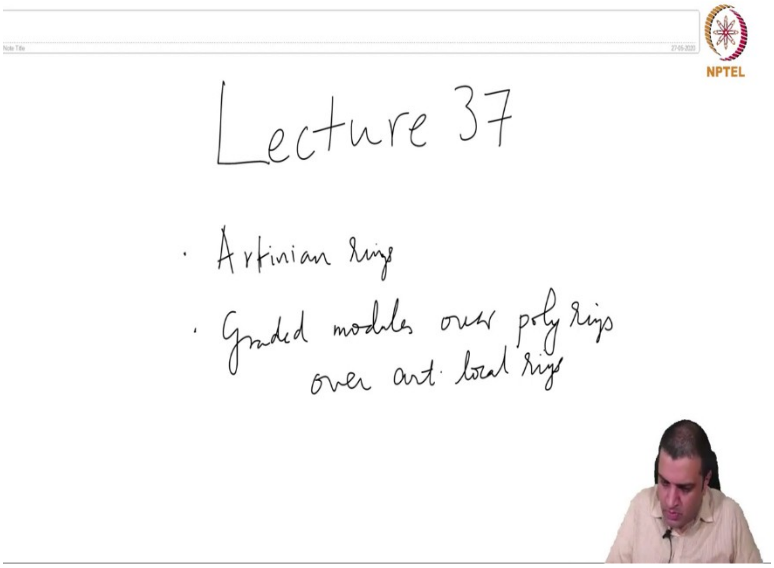
Computational Commutative Algebra
Prof. Manoj Kummini
Department of Mathematics
Chennai Mathematical Institute
Indian Institute of Technology, Madras

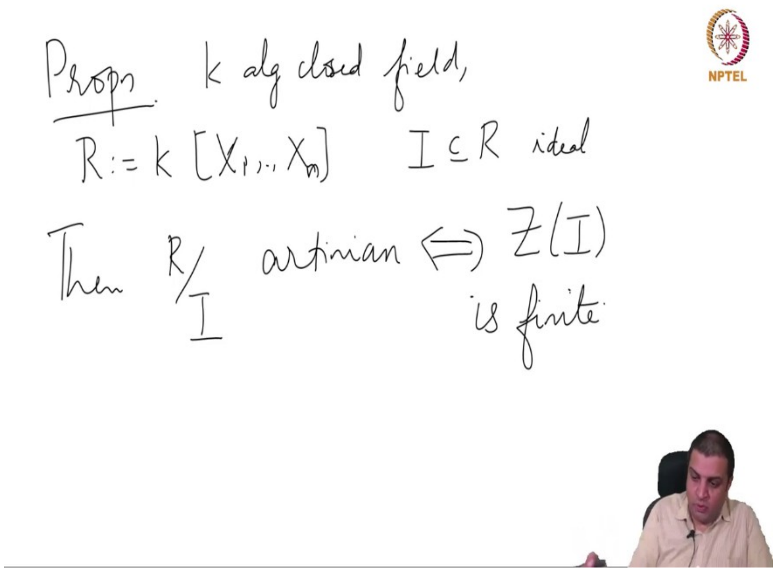
Lecture – 37
Graded modules

(Refer Slide Time: 00:16)



Welcome, this is lecture 37. So, in this we continue our discussion about Artinian rings and then we would like to discuss about Graded modules over Artinian local rings. This is graded modules over polynomial rings over Artinian local rings. So, this is the plan for this lecture. And so, Artinian rings we are just continuing our discussion from the previous just to give a specific example or one place where they show up.

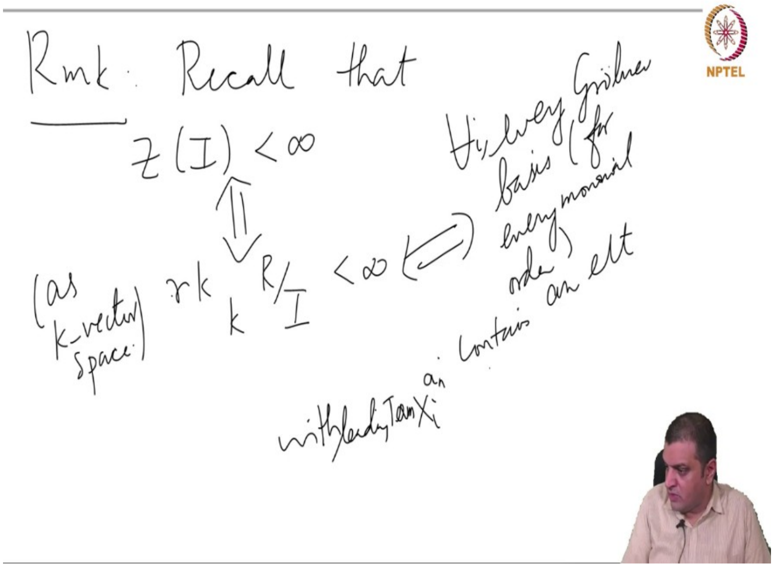
(Refer Slide Time: 01:15)



So, proposition: let k be an algebraically closed field. So, this assumption is only to say we can visualize the solution set field. $R := [x_1, \ldots, x_n]$, I an ideal of R, then R/I is Artinian if and only if Z(I) which is the solution set of I, is finite. So, we can remove this hypothesis the case algebraically closed.

Again that is used only to say that if Z(I) is finite then there are only finitely many maximal ideals in that ring. It may not be true if k is not algebraically closed. It is possible that Z(I) is infinite Z(I) is empty, but the ring is not is non-trivial . So, it is to avoid those situations that we are assuming case algebraically closed.

(Refer Slide Time: 02:52)

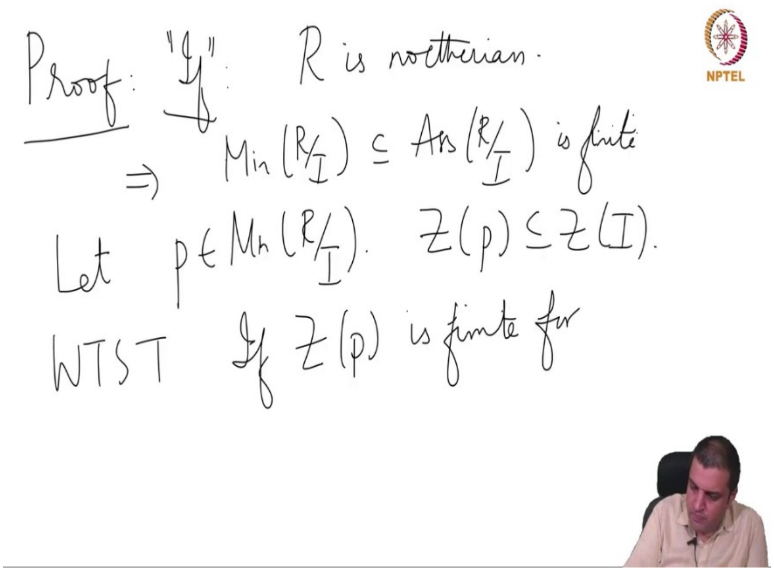


So, let us so, just a remark before we prove we already know this we have already seen an equivalence of this condition recall that Z(I) is finite if and only if this we proved little while ago that $rk_k\left(\frac{R}{I}\right)$ as a k- vector space. This is this k as k -vector space. So, as I mentioned in an earlier lecture we have to use dimension in two different ways; one is a vector space dimension and the other one is (Refer Time: 03:39) dimension. So, we will try to use the word rank whenever we talk about vector spaces.

So, this is finite and then we also saw that this is equivalent to the condition that Grobner base every Grobner basis. Meaning that is for every monomial order contains a term with $x_i^{a_i}$ for all i contains an element; contains an element with this as the leading term. So, sorry, every Grobner basis in any monomial order; in any monomial order contains an element with the leading term $x_i^{a_i}$ for every i, this is true. So, for all i and for all Grobner basis there is a term with this leading.

And, this I mean equivalence of all this we had proved earlier. So, again it does not one does not need Z(I) to be fine case may be algebraically closed one can reformulate it in terms of Spec(R/I) and, but first to visualize it, it is better to state in this geometric fashion. So, let us prove the proposition.

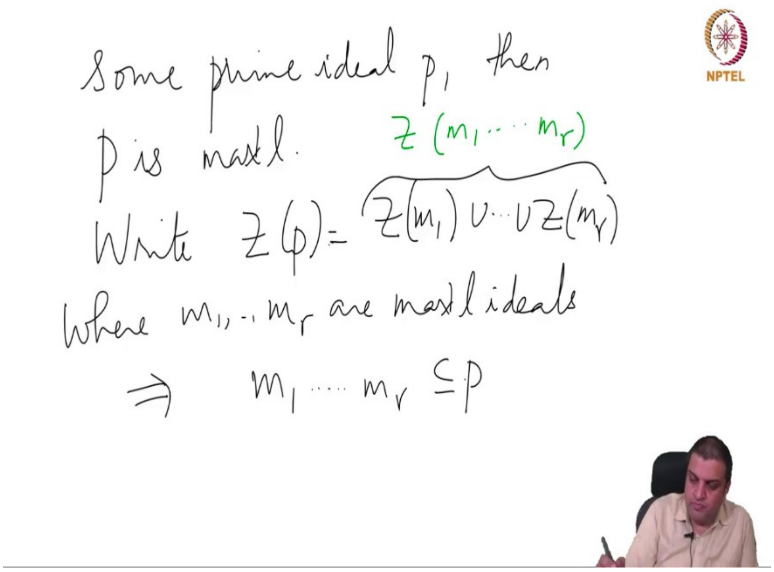
(Refer Slide Time: 05:26)



If so, if Z(I) is finite. So, notice that R is Noetherian. So, therefore $Min\left(\frac{R}{I}\right) \subseteq Ass\left(\frac{R}{I}\right)$ is finite. Till now, we had not seen a proof or we did not know that minimal primes over an ideal in a Noetherian ring is finite. So, this is a proof that we know that associated primes are finite because associated primes are precisely the primes that appear in an irreducible decomposition.

Starting with an irreducible decomposition we can make it irredundant primary decomposition and associated primes appear as associated to one of those things and therefore, this list is finite and hence this is finite. So, this is. So, minimal primes are finite. So, let p be a minimal prime over I. Then, Z(p) ⊆Z(I). So, this is also finite.

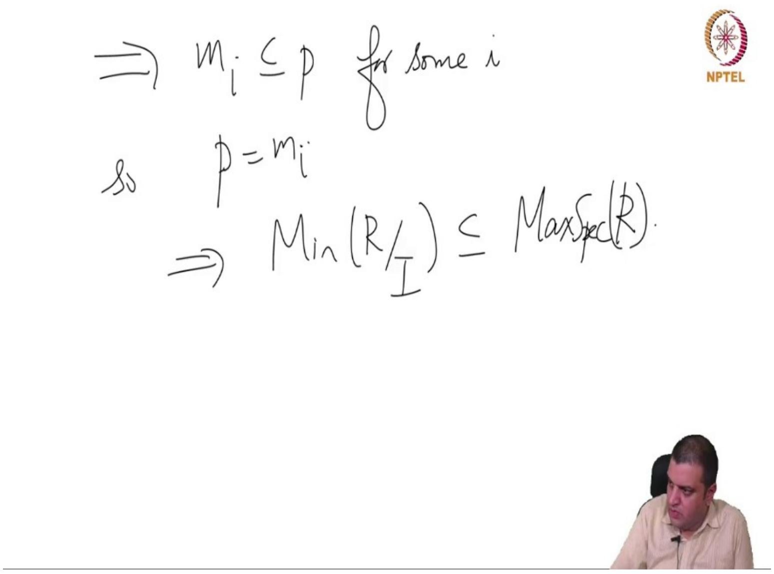
(Refer Slide Time: 07:12)



So, we want to show that if Z(p) is finite for some prime ideal p, then p is maximal. So, write $Z(p)=Z(m_1)\cup\ldots\cup Z(m_r)$ where $m_1,\ldots,m_r$are maximal ideals and Z (1) is the point where the point corresponding to that maximal ideal. So, we can write it like this. But remember that this is equal to also equal to $Z(m_1\ldots m_r)$ if the product or the intersection whichever one.

In other words, these two ideals have the same radical or since p is prime this is a radical ideal. So, now, this implies that this implies that $Z(m_1\ldots m_r)\subseteq I$ mean it says that these two have the same radical, but since p is prime it is shown radical. So, this is inside p.

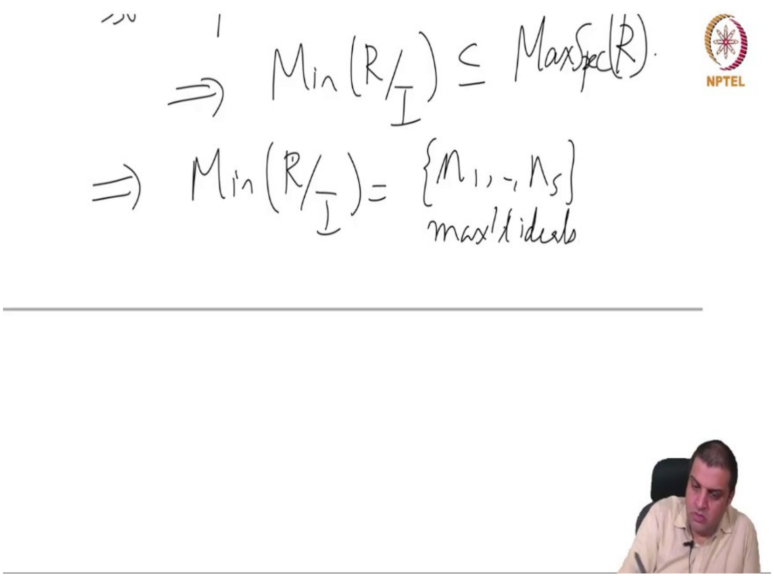
(Refer Slide Time: 09:10)



Which now implies that $m_i \subseteq p$ for some i, but this is maximal. So, p = m_i. So, prime ideals in which if p is a prime ideal such that Z (p) is finite then p is maximal. So, it is in this thing that we use that k is algebraically closed. I mean it is possible that p is a non-maximal over a non-algebraically closed field it is possible that p is a non -maximal ideal, but it has no solutions in that over that field.

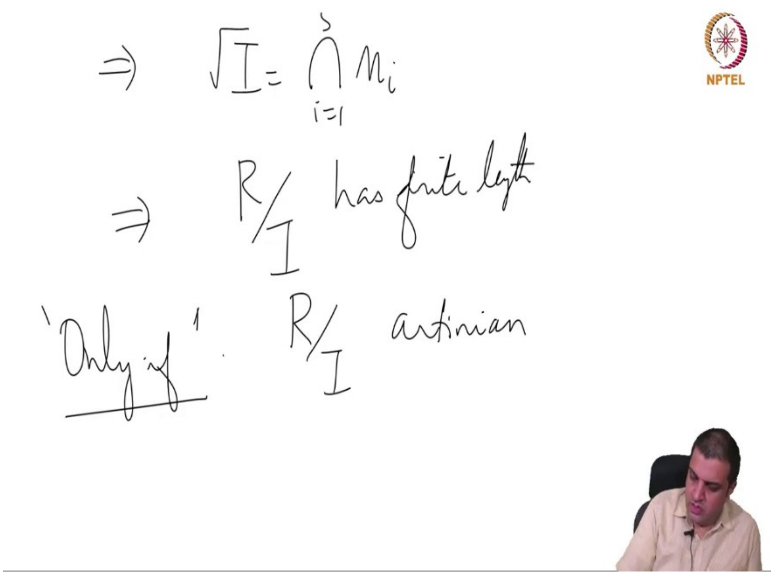
So, conversely this is the only if. So, this proves that p is maximal, but then remember where we chose p from. p was from minimal primes. So, every minimal prime is a maximal ideal you let us just write that and conclude. This now implies that $Min\left(\frac{R}{I}\right) \subseteq maxSpec(R)$ which is what we wanted to prove. So, sorry, let us add the details.

(Refer Slide Time: 10:44)



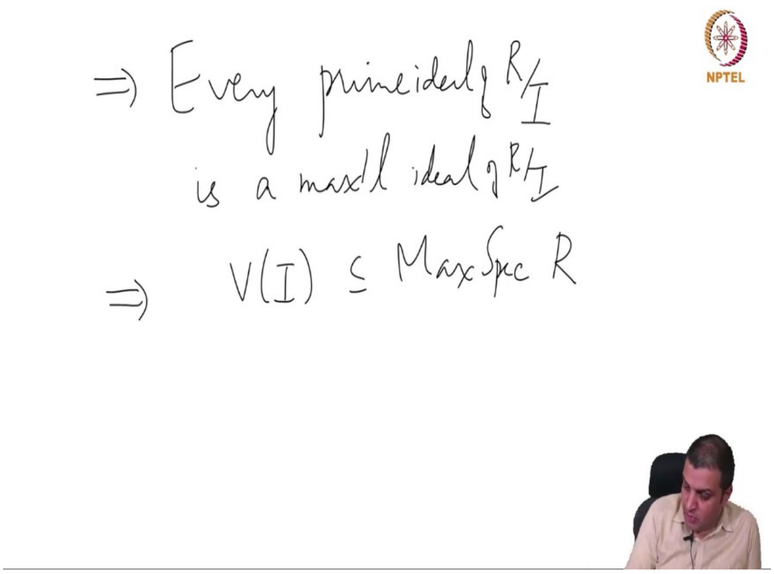
So but, this is finite. So, then this implies that $Min\left(\frac{R}{I}\right)$ is some set let us say $\{m_1, \ldots, m_s\}$ not necessarily the one that we saw some other $\{m_1, \ldots, m_s\}$ maybe just to be safe let us call it $\{n_1, \ldots, n_s\}$ maximal ideals.

(Refer Slide Time: 11:12)



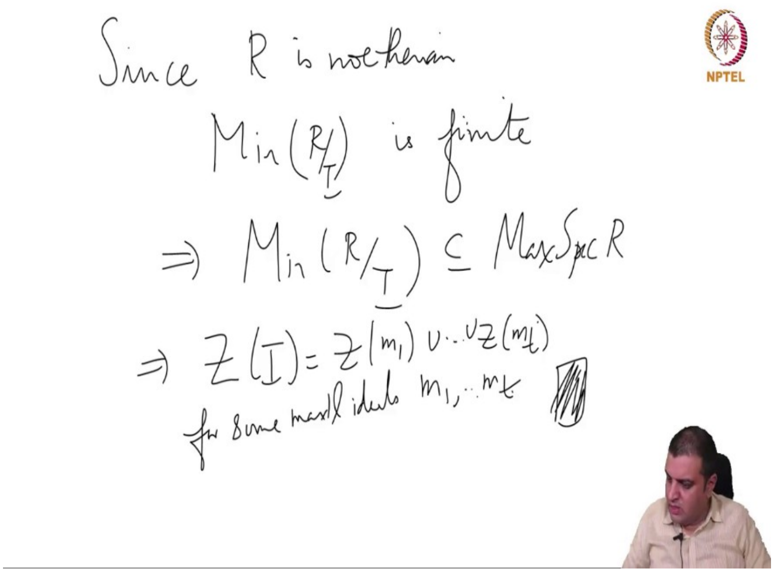
So, now this implies that $\sqrt{I}=.\bigcap_{i=1}^{s} n_i$, but from which we can conclude that R/I is finite length because some power of this will be inside I and these all have finite. So, R/I has finite length in other words it is Artinian. So, this is the direction of if and only if. Now, it is Artinian. R/I Artinian.

(Refer Slide Time: 12:04)



Which now implies that every prime ideal in R/I is a maximal ideal of R/I. This now implies that $V(I) \subseteq Max\ Spec(R)$. This is primes containing I and maximal ideal of R/I. So, they pull back to maximal ideals in R. So, we get this condition, but we need to show that it is actually finitely many points.

(Refer Slide Time: 13:00)

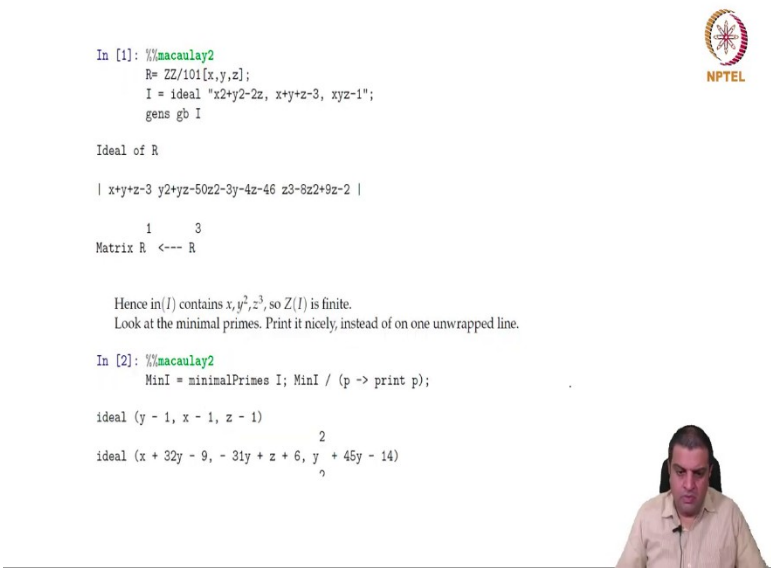


Since R is Noetherian, Min (R/I) is finite, but what is in V(I)? V(I) is prime which contains some prime in Min I, but then they are all maximal which now mean says that the minimal primes of R/I is a finite subset of Max Spec R. Remember, minimal primes are the minimal elements in these are the minimal elements in V (I), but they are all inside here and among them there is no comparison. So, it they are all in pairwise incomparable.

So, this is inside here and so, $Z(I)=Z(m_1)\cup\ldots\cup Z(m_t)$for some maximal ideals$m_1,\ldots,m_t$. n was the number of variables this list as well collection cardinality this could be more or has no relation to the number of variables. So, this is. So, this is the proof of this.

So, now, let us try to take a look at Macaulay an example. So, I will also try to. So, it is a relatively simple example, but we will try to sort of use things that we have learnt so far to understand that ideal I mean various things that we have learnt so far. We will try to use on that ideal.

(Refer Slide Time: 15:04)



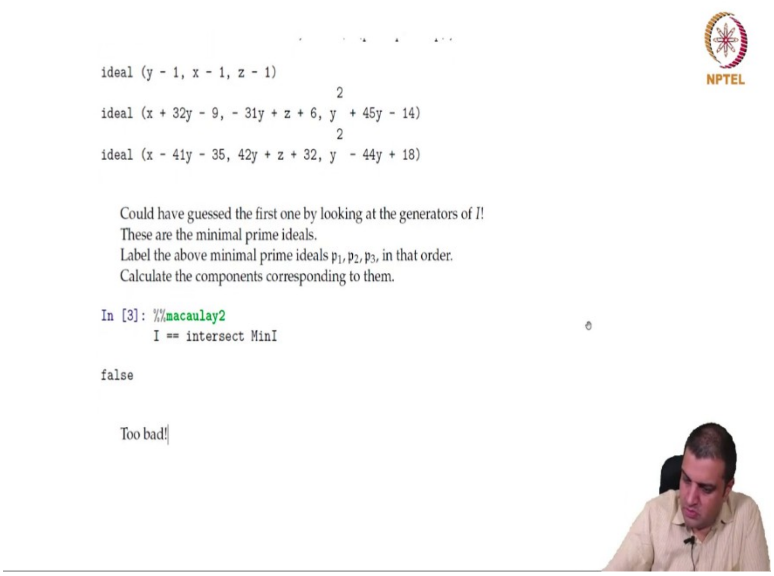
So, here is an example. So, we take a polynomial so, again just let me just remind you this is the first line ignore the first line; that is not part of the code. Just these three lines that are offset; Z just ignore that line. So, polynomial ring in three variables and I is an ideal generated by three things. So, let us just quickly look at what it is in the if we were looking at in R 3.

So, this is a cone at least in the positive part; when Z is positive this is a cone. This is a plane which will intersect the cone because well 111 it is a solution and so, is for this one. So, just by looking at itself, we know one point in the intersection even. So, and since these coefficients here are substantially less than 101 whatever calculation that happened in R will also happen identically in this field. So, 111 is going to be a solution, because the coefficients are integer.

So, in principle one can ask those relations over any field; not necessarily in over R or over the field that was defined. So, here we ask for generators of it is Grobner basis and it produces something and so, if you look there is a term here with x, there is a term here with $y^2$and then there is a term there is a polynomial here with leading termz^3.

So, from the test that we have learned earlier this will have finitely many solutions even over the algebraic closure; this is of course, a finite field. So, you will have only finitely many solutions. But, even over its algebraic closure, it will only have finitely many solutions.

(Refer Slide Time: 17:33)



So, now let us look at the minimal primes. So, if you ask for the minimal primes what will happen is it will produce some long list which will be unwrapped you know unwrapped line going out of this page. So, I just try to print one print them one by one ok. So, I ask Macaulay for minimal there is a command called minimal primes. So, take a look at its help, take a look at its methods.

So, if you ask methods minimal primes it will tell you the ways in which methods minimal primes can be used, the types of the arguments. So, call it Min I then I ask. So, this is a this we have seen this is an abbreviation for the apply function. So, this is same thing as apply to Min I this function p maps to print p. So, it just prints and does nothing else.

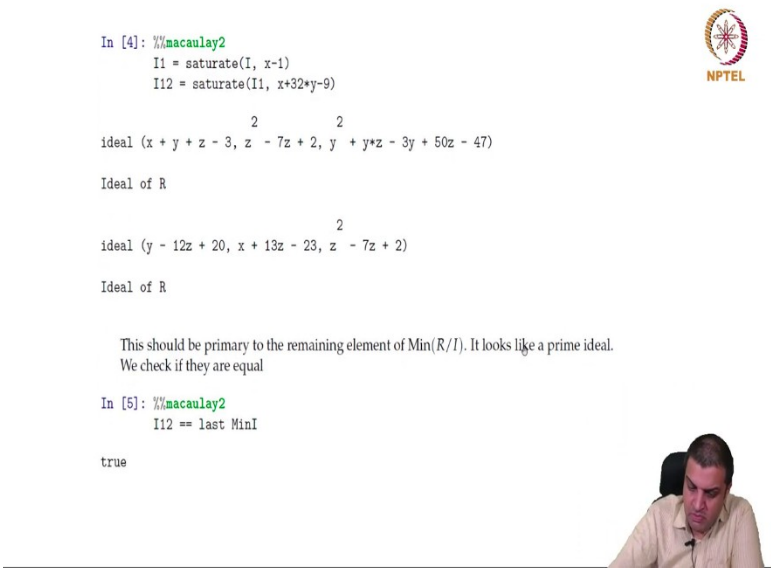
So, these are the things and as I mentioned earlier. We know that; we know that 111 is a solution to this I mean. So, it has some solution even over any field. So, we ask (xy - 1, x -1, z- 1) is an ideal which contains this and then it produces something and of course, depending on what field you put these coefficients might be different.

So, if you put instead of 111 you put some other larger field this might be these numbers might be different. So, that of course, depends on the. So, I mean these numbers will depend on what field we are working over. It may not actually have any solution over R. I did not check, but it may not have a solution over R it. Over R maybe one would just see this and some other thing.

So, how even if the equations involve only integers what the primary decomposition will look like will depend on the field. So, please keep that in mind. So, as I said we could have guessed the first one by looking at the generators of I. These three other minimal primes . Label the above minimal primes p_1, p_2, p_3in that order $p_1$is that one $p_2$is a one with this and $p_3$in this order.

So, we ask we would like to understand what is the primary decomposition of I? Meaning I = $j_1 \cap j_2 \cap j_3$; $j_1$associated top_1, $j_2$associated to $p_{2,}$ $j_3$associated to p_3.We would like to understand this, but. So, we will see if we are lucky enough and we ask just intersect these three prime ideals and check is it equal to I? So, then it says no.

(Refer Slide Time: 20:34)

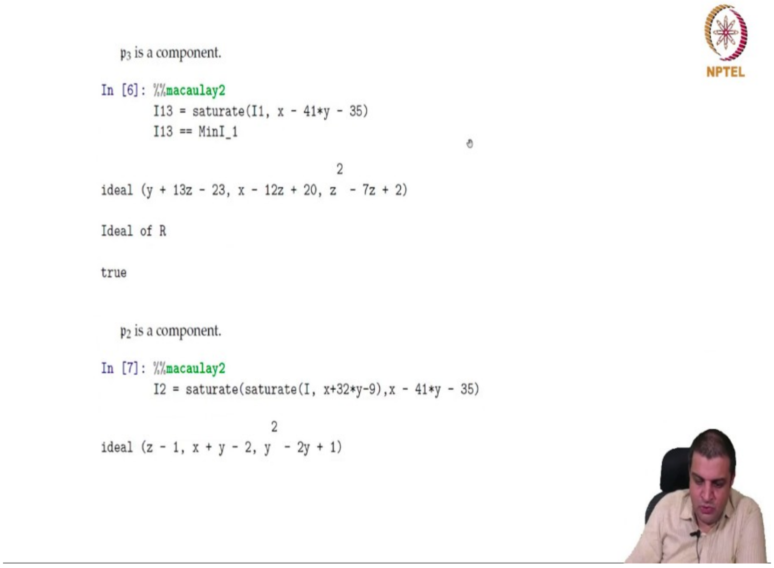


So, that is too bad and then let us look at the next. So, we saturate. So, now, we try what we know if you want to find associated or primary components that are that do not contain a specific element saturate with respect to that element and then try to find a primary decomposition. So, let us saturate I by x-1. So, it is it will try to find components corresponding to these two primes.

So, first I saturate that and the second time I saturate with respect to x + 32y - 9. So, we are looking for this prime component corresponding to this prime. So, then it produces some ideal ok. So, now, this would be this ideal would be primary to p_3. So, this is this way this will get rid of p_1, this will get rid of $p_2$and so, what is left over will be primary to $p_3$since that is the only other prime ideal.

And if you look at it you know this looks very prime like because there is some linear thing here there is some linear thing here and then some z^2. So, this might be prime I mean we by looking at it we just feel this might be prime .

(Refer Slide Time: 22:02)

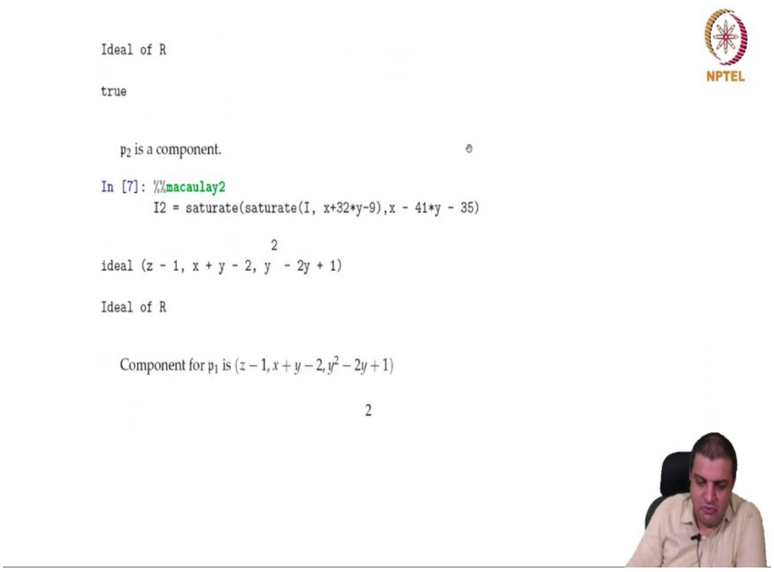


So, then we ask, is this prime? Is this ideal that we got which I called I12, I12 is that equal to the last one in that list Min I? So, what was Min I? Min I was these three Min I was these three p_1, p_2, p_3. So, we remove p_1, we remove p_2 and we got some ideal and since that thing looks very prime like we see if we get lucky and ask is it equal to is I12 equal to last Min I and it says it is true. So, I12 is p_3 and it is the component corresponding to p_3.

So, now we will remove p we have already removed p_1 we will remove p_2 we will now remove p_3. So, this is the this is what we did. So, remember p_3 has this x- 41y - 35. So, we remove that now. So, we remove that now here and then we get. So, then we ask is this the middle one? Middle one is p_2.

So, remember list is numbered from 0 to length minus 1. So, 1 means the second one in the list and so, we ask is this saturation. So, we get some ideal here then we ask is it equal to Min I and it says true. So, therefore, p_2 is also a component .

(Refer Slide Time: 23:43)

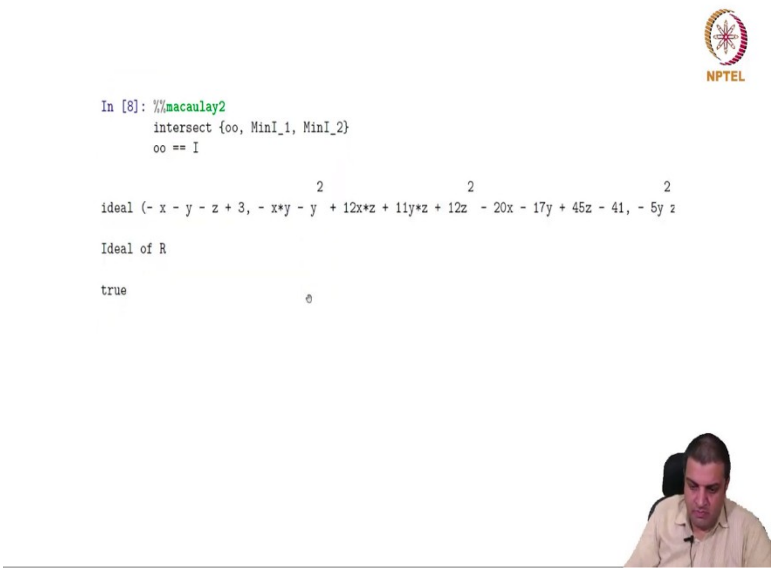


So, then we become sort of hopeful and then we ask let us now saturate . So, I just put this in a you know in one line. First, we saturate that linear form in $p_2$of that thing we saturate the linear form in$p_3$and then we ask we get some ideal then we ask is this equal to. So, this is the component forp_1. So, $(z-1, x+y-2, y^2-2y+1)$; if you notice this itself is not over that field this is this itself is not a prime ideal. This is not same thing asp_1.

But if you notice if you go with algebraic closure it has only one I mean z of this ideal is just one point. The zero set of the side is just one point, z coordinate is 1 and if we solve this one says y coordinate is 1, but with multiplicity 2 and that will solve x coordinate also to be 1.

So, this is I mean the component of p_1; $p_1$corresponds to the maximal ideal at 1,1,1. The component correspond component is an ideal which is not exactly p_1, but something which is just prime only primary to it.

(Refer Slide Time: 25:06)

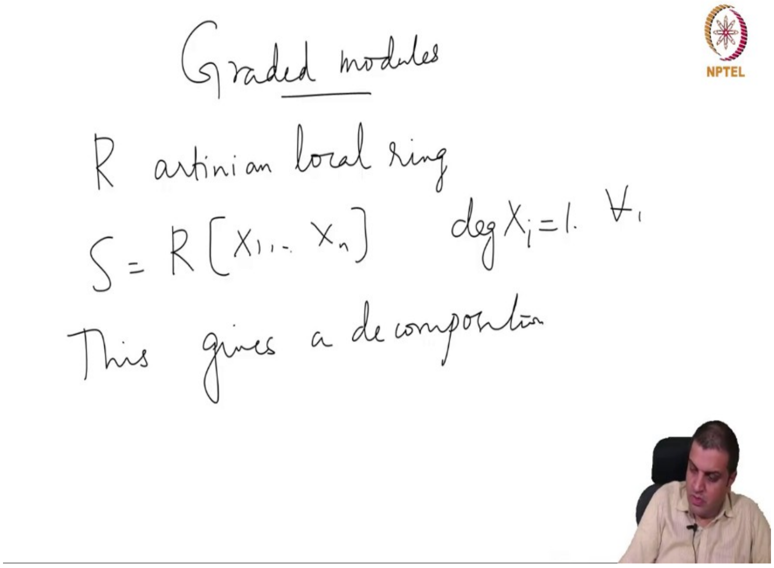


So, this is not the same as $p_1$and then we just check that we got the correct answer. We intersect the last one we just got output and then MinI and we already concluded that Min (I_1) and Min(I_2) are already the associated components associate corresponding to the p_2 and p_3. So, we just intersect. So, of course, we will get some because intersection remember involves extending to some bigger ring. So, ti + (1- t)j and then contracting it back to R.

So, we get some really long list of generators and then we just ask is this same as I and the answer is yes. So, this is an example where we work through an Artinian ring using various techniques that we have learned so far about saturation finding minimal components and so on.

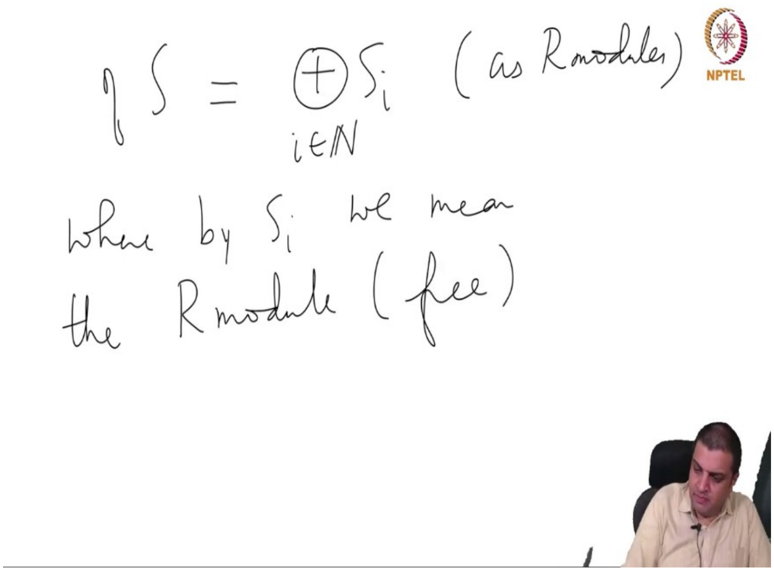
So, the only thing that we actually I mean in this example only thing whose algorithm that we have not studied is the minimal primes. So, that I mean one has to I mean I use that to get this list and then proceed from there.

(Refer Slide Time: 26:25)



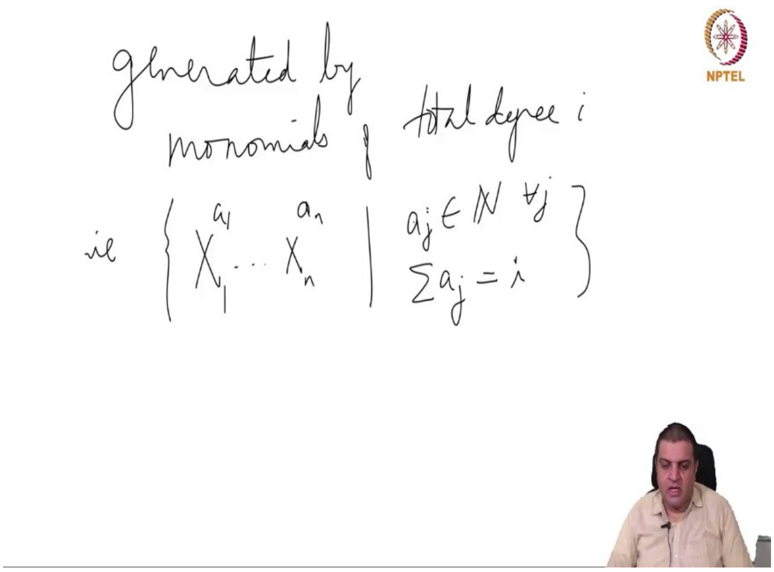
So, now we want to discuss a new topic called graded modules this is. We will not discuss them in more generality than what we need for the present which is the following. So, R is an Artinian local ring. $S=R[X_1,\ldots,X_n]$. But, not just I mean we put an extra structure which is natural to this one which is the $deg(X_i)=1$ i for all i.

(Refer Slide Time: 27:17)



So, this gives a decomposition of S = $\oplus_{i \in N} S_i$; So, this is as R modules not as a not as a ring, but as R modules.

(Refer Slide Time: 27:56)

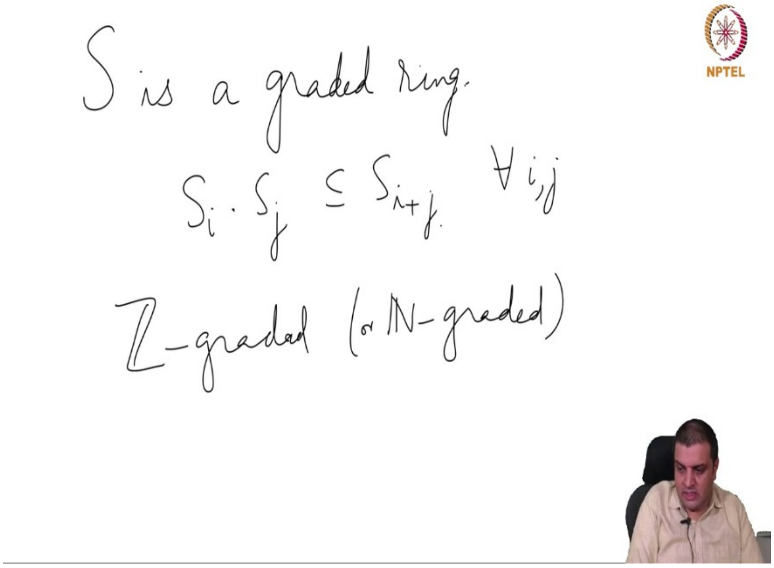


By S_i we mean the R module actually free generated by monomials of total degree i that $\{X^{a_1}\ldots X^{a_n} : a_i \in N\ \forall\, j, \sum a_j = i\}$. So, this is; so, this is a free module this is just a basis.

So, it is exactly like a polynomial ring over a field except that the coefficients we now allow for Artinian rings. And, polynomial ring always has a I mean depending on what degree one wants to give to these variables and for our I think all that we need is this.

So, we will just stick with that thing. So, only point that we allow for some freedom for here to be Artinian local rings . So, this is the this is. So, this is called a graded ring. So, in this particular case it is graded by Z, although it lives only in the non-negative degrees.

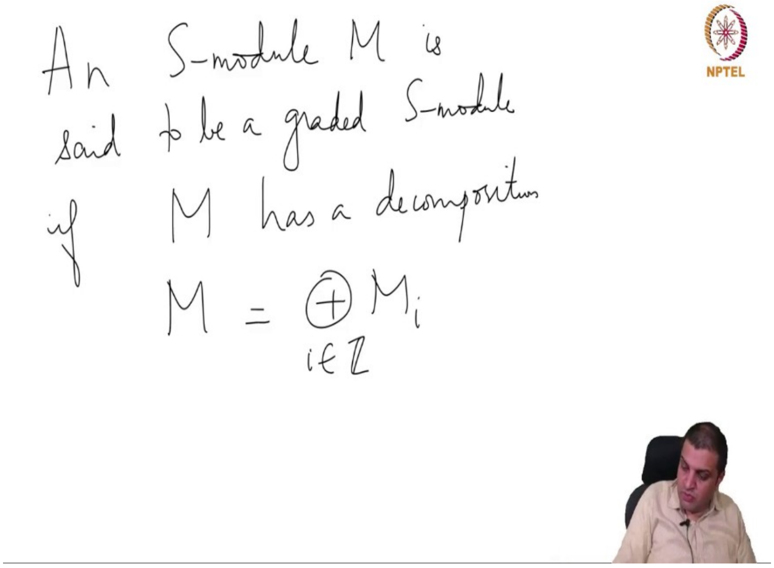
(Refer Slide Time: 29:18)



So, S is a graded ring we will come back to graded rings later after we prove these somewhat technical things and use them extensively with Macaulay tool. But, I right now we just develop enough for us to go through this just to you have to understand how to discuss dimension of Noetherian rings. So, this is a graded ring and what is the other property? $S_i S_j \subseteq S_{i+j}$ for all I, j. If you take polynomials elements of $S_i$and multiply them with elements of $S_j$this is inside $S_{i+j}$for all I and j.

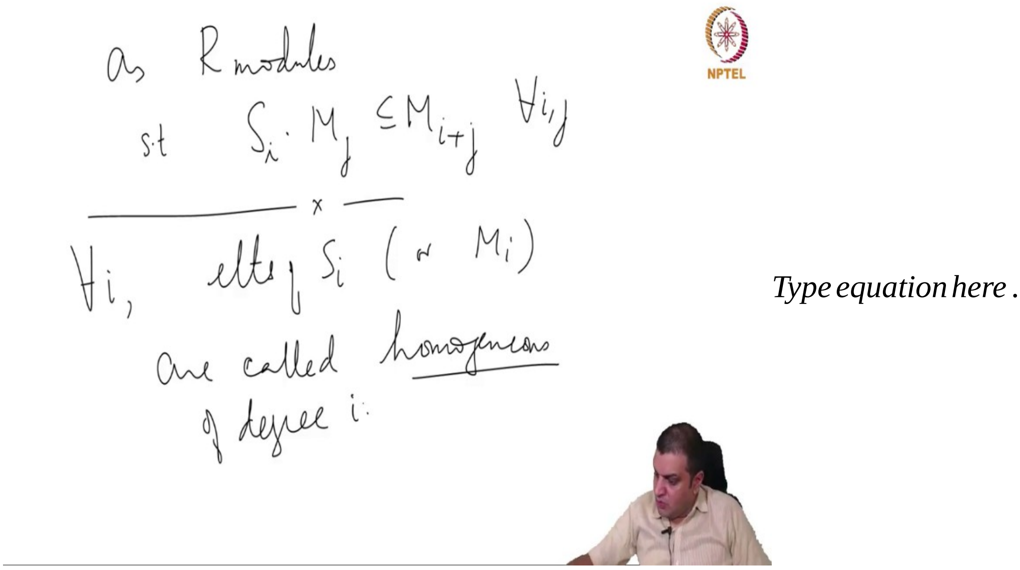
And, we will say that this is Z graded or we will just say non-negatively graded because in this case this is. So, first Z-graded means that the grading is controlled by integers like this N-graded in particular says that it is 0 the graded parts are 0 in the negative degrees. So, that is all.

(Refer Slide Time: 30:38)



Now, an S-module M is said to be a graded S-module if M has a decomposition $\oplus_{i \in Z} M_i$ and here we would allow for all elements in Z all index by integers as. So, remember any S module is also an R module.

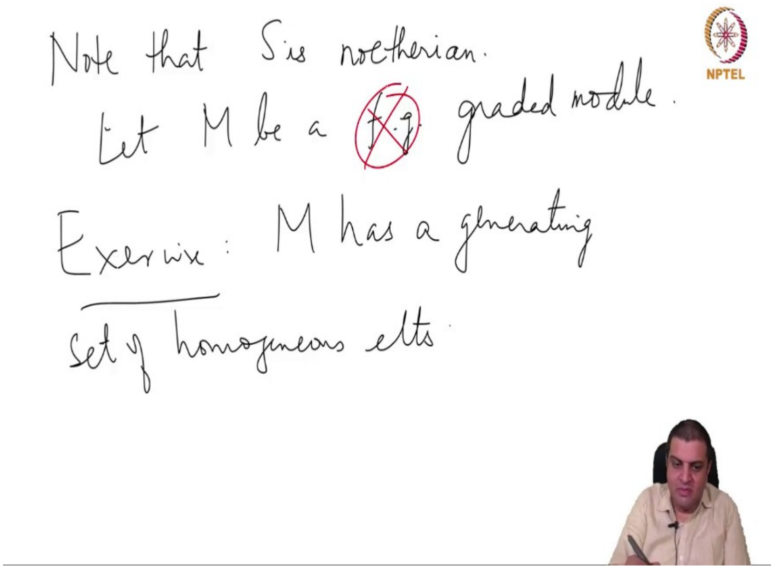
(Refer Slide Time: 31:25)



So, as R - modules such that $S_i . M_j \subseteq M_{i+j}$ for all i and j. So, elements of degree i will multiply elements of degree j to elements of degree and the result would be elements of degree i + j. So, this is what we mean and for any i for all. So, just one piece of notation for all i elements of $S_i$and M_i are called homogeneous of degree i.

I mean technically the union of S_i not the direct sum the union of $S_i$element are called the homogeneous elements and if f belongs to S_i we say it is degree is i, but I hope this is clear homogeneous means it belongs to one degree piece not a sum of two things from two different degree pieces.

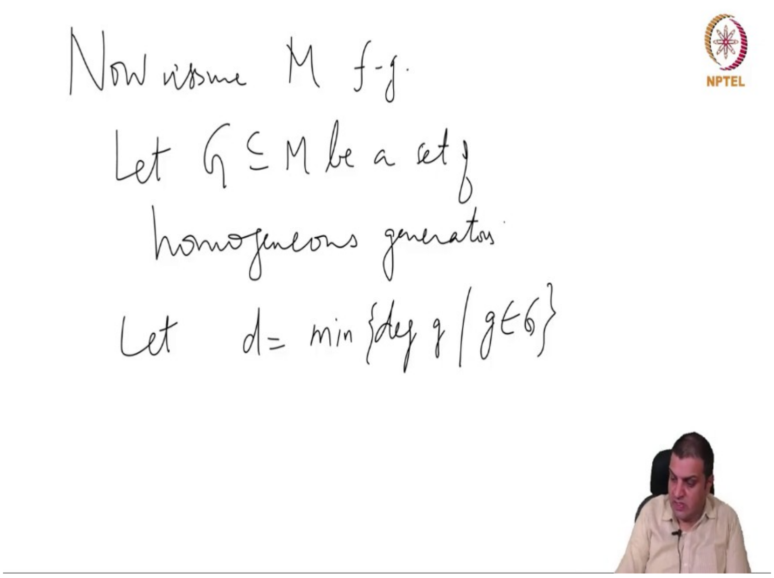
(Refer Slide Time: 32:42)



So, one observation that we want, note that S is Noetherian. Now, we would like to understand what M. So, let M be a graded module. So, then note or maybe exercise.

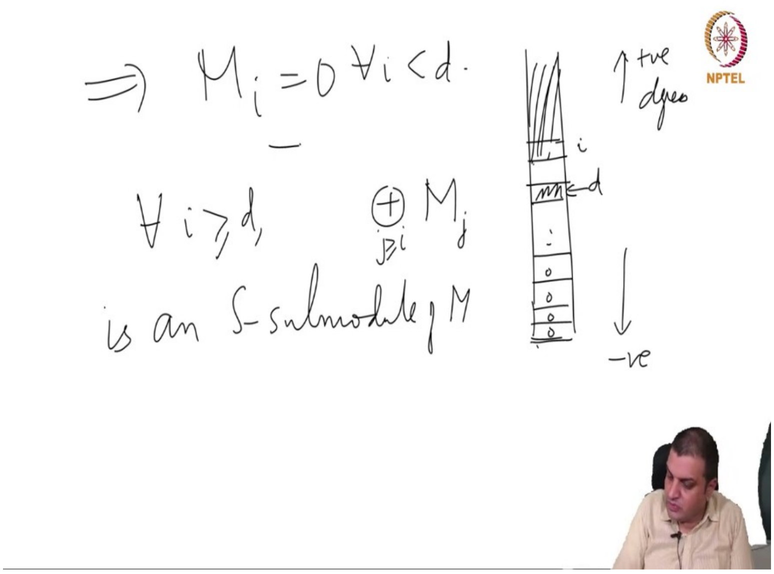
So, just by saying that it is not asserted that it has a generating set of homogeneous elements, but the exercises to show that M has a generating set of homogeneous elements sorry, that does not need a finitely generated you can just you need this just graded module means it has a generating set of homogeneous elements.

(Refer Slide Time: 34:11)



Now, assume that M is finitely generated then there is a least degree of a generating set. So, let G ⊆M be a set of homogeneous generators. Let so, this is finite. So, let d = min $\{deg(g): g \in G\}$. So, among the generators take the among the degrees of the generators take the least number.

(Refer Slide Time: 34:57)

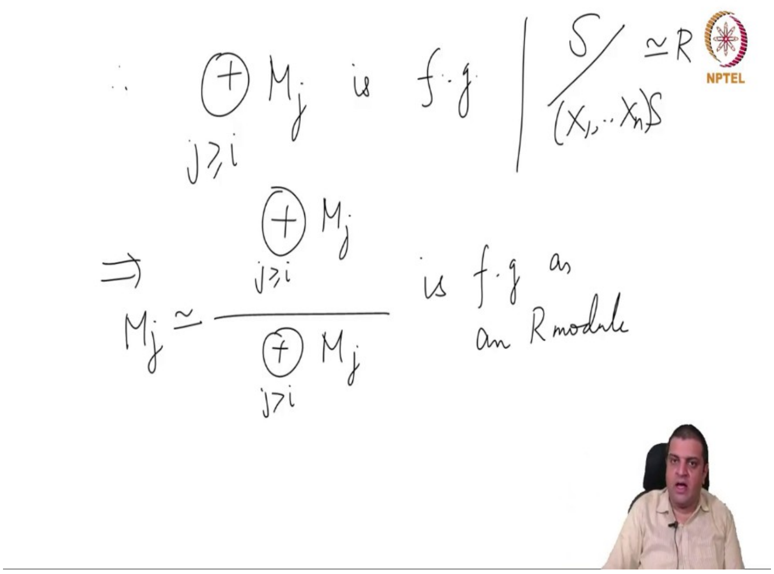


So, now, this implies that $M_i = 0$ for all i < d. So, in so that is the first observation if you have a finitely generated module, then in sufficiently negative degrees the module is 0. So, that is one observation. The next observation that we would like to make is. So, now, by so let us say that so let us take such a d then for all i; for all $i \geq d$, if you look at j direct sum j greater than or equal to i.

So, maybe I will just draw a sketch of what so, there is. So, let us say this is negative degrees, this is positive degrees at some sufficiently negative degree we have 0's then something happens. Then here is M_d. It does not say d is positive or negatives, this is some d. So, here there is stuff.

So, now, take some$i \geq d$. So, take this module $\cdot \bigoplus_{j \geq d} M_j$ So, this is an S-submodule. So, this is a graded submodule of M.

(Refer Slide Time: 36:56)



Therefore, it is finitely generated because S is Noetherian this is modulus finite M is finitely generated. So, which now implies that if you take the $\frac{\oplus_{j \geq i} M_j}{\oplus_{j > i} M_j}$ is finitely generated S module of course, because it is quotient of an S module, but it is finitely generated as an R module.

So, the observation that we want to make is if you take S and go modulo the ideal generated by the X is which is the entire positive part of S this is isomorphic to R. So, if you use that thing we conclude that so, but the what is this? This is just as an R module this is same as M_j. So, if M is finitely generated then in sufficiently negative degrees it is 0 and moreover every homogeneous part is a finitely generated R- module. So, this is an observation that we would need to use.

So, we will continue discussing graded modules in the next lecture and we will prove a theorem about their lengths.