Computational Commutative Algebra

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Lecture – 36
Artinian rings

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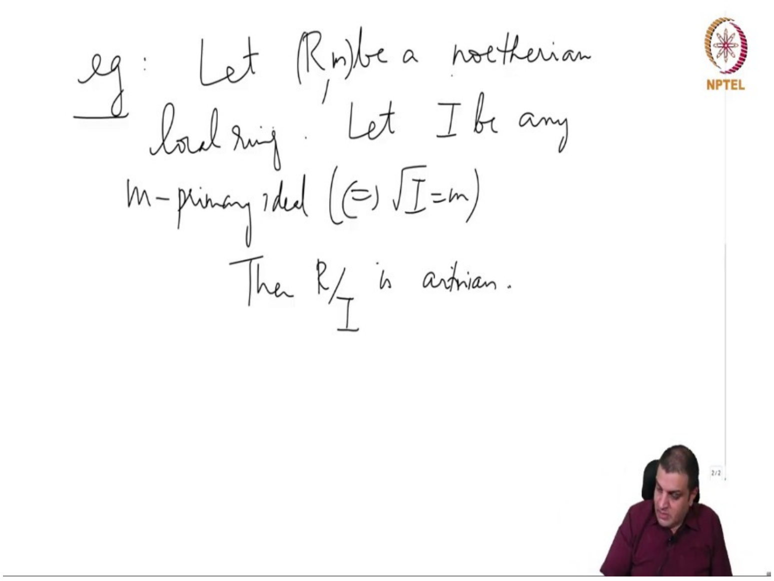

So, we continue our discussion about Artinian rings. So, maybe one more example which we did not discuss last time. Suppose that R is a let us say R is a Noetherian local ring or take any Noetherian ring localizing let us say (R, m). Let I be any m- primary ideal i.e radical of I is m then R/I is Artinian.

That is because if you keep a descending chain of ideals because R is Noetherian some power of the maximal ideal is inside I and if you keep a descending chain of ideals one can so, this requires a little bit of working, but then one sees that it must be inside eventually it will go into larger and larger process of the maximal ideal and hence also inside I and therefore, it would be 0.

So, every descending chain stabilizes we will understand this in greater detail later, but this is just I mean may be just to give an idea and it is because of this. So, that we will have to worry about such ideals and we will see what is called filtration by ideals which is why we have to first understand Artinian rings a little bit.

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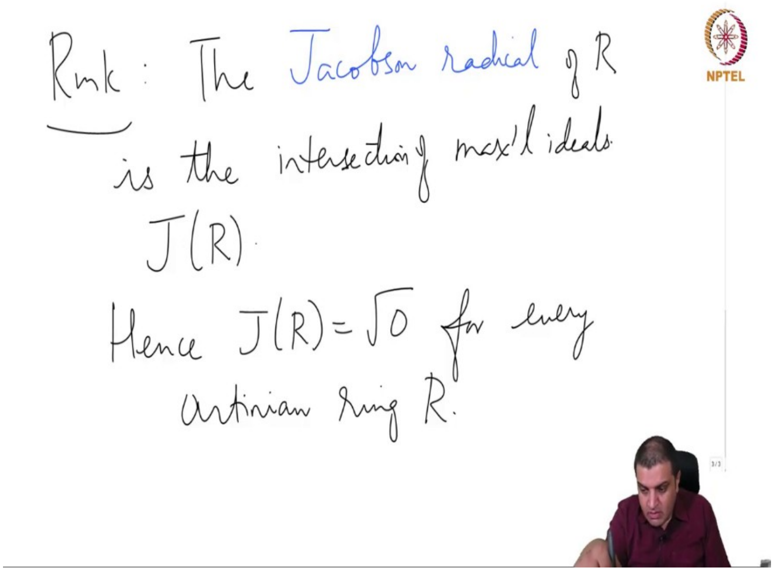

So, remark from the continuing from the discussion from the previous the final proposition that maximal ideal is same as prime ideals in Artinian rings. So, first is the definition, Jacobson radical of R is the intersection of maximal ideals.

So, we will denote it by J(R). So, hence J(R) equals the nil radical for every Artinian ring R. So, this is an observation we will need to use this later, but I mean we will need to use the fact that maximal ideals and prime ideals are the same in any Artinian ring and we will use it in this fashion.

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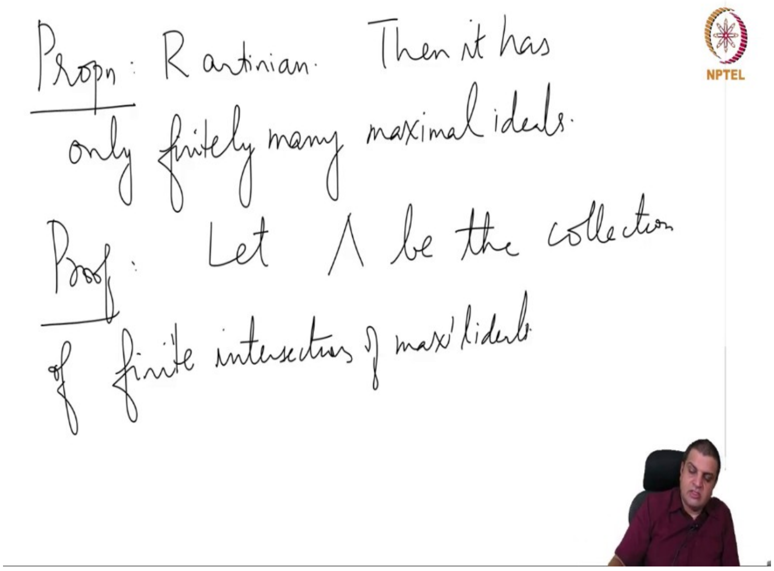

Proposition: Let R be an Artinian ring. Then it has only finitely many maximal ideals. Proof: let ∧ be the collection of finite intersection of maximal ideals. So, take finitely many maximal ideals intersect them that ideal. So, take such things and call it ∧.

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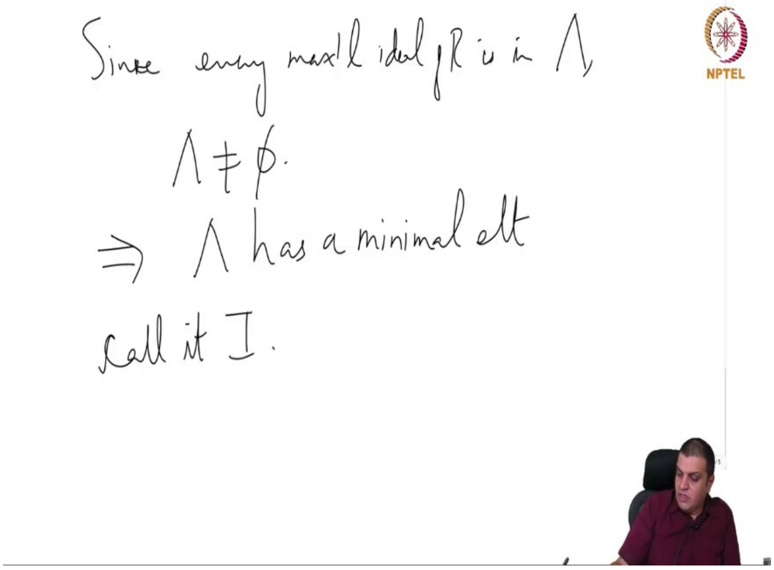

Since every maximal ideal of R is in ∧because this is just trivial intersection of one maximal ideal ∧ which is not empty which means that ∧ has a minimal element. DCC for ideals the same thing as saying that every non-empty family of ideals has a minimal element, call it I.

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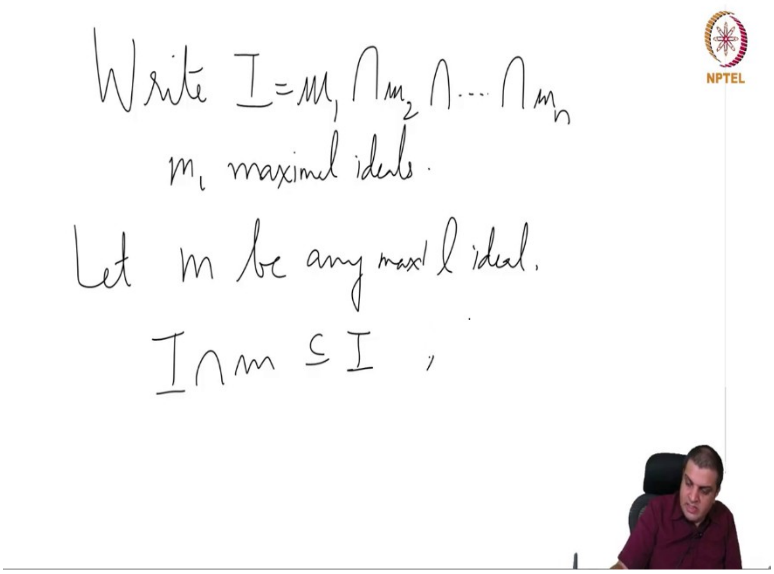

Write I = $m_1 \cap \ldots \cap m_n$, *where* m_i are maximal ideals. Now, let m be any maximal ideal then I $\cap$m $\subseteq$I. So, I$\cap$m is an intersection of finitely many maximal ideals.

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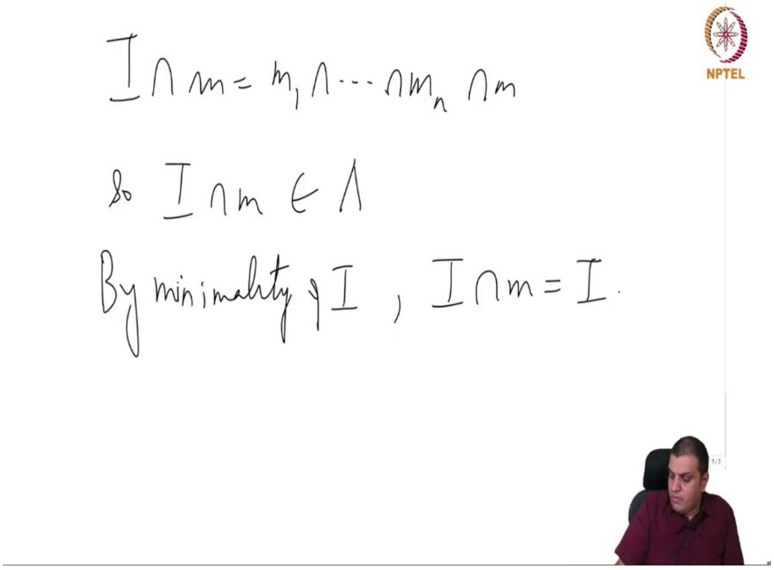

So, I $\cap$m $\in \wedge$.By minimality of I, $I \cap m = I$.

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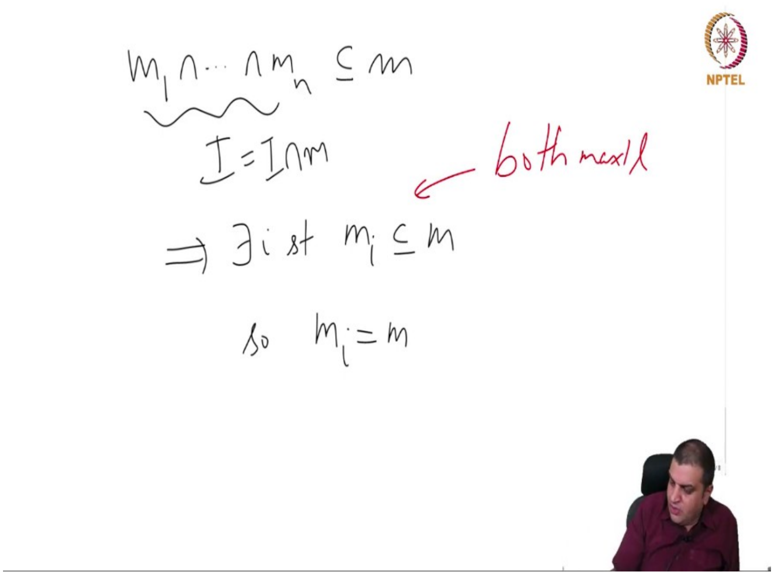

In other words, $m_1 \cap \ldots \cap m_n \subseteq m . I$ = I $\cap$m which is inside m. But, here again same argument a bunch of ideals whose intersection is inside some prime ideal so, this now implies that there exist i such that $m_i \subseteq m$, but both are maximal.

So,$m_i = m$. So, there is a minimal element and only those prime those maximal ideals that appear as an intersect in that intersection expression are the maximal ideals, there is nothing else.

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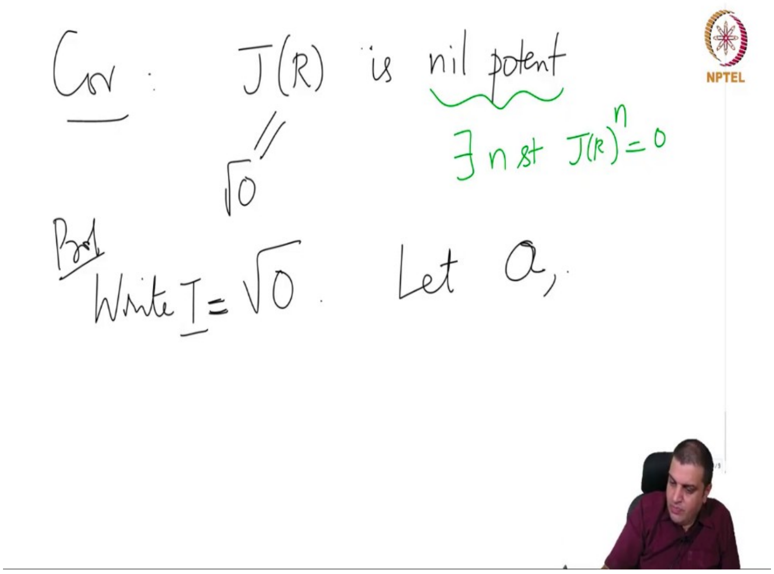

So, corollary of this is that Jacobson radical of R is nilpotent. So, remember this is also the nil radical. So, just there is some content in the statement in the statement because nil radical consists of elements that are themselves nilpotent, but what do we mean by nilpotent? Nilpotent for an ideal what does this mean? So, there exists an n such that $J(R)^n = 0$.

Of course, l because J(R) is equal to the nil radical. For every element there is an n with this property, but we do not know whether any of these ideals is finitely generated. So, by knowing that it is true for every ideal every element of the ideal, one cannot conclude that it is true for the ideal it is ideal itself. I mean at the end it will be true because we will prove that an Artinian ring is Noetherian. So, these are finitely generated, but that is after all of this is proved.

So, let us prove this. So, let us write I. So, let us write I = $\sqrt{0}$. One could just keep writing J R, but it is really the argument that the nil radical is nilpotent is what we are going to prove and write I for this. So, we want to show that there exists some power of I which is 0.

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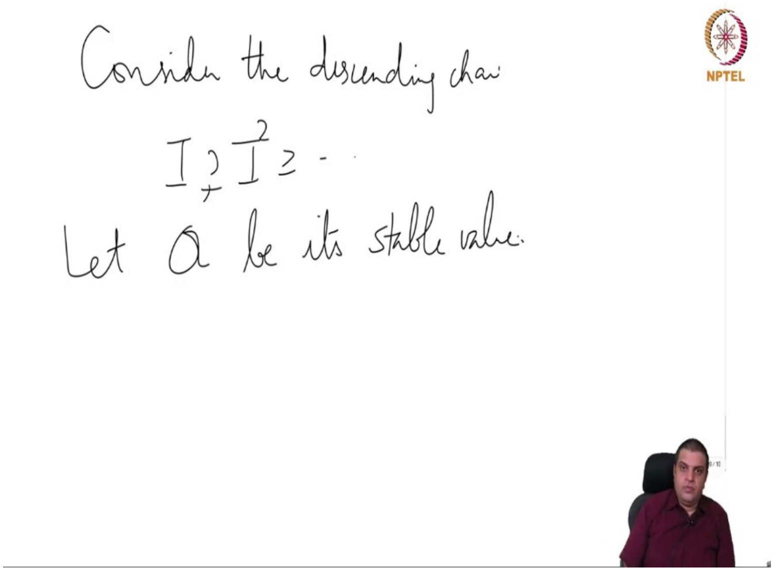

Consider the descending family chain $I \supset I^2 \supset \ldots$. Let Q be its stable value. What is if Q is 0, then we are.

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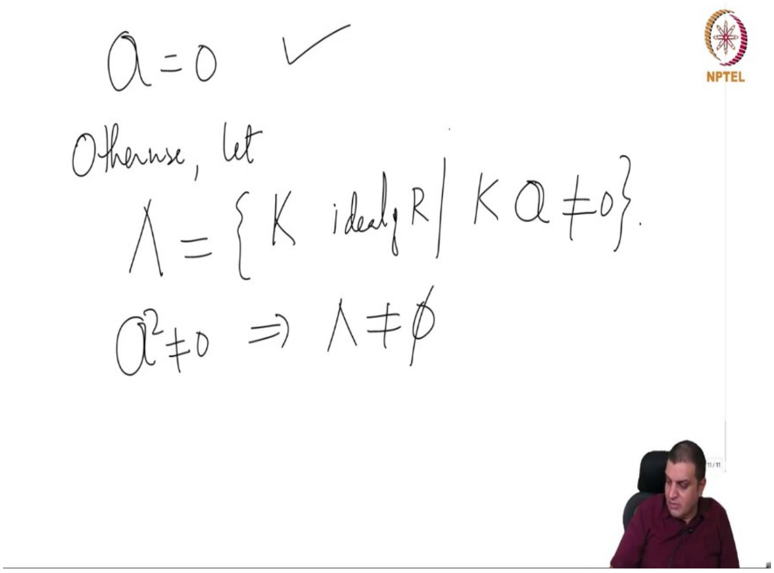

because if Q = 0, then some power of I is 0 that is all that we wanted to prove. So, we want to assume a is nonzero and get a contradiction. Otherwise, let $\wedge$ = {K ideals of R $|KQ \neq 0|$.Note that $Q^2 \neq 0$because if $Q^2 = 0$ means some$I^{2n} = 0$, but then this is the stable value. So, $I^{2n} = I^n$.

So, anyway we will be done. I mean for some large n. So, $Q^2 \neq 0$. So, this implies that $\wedge \neq \varnothing$ right this a itself is in $\wedge$. So, it has a minimal element.

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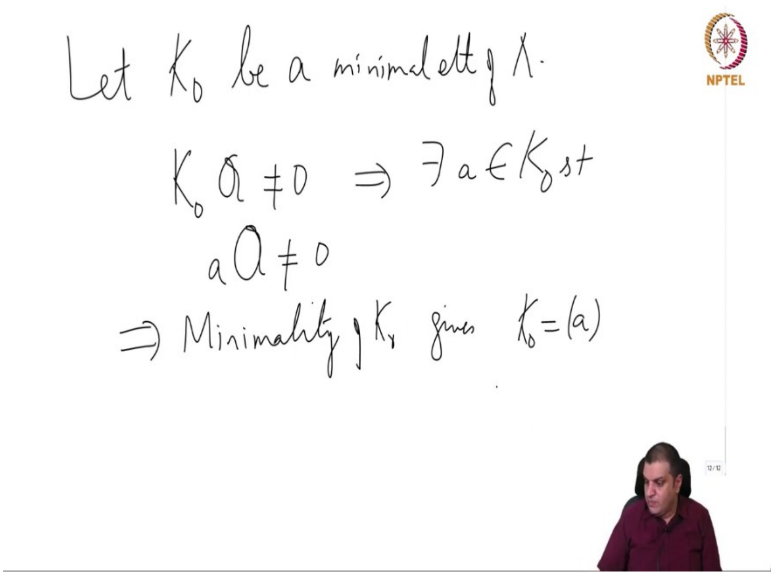

So, let K_0 be a minimal element of $\wedge$. Now K_0Q $\neq 0$ this implies that there exists $a \in K_0 \, s.t$ $aQ \neq 0$ Because if this is true for principal sub ideals of K_0, then it would be true I mean it would not be true for K_0also. So, this by minimality of K_0gives that K_0itself is a principal ideal because a is principal ideal. So, this is some a.

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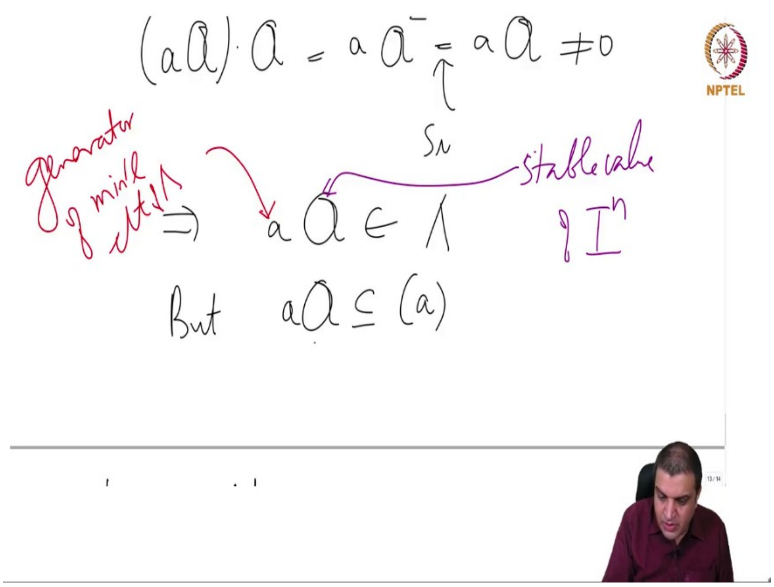

So, now let us look at it this. (a Q) Q = a Q^2 = $aQ \neq 0$. It implies that $aQ \in \wedge$ but $aQ \subseteq (a)$.

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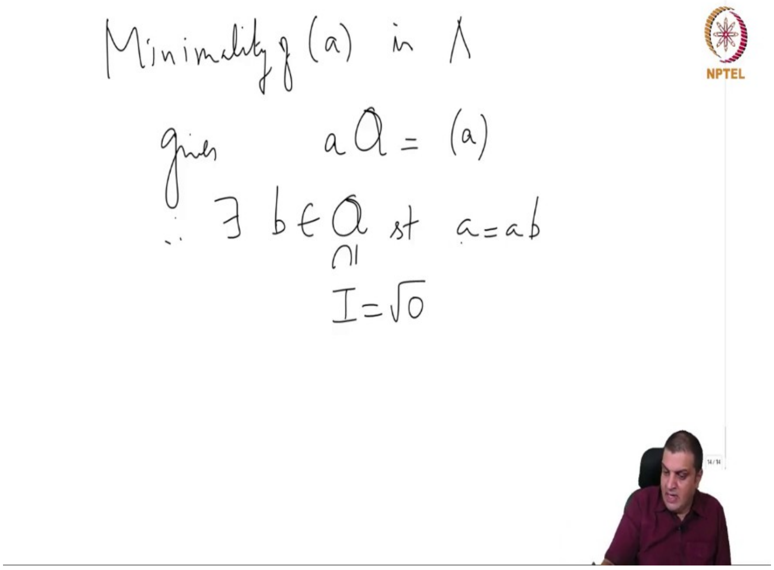

But, aQ is inside ideal a minimality of ideal a sorry, sorry that is not correct not it will a is inside I ideal generator by the element. Sorry, just notation is a little confusing at this point which is what.

So, let us just make sure we let us just recall what these two things are this is the generator of minimal element of ∧ and this is the stable value of I^n. So, this is this is the minimal element in∧. So, this is also inside ∧. So, this is inside here.

So, minimality of (a) in ∧gives that a Q = (a). So, hence therefore, there exist $b \in Q$ such that a = ab. Remember, this is inside the I which is the nil radical. This is the power of I this a is a power of I. So, it is in the nil radical.

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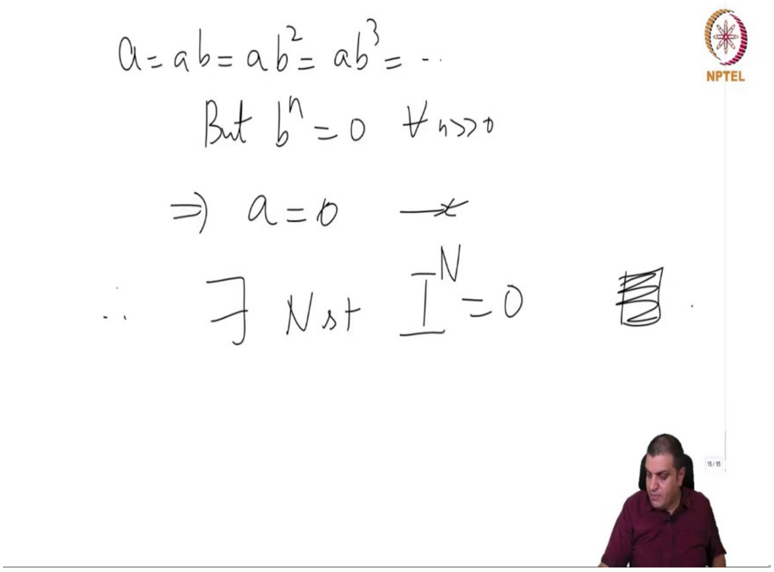

So, now, $a=ab=ab^2=ab^3=\ldots$ But $b^n=0$ for all sufficiently large n which means that a is 0, but that was a contradiction.

So, the contradiction came when we go backwards contradiction came by assuming that this set is not empty. So, therefore, there exists some N such that $I^N=0$. So, now we define something called a composition series. So, this is the same thing one would see in group theory.

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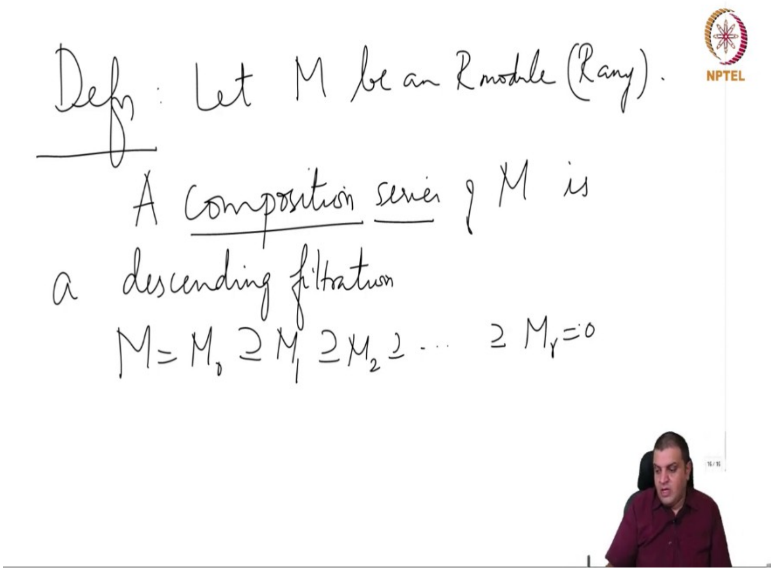

So, definition let M be an R- module, R is not necessarily Artinian just a many ring be an R module R any ring not necessarily Artinian. Composition series of M is a descending filtration $M = M_0 \supseteq M_1 \supseteq \ldots \supseteq M_r = 0$.

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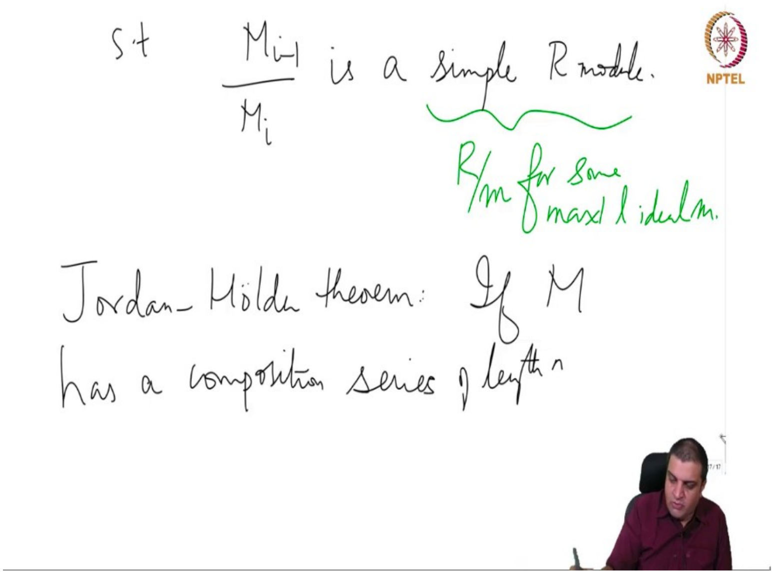

such that $\frac{M_{i-1}}{M_i}$is a simple R- module over a commutative ring the only simple R -modules are quotients by. So, a simple R -module means that there is no proper submodule in it non-zero proper submodule in it and over commutative ring the only simple R modules are quotients by maximal ideals. So, this is really R/m for some maximal ideal m. So, this is so, I mean M need not admit a composition series.

So, then fact this is called Jordan – Holder theorem which we will not prove we will not prove in this course is that if M has a composition series of length n; length here is this number here. So, it starts from 0, length 1 means there is just one inside it and that is a 0 length 2 means in when you go down two steps it becomes 0 and so on.

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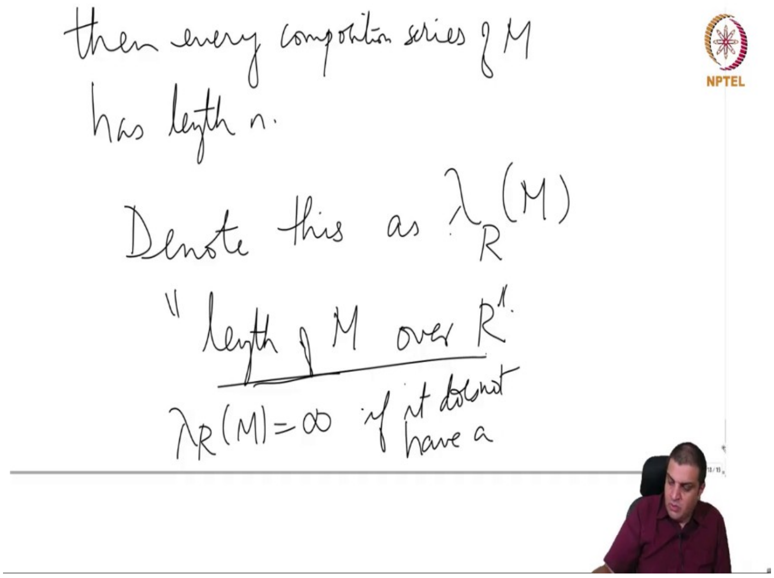

So, as we mentioned it is not necessary that every module has a composition series, but if it has a composition series of length n then every composition series of M has length n.

So, this we would not prove we will just accept this thing and we will denote this number; denote this number n as $\lambda_R(M)$because the quotients have to be simple R modules. So, it is in the thing and call it the length of M over R or as an R module or something like that. So, this is what we will call this thing.

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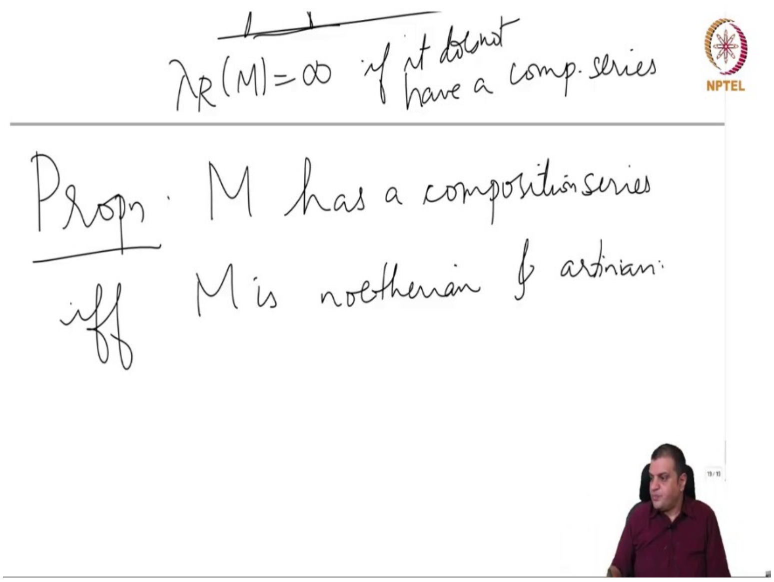

So, here is the here is important property about existence of a composition series. M has a composition series in other words the length. So, if it has a composition series we will denote that length by $\lambda(M)$. If it does not admit a composition series we will just say $\lambda_R(M)$is infinite if it does not have just a notational convention composition series.

We at least we are discussing Artinian modules and rings, we are looking for things that have finite length. So, has a composition series if and only if M is Noetherian and Artinian.

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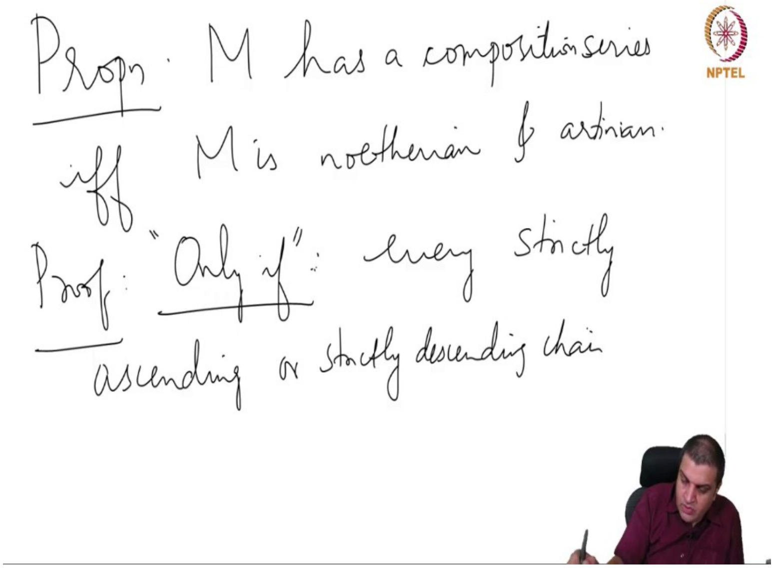

So, proof : So, let us say only if . So, we assuming that it has a composition series well, if it has a composition series then every ascending chain has to strictly ascending chain has to stop somewhere or strictly descending chain also has to stop somewhere. Because when you intersect a chain with the composition series we would just get strictly we if you take a strictly ascending chain and intersect with the composition series we would just get a weakly ascending chain.

But then it cannot be weakly ascending infinitely long if that happens then it becomes stable and similarly for descending. So, every strictly ascending or strictly descending chain of ideals is finite that is because the composition series itself is finite.

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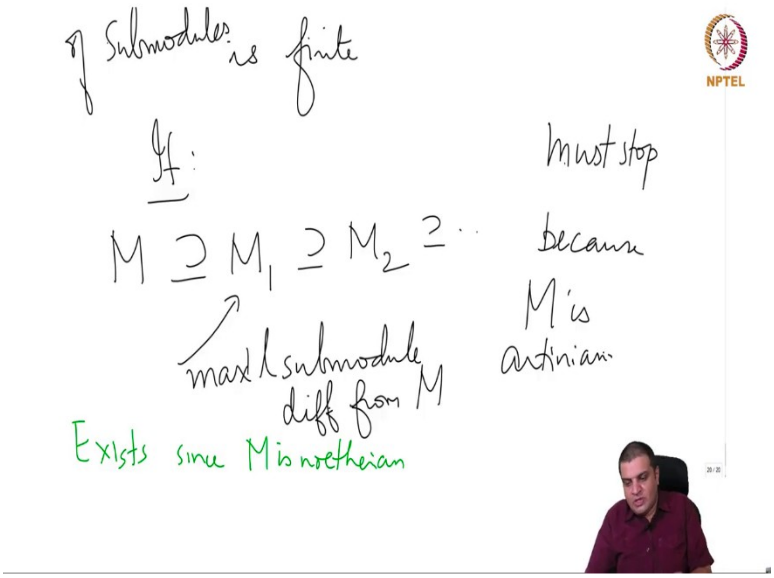

Sorry, not ideal submodules is finite. And, if so, we want to now exhibit a composition series ok. So, this again goes like the proof of existence of maximal ideals in a ring by. So, start with M then M_1 is a maximal submodule different from M different from M such a thing exists because it is Noetherian this exists since M is Noetherian.

Now, M_1 is also Noetherian. So, continue doing this at every stage pick a maximal submodule. So, the quotients will be simple because you cannot put anything else in between. So, quotients would be simple and this must stop because M is Artinian. And, that is and if it stops it must stop at 0 because if it does not stop at 0, you can find a submodule in it and hence a maximal submodule.

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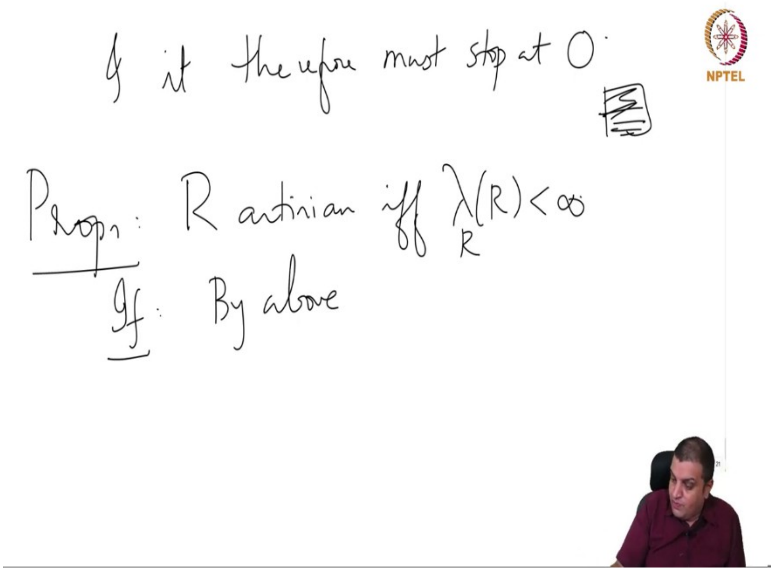

So, and it therefore, must stop at 0. So, therefore, it has a that is the composition series. So, this proves that existence of a composition series is a very special property, and we will the characterization of Artinian rings is that it admits a composition series.

It has finite length as a module over itself which therefore, also proves that they are Noetherian. It is not true for arbitrary modules, there are Artinian modules that are not Noetherian; the Artinian rings are Noetherian rings.

So, proposition R is Artinian if and only if length of R is as an R module is finite. In other words, it has a composition series. So, by above if is by above in fact, by above R is also Noetherian. So, it is this direction that is nu.

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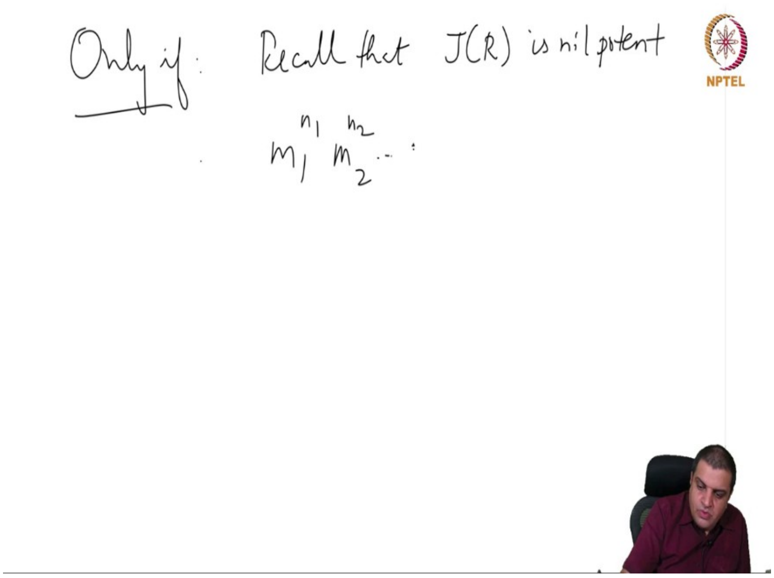

So, only if. So, we so, recall that the Jacobson radical is nilpotent. So, what does that say? So, then so, we can consider the descending chain. So, look at this $m_1^{n_1} m_2^{n_2} \dots m_t^{n_t}$ not in product intersect .

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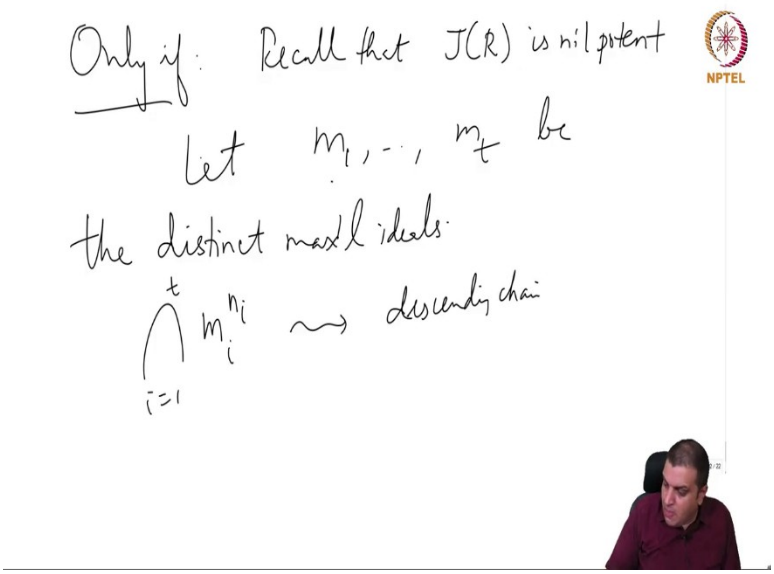

So, as a vary n so, this goes sorry the word are m_1. So, let $m_1, \dots, m_t$ be the distinct maximal ideals finitely many that we already said maximal ideals. So, then we can look at this one $m_i^{n_i}$, but intersect then and as we vary n_i I mean maybe fix all, but one $n_i$and then vary this $n_i$this can give a not all n_i's, but this can give a descending chain.

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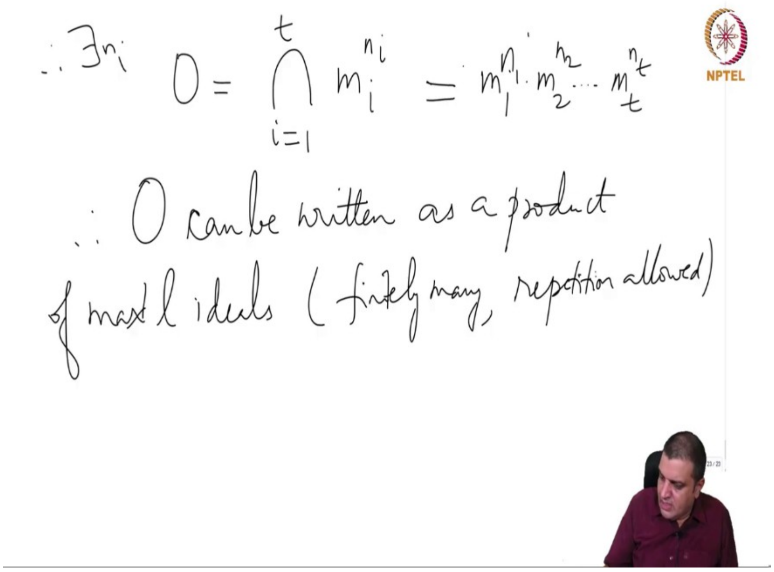

And, therefore, we see that 0 is equal to some intersection of maximal ideals. Therefore, there exists $n_i$such that this is true, but these are pairwise co-maximal so, but say therefore, this is also equal to $m_1^{n_1} m_2^{n_2} \dots m_t^{n_t}$.

So, in other words therefore, 0 can be written as a product of maximal ideals product of finitely many copies, repetition allowed. So, this is an observation that we know about Artinian rings.

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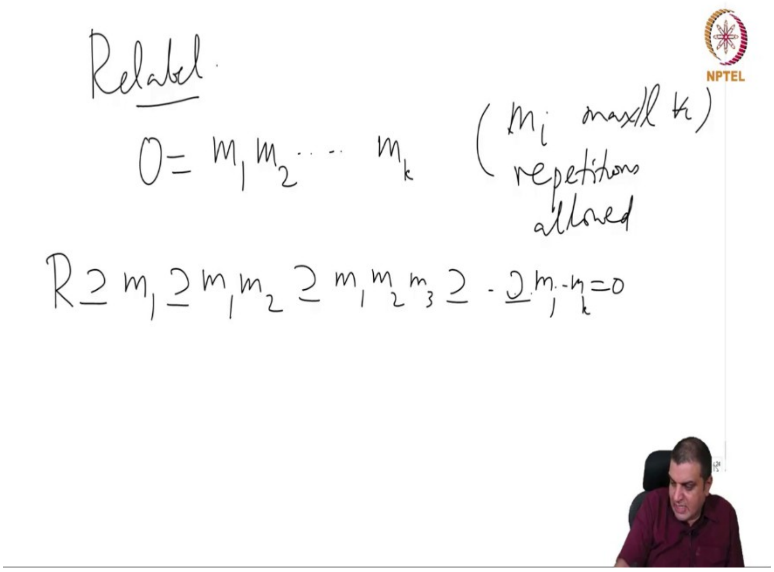

So, now let us take a descending chain. So, it is let us re-label and call them 0 = $m_1 \ldots m_k$. So, repetitions are allowed not m_i is maximal m_i maximal for all i and repetitions allowed. That means m_i is not different from m_j if i is different from j these are not just the distinct maximal ideals.

So, we just allowed relabeling to. So, that is fine. So, now, let us look at this filtration $R \supseteq m_1 \supseteq m_1 m_2 \supseteq m_1 m_2 \ldots m_k = 0$. So, there is this.

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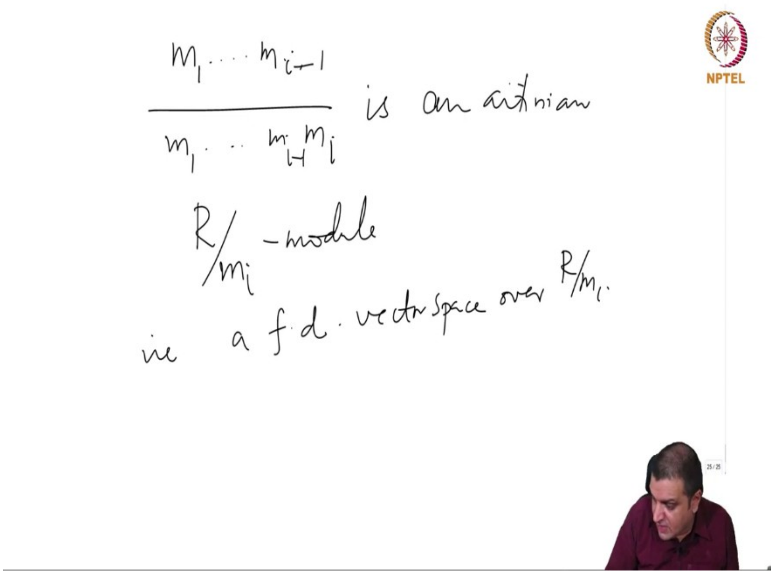

Now, if you take $\frac{m_1 \ldots m_{i-1}}{m_1 \ldots m_i}$. So, this is an Artinian R/m_i module. So, this quotient is a submodule of R modulo this product ideal and R is an Artinian or quotients are also Artinian submodules of Artinian and Artinian.

So, this is an Artinian module. It is an Artinian R module, but it is killed by m_i. So, it is an Artinian module over R/m_i, but this just means that it that is a finite dimensional vector space over R/m_i. This is a field over a field a modulus Artinian if and only if the rank is finite.

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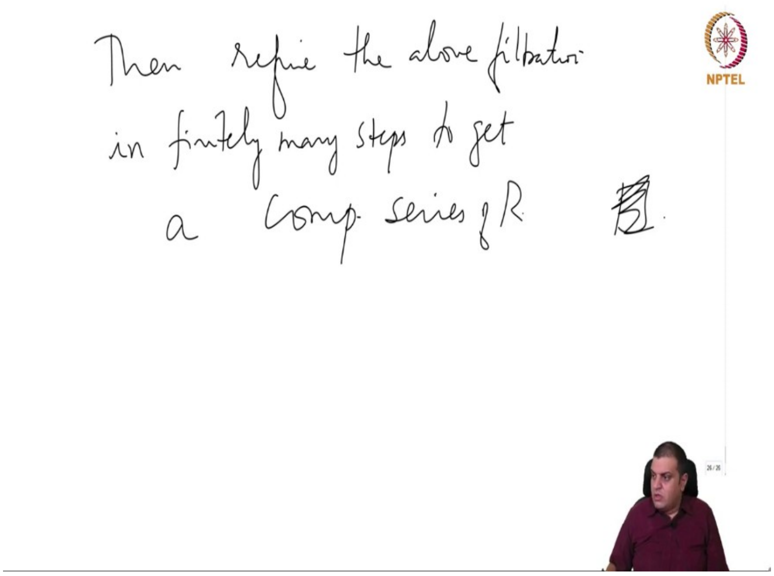

So, then therefore, we can take this filtration. So, then refine the above filtration infinitely means in finitely many steps to get that to get a composition series of R that is.

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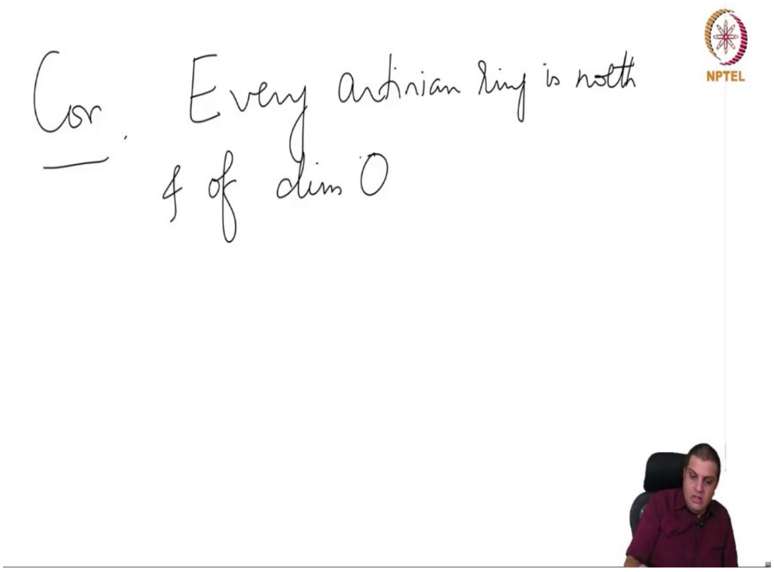

So, we will do one corollary of the statement which is that every Artinian ring is Noetherian that is because the length is finite implies Noetherian and of dimension 0. That is because every maximal ideal is every prime ideal is maximal. There are no chains of primes of length at more than one.

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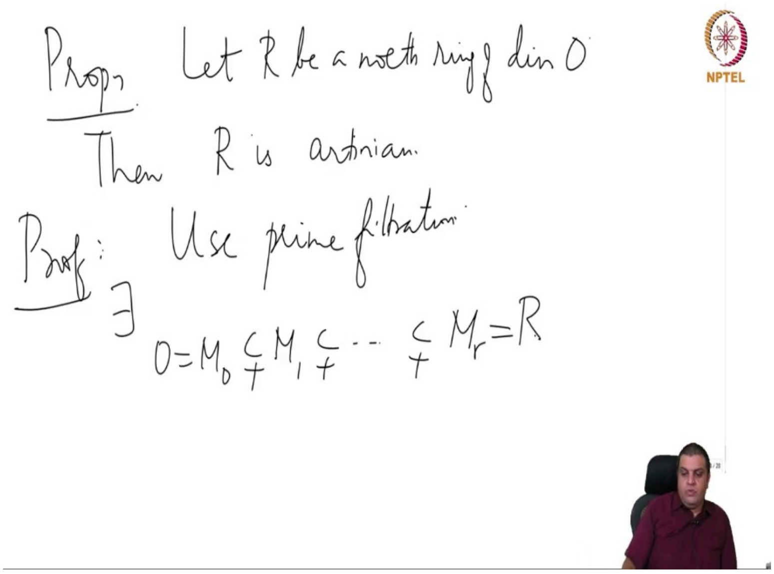

Now, the converse is also true proposition. Let R the Noetherian ring of dimension 0, then R is Artinian. So, proof: we have seen this for Noetherian modules we know that there exists something called prime filtration which we saw earlier ok. So, use prime filtration. So, what is that? There exist $0 = M_0 \subset M_1 \subset \ldots \subset M_r = R$ and these are strict inclusions.

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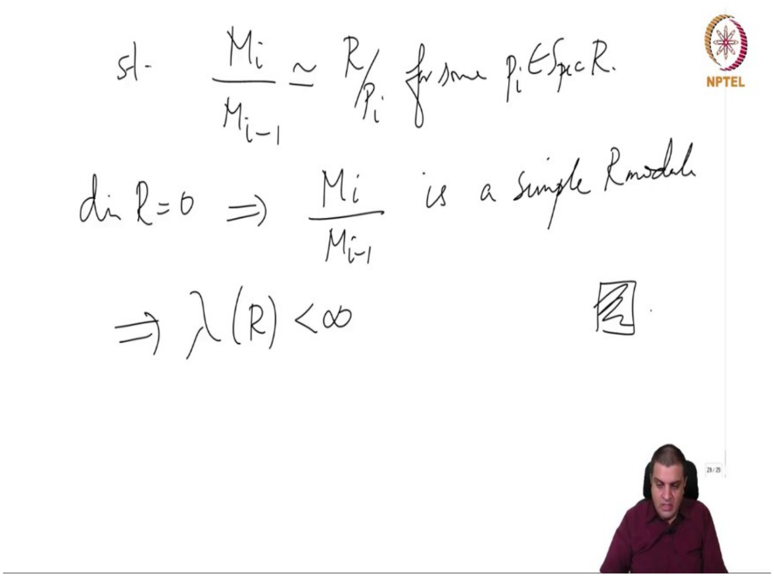

Such that $\frac{M_i}{M_{i-1}} \cong \frac{R}{p_i}$ for some $p_i \in$spec(R). But, dimension of R is 0 that is. So, this uses the fact that R is Noetherian this assertion so far. Now, we use a fact that dimension is 0 which now implies that $\frac{M_i}{M_{i-1}}$ is a simple R -module. These are quotients by maximal ideals which now implies that length of R is finite and therefore, it is Artinian.

So, this is a short discussion about Artinian rings. We will see we will come back and see more examples little later, right now our immediate goal is to understand proof we understand dimension a little bit better. So, in the next lecture we will look at some graded modules and then from there we will develop a notion of a notion of another way of estimating dimension and we will prove that these numbers are the same.