppose $G_i \subseteq I_i$ is generating set, this is for all i. For each ideal take a generating set then we do not need to take; so, in this thing here it is here we said take one element at a time from each in the family of ideal and then take the product. So, we do not need to take from all of them, we just need to take a product of elements.

Product of elements from here, where a_i is in the generating set G_i for each i, also generates the product. That is because orbit, if you have an element from an arbitrary element from I_i then one can express it in terms of the elements from the generating set for I_i and then one can rewrite it in terms of these .

So, this is the remark and a definition here ; for $m \geq 2$, define the m-th power of the product of I with itself m times and this is the m-th power of I.

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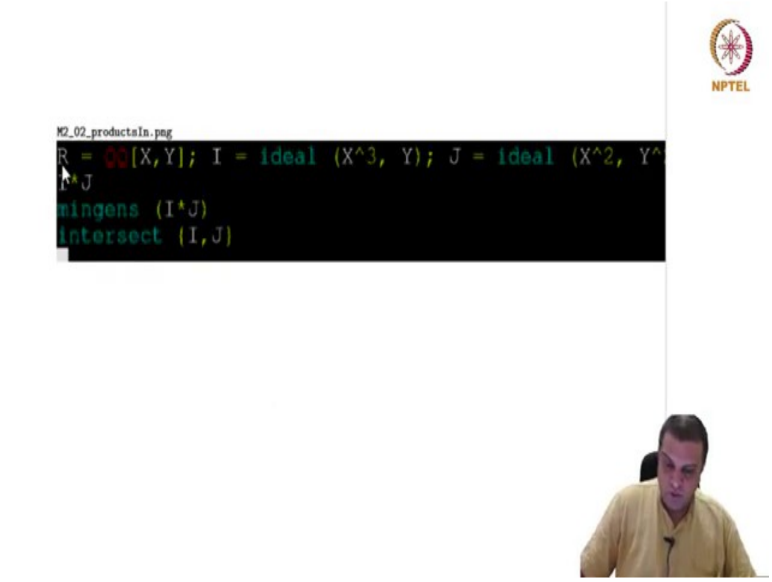

So, let us look at a couple of examples. So, R is a ring in 2 variables; I is an ideal generated by X^3, Y and J is an ideal generated by X^2, Y^2. Then we take the product I times J, then at that point you will realize at we have not obtained the minimal generators. And, then we are also asked to intersect these issue and let us see why that is relevant.

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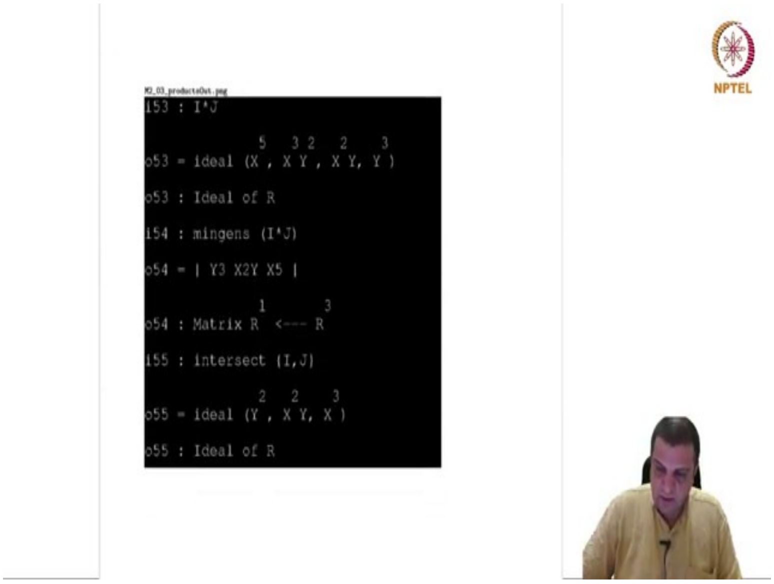

So, I just show you the output that is relevant; IJ is the ideal generated by X^5, X^3Y^2, X^2Y, Y^3; so, that is just. So, what it has done is taken this generating set and then multiplying them element by element. So, X^3 times X^2, X^3 times Y^2, then Y times X^2 and then Y times Y^2. So, we get this product and it is just an ideal of R. Then when we now lo at this thing, we see that X^2Y already divides X^3Y^2. So, anything that is a multiple of this is already a multiple of this one.

So, this is not relevant, it is not it is a redundant generator. So, we ask Macaulay to compute it using the mingens command; it outputs as a matrix, again we will not worry we do not need to worry about these issues now.

So, Y^3 is necessary, X^2Y is necessary, X^5 is necessary, but just we observed $X^3Y^2$is unnecessary. So, it remove that thing; then we ask take the intersection of these two ideals I and J . So, notice that suggests a remark.

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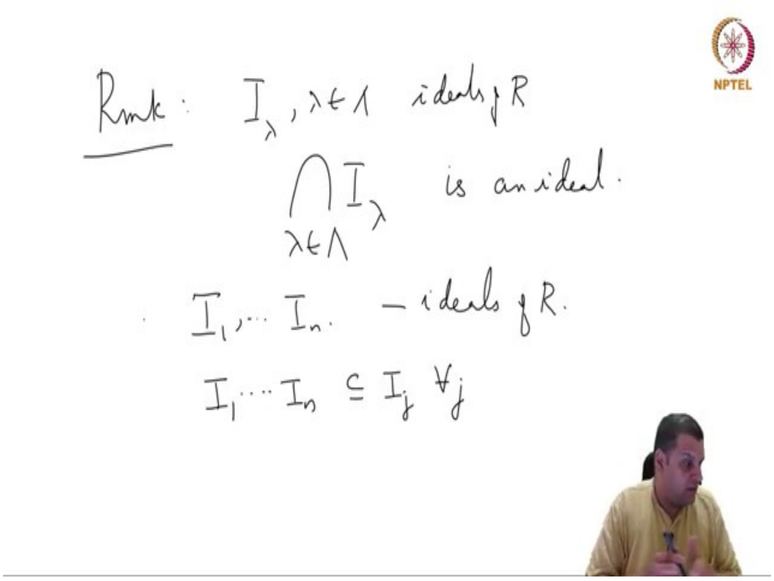

So, if you have a family of ideals not necessarily finite family. Then if you take the intersection is an ideal. Now, suppose we have $I_1, I_2, \dots, I_n$, say this are ideals and then if you intersect if you take the product ; this is inside I_j for every j. Therefore, the product of an ideal is always inside the intersection of the ideals and what we were checking in this example is what is the relation between these two things.

So, the product ideal is generated by Y^3, X^2Y, X^5. While the intersection and again at this point we just know that Macaulay to an computational intersection not exactly how which is done. The intersection of this ideal is ¿ ¿,X^3¿ and this is bigger than that because Y^2 is in in the intersection, but not in the product. And, similarly X^3 is in the intersection, but not in the product.

So, in general the intersection could be bigger than the product, but they are both relevant. So now, we do we have done with basic definitions about rings and ideals. And, now we lo at an important class of rings called Noetherian rings.

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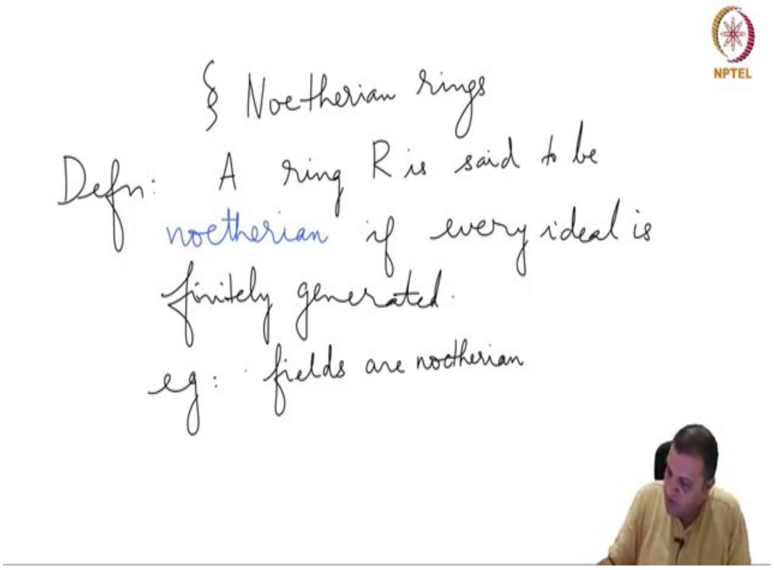

So, this is going to be somewhat longer section on Noetherian rings. It is a definition. A ring commutative ring, we said that hereafter when we say ring we mean commutative ring; R is said to be Noetherian if every ideal is finitely generated. So, it is not our first examples, that we know are Noetherian rings which is why this is a we have not yet come across a non-Noetherian ring.

Of course, we have seen rings which we have not yet proved this Noetherian, but so far we have not seen any non-Noetherian ring. For example, the fields are Noetherian. Why? A field has only two ideals: 0 and the full ring. The 0 ideal is generated by singleton 0 and the full ring is generated by the singleton 1 or any invertible element. So, hence fields are Noetherian.

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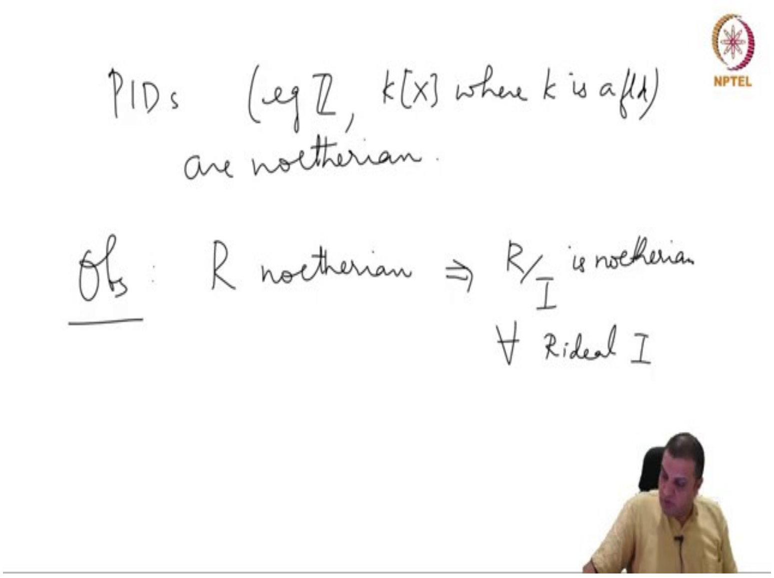

PIDs; so, for example, Z, $k[X]$ where k is a field are Noetherian, that is when every ideal is principle. So, it is finitely generated ; sorry when you say finitely generated, it means that it has a finite generating set; we already know and we already saw the notion of a generating set. And, now we are saying that an ideal has generating set which is a finite set. So, this is also that we have seen.

So, here is one more observation then which we will keep using ; if R is Noetherian then so is $\frac{R}{I}$ for every R ideal I. So, the property of being Noetherian is transferred to the quotient rings. So, we are interested not just in Noetherian rings or rings in general, but also about their morphisms, homomorphisms; sorry before that before I discuss maps, let me just do sorry do a proposition.

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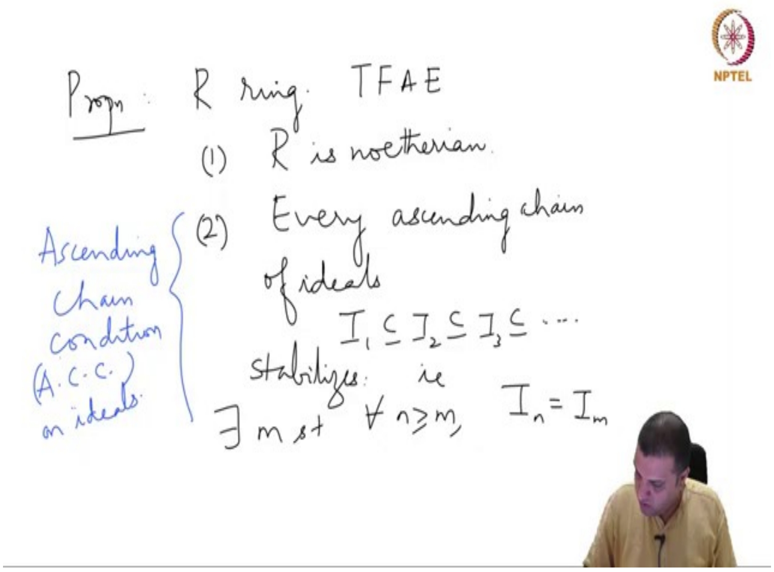

Let R be a ring, then the following are equivalent: (1) R is Noetherian. In other words, every ideal is finitely generated. (2) Every ascending chain of ideals. So, what do mean by an ascending chain? $I_1 \subseteq I_2 \subseteq I_3 \subseteq \ldots$. So, it is a its a chain ascending by inclusion stabilizes, that is there exists an m such that for all $n \geq m$, the n-th ideal is same as the m-th ideal. So, after while its same ideal that repeats infinite time.

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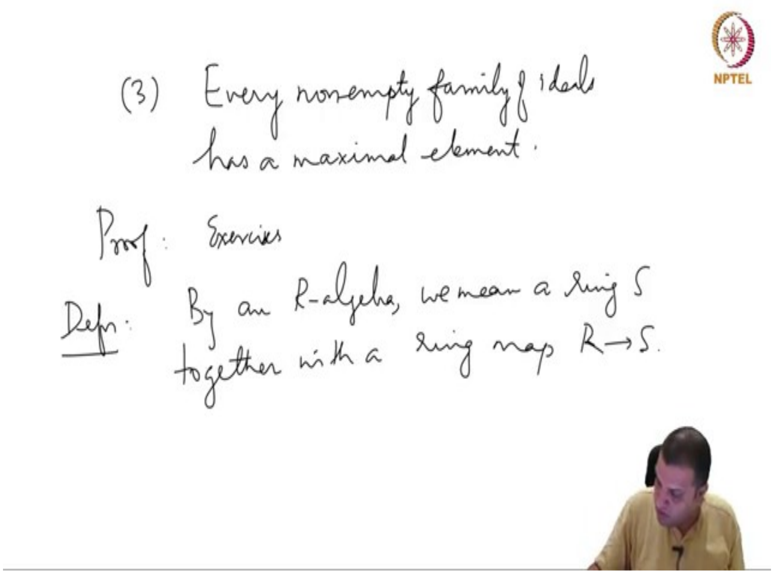

And, let us look at the 3rd condition; (3) Every non-empty family of ideals has a maximal element. So, just a maximal element in that family not necessarily of maximal ideal, but a family has a maximal element. So, this is the proof. It is not very difficult. So, I leave that I will sketch the steps in the proof of in the exercises.

And, just condition 2 which say that every ascending chain of ideal stabilizes is often referred to as Ascending Chain Condition, another sometimes ACC on ideals. So, we say I mean a non-Noetherian ring is characterized by this ascending chain condition on ideals; proof we done on this exercise.

So, now we want to talk about algebra homomorphisms definition; by an R algebra we mean a ring S together with a ring map from R to S. So, what makes S and R algebra is the specification of a ring map.

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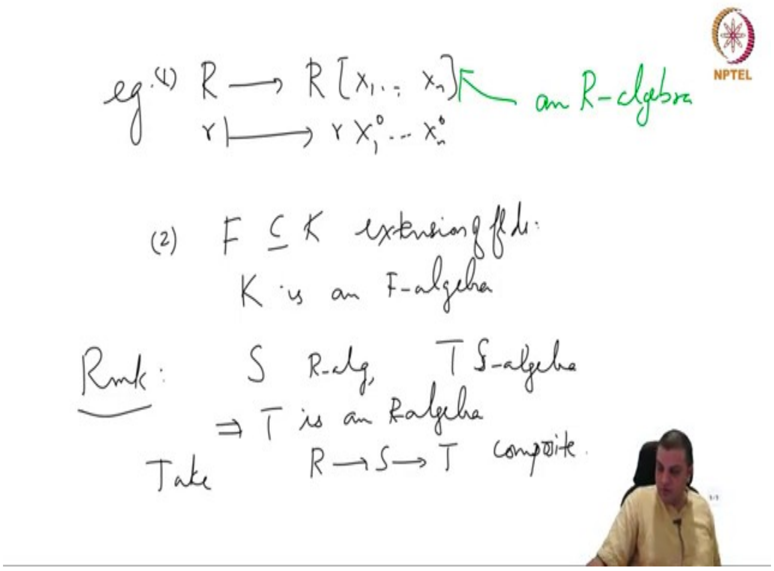

So, for example, R is a ring and consider the polynomial ring $R[X_1, X_2, \ldots, X_n]$. So, here any $r \in R$ goes to the constant polynomial . So, the polynomial ring is an R-algebra. Another example, suppose we have two fields $F \subseteq K$ that is K is an extension field of F, then K is an F algebra.

And, here is this map here is a usual inclusion map and a remark is that, if S is an R algebra and T is an S algebra; this implies that T is an R algebra in a natural way. And what is that? So, we have to it is a ring; what we need to specify is a ring map from R to T and for that take the ring map R to the composite map , take the composite . So, that makes T an R algebra.

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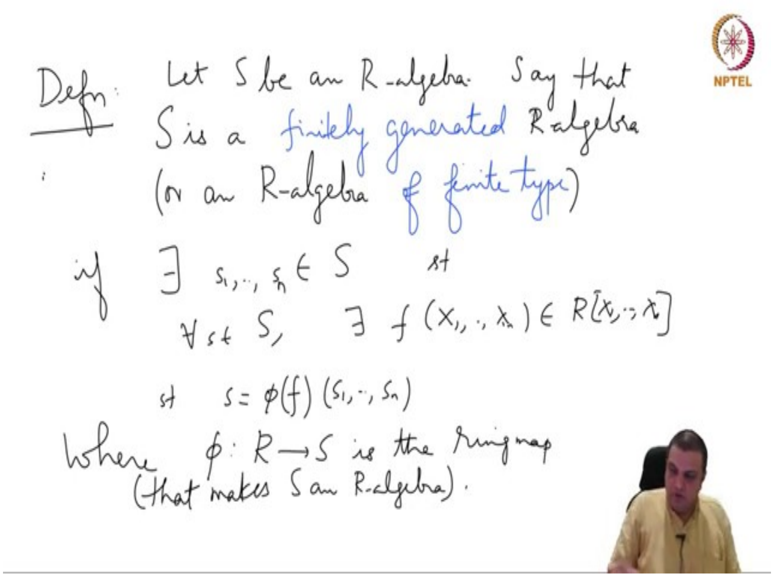

So, now we come to an important definition, in the values for the study of Noetherian rings. Let S be an R-algebra. Say that S is finitely generated R-algebra or an R-algebra of finite type if there exists finitely many elements inside S such that for every $s \in S$ there exist a polynomial in n variables, such that $s = \phi(f)(s_1, s_2, \ldots, s_n)$.

So, I apologize I forgot a piece of notation. And what is it? So, when we say S is an R algebra, it comes with the information about an ring map from R to S; often we will not label it or we will not need it, but here we do. So, let me just add that somewhere and let us say that I mean (Refer Time: 18:58) you should just write, such that $s = \phi(f)(s_1, s_2, \ldots, s_n)$, where ϕ is the ring map that makes S an R algebra; sorry I should have said this as part of the setup itself.

So, let me just reread this. So, S is an R-algebra So, to say it is an R algebra, it has to be a ring and they must be a ring homomorphism from R to S and call that thing ϕ. Now, say that S is finitely generated R-algebra or it is an R-algebra of finite type if there is a finitely many elements inside S such that for every element $s \in S$ there is a polynomial f. So, what does this mean $\phi(f)$? .

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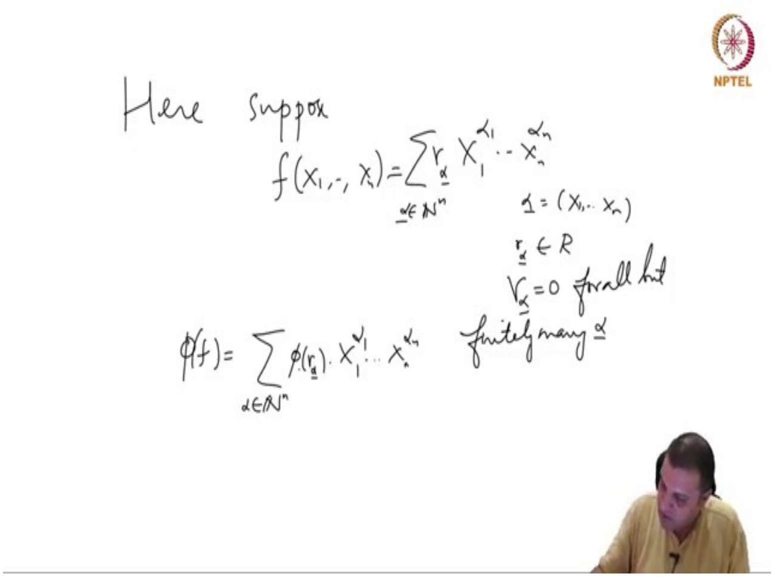

Here $\phi(f)(X_1, X_2, \ldots, X_n)$; sorry I will write it in; suppose $f(X_1, X_2, \ldots, X_n) = \sum_{\alpha \in N^n} r_{\underline{\alpha}} X_1^{\alpha_1} \ldots X_n^{\alpha_n}$. So, α is n tuple. So this is a polynomial.

So, the coefficients are from the ring; coefficients corresponds to that monomial. So, then $r_{\underline{\alpha}} = 0$ for all but finitely many $\underline{\alpha}$. So, one can write like this. So, then $\phi(f)$ is a same polynomial, but except the coefficients are now substituted inside S. So, let us go back here.

So, what we are saying is for any element s of S there is a polynomial which if you evaluate on these $s_1, s_2, \ldots, s_n$ will give s, but to make sense of it. This must be a polynomial not with coefficients inside R, but coefficients inside S. And this how which is made sense, that the coefficients themselves are just substituted through ϕ.

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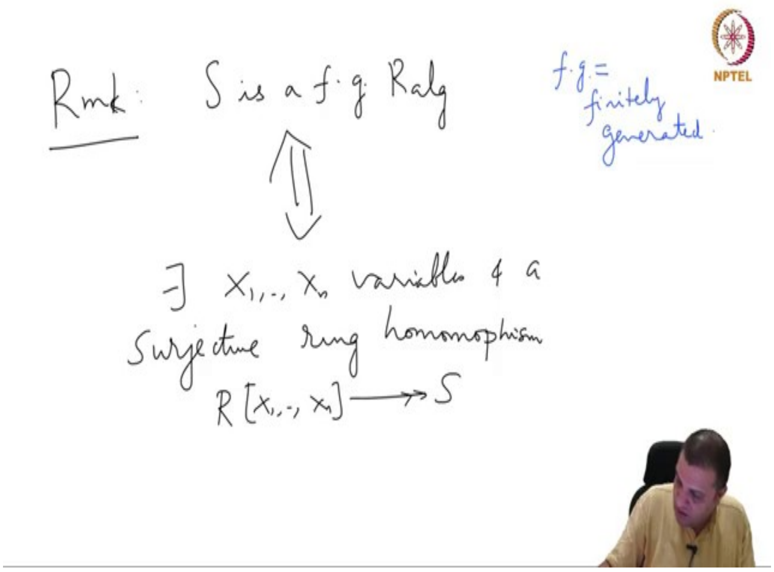

So, this is what we mean by a finitely generated R-algebra and here is a remark which I will sketch in the exercises; S is finitely generated. So, we are to write finitely generated a lot in these notes. So, fg denote is whether it is an algebra or it is it is ideal or later modules, fg is for finitely generated; so you have to write that a lot.

So, S is a finitely generated R-algebra if and only if there exists $X_1, X_2, \ldots, X_n$ variables and a surjective ring homomorphism from the polynomial ring in that many variables to S. So, this is not at all very difficult to prove. So, I will describe one observation that we will need, but the point is that whatever was used in the definition just plug it in.

So, I will give one direction. So, if you use that as a definition, it will give the direction from the top to bottom and for the other direction; assuming this is true, that is true.

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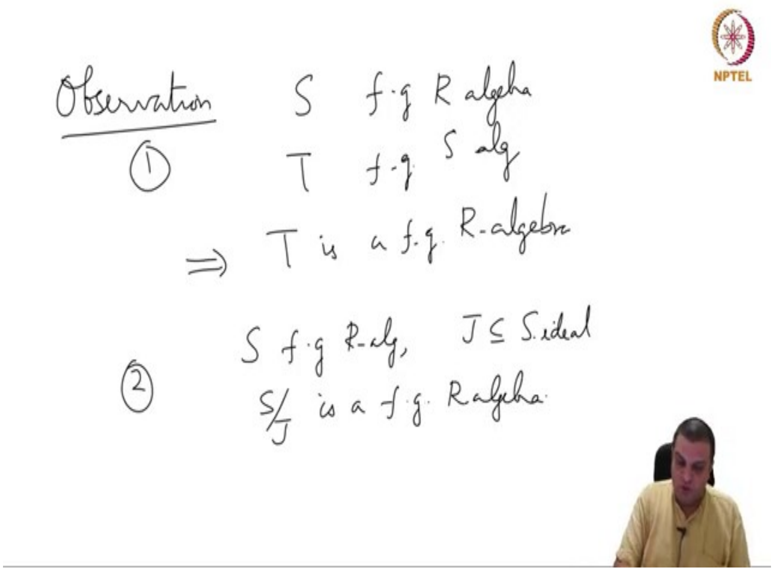

So, for that we need the following observation ; suppose we have S finitely generated R-algebra and T a finitely generated S algebra. This implies that T is finitely generated R algebra. So, this is one observation that we need. So, the second observation that we one can use is S finitely generated R algebra and J an ideal of S , then $\frac{S}{J}$ is finitely generated R-algebra.

So, using observation 2 one can prove the other direction which is; so, this is finitely generated because the generating set are the variables, S is a quotient. So, it is going to be a quotient of a polynomial ring; this finitely generated thing modulo an ideal. And, hence the quotient is also finitely generated. So, observation 2 can be used to prove the remark earlier and observation 1 is more general. In fact, it is better to prove the remark earlier and then prove observation 1.

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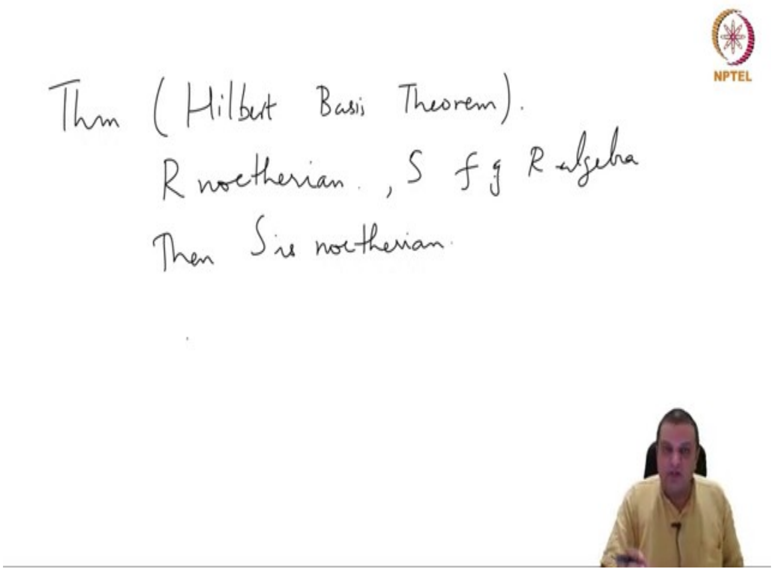

So, here now we come to the one of the most important basic theorems in commutative algebra called the Hilbert basis theorem. So, what is the statement of the theorem? R Noetherian, S finitely generated R-algebra. Then S is Noetherian and we will look at and in this exercise. So, for example, exercises we will one would be able to show a non-Noetherian ring now, which I will do it in the exercise.

So now, I want to give a we are now going to prove it now, but I would like to give some general discussion about this remarks which using which we will reduce it to some apparently special cases for which it does not . So, that is what I will do now.

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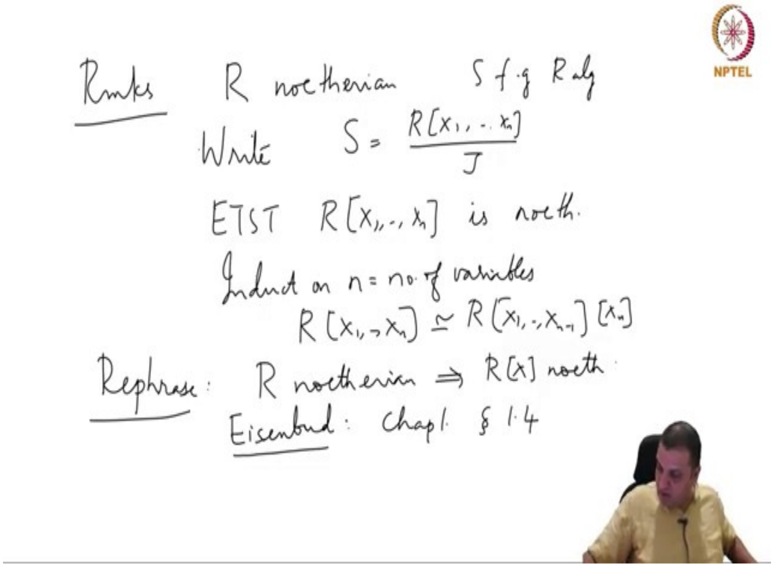

So, general remarks. So, we know that R is Noetherian, this is the hypothesis. S is finitely generated R algebra. We want to show that S is Noetherian. So, then write S as some polynomial ring, this we this was the remark earlier where J is an ideal of that ring of the polynomial ring.

To show that S is Noetherian, it is enough to show that the polynomial ring is Noetherian. We noticed earlier that quotients of Noetherian rings are Noetherian. This we can do on the number of variables, that is because the polynomial ring in n variables is isomorphic as a ring to polynomial ring in $n-1$ variables first and then the last variable that is $R[X ¿¿ 1, X_2, \ldots, X_n] \simeq R[X ¿¿ 1, X_2, \ldots, X_{n-1}][X_n] ¿¿$. And, by induction if we know this is Noetherian and then by induction again one variable case we can prove, then we know that this is true.

So, then we can restate the theorem to the following, we just have to prove it for one variable. So, we start from an arbitrary finitely generated as ring from which we said, we generally do for polynomial rings. And, then we do induction and say we just have to do one variable.

So, this is the more this is equivalent to the general version of a Hilbert basis theorem we stated earlier, because we can prove this, we can go back. So, this we will not prove partly because we would like to become familiar with the different proof which works only for fields; I will clarify in a minute.

But, for computational purposes that is on one case its more than enough. Secondly, it should be more importantly it gives us a family, it introduces us to certain to some of the basic concepts of computation commutative algebra. So in fact, this proof is much simpler much shorter, the other proofs somewhat longer. But, it introduces us to newer notions and things that are cleared towards understanding computational commutative algebra.

So, we will only prove that in special case. So, for a proof of this, this is proved in this generality in Eisenberg's book which I mentioned in the first lecture. Commutative Algebra with a View Towards Algebraic Geometry, Chapter 1 section 1.4 in this generality , it just proved here. So, it would be an exercise for you to read that proof and understand it. So, I will just say what we are going to do now and in this lecture.

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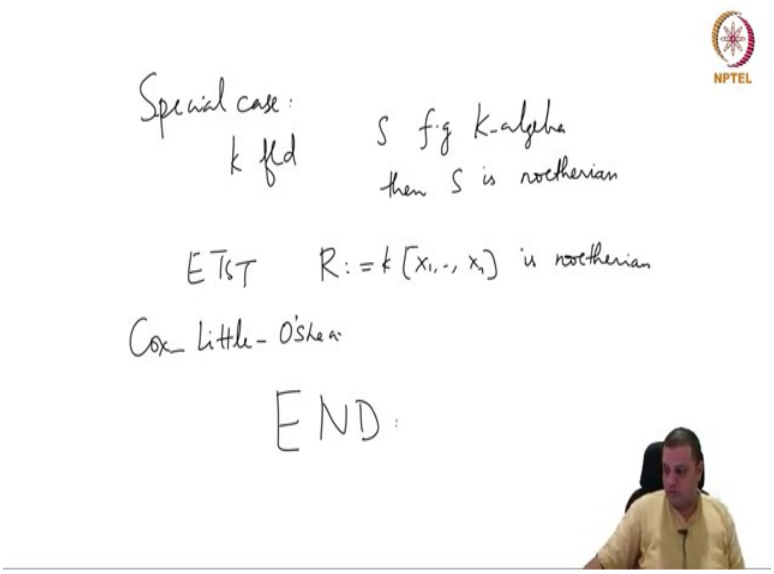

So, we will prove a special case, which is that k is a field and let us say S is a finitely generated k-algebra, then S is Noetherian. And, arguing as earlier S can be written as a quotient of a polynomial ring over k. So, again it is enough to show that R which is a polynomial ring in n variables for arbitrary n finite is Noetherian.

And, in improving this we are following the book by Cox-Little-O'Shea. It is spread in a few sections and we will go over this proof carefully. Partly, our motivation is to understand the points relevant to computation of questions. So, this is the end of this lecture. And, we will continue with developing the ideas to prove this theorem in the next lecture.