**Computational Commutative Algebra**
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**Lecture – 35**
**Going up theorem**

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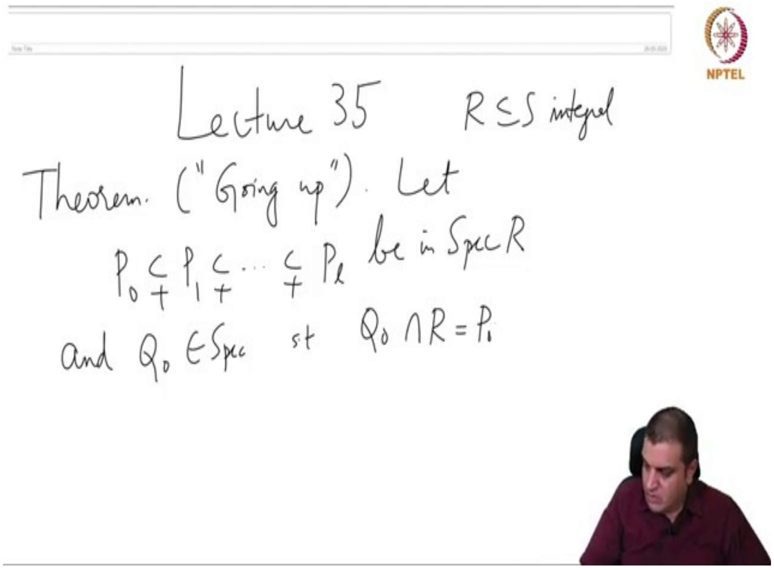


So, we continue our discussion of integral extensions by proving. So, the next statement that we want to prove is the following called the Going up theorem. There is a similar going down theorem also, which we will not prove now. At the end of the course if we have time we may consider it, but that depends purely whether we have time or not. And we will try to develop things that are of use in computational situations.

So, so recall our setup is that, $R \subseteq S$. Let $P_0 \subsetneq P_1 \subsetneq \ldots \subsetneq P_l$ a chain of primes of length $l$ in Spec R and $Q_0$ in Spec S such that $Q_0 \cap R = P_1$.

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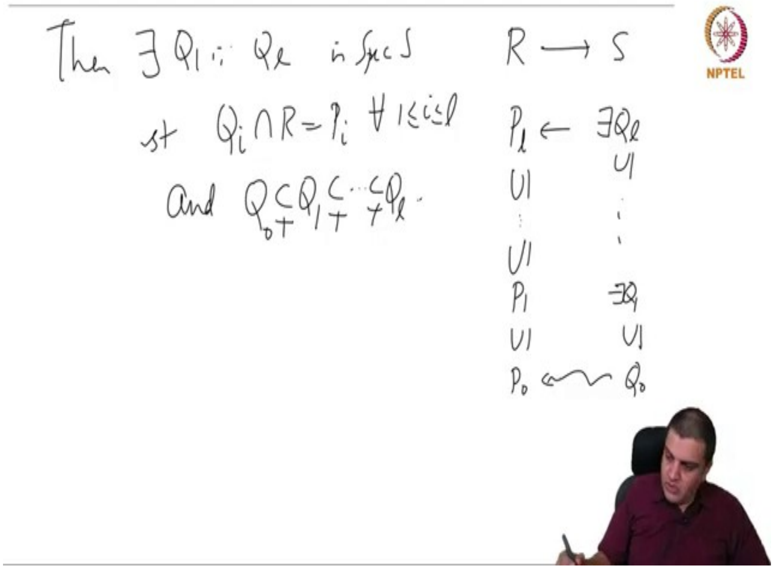


Then, there exist $Q_1, \ldots, Q_l$ in Spec S such that $Q_i \cap R = P_i$ for all $1 \leq i \leq l$. So, we can think of it as follows, there is an integral extension $R \subseteq S$, and then there are primes here contracting to this.

So, in other words the map of Spec, $Q_0$ is in the fiber over $P_0$, then the theorem asserts that, there exist $Q_1, \ldots, Q_l$ such that, this and they form a chain $Q_0 \subsetneq Q_1 \subsetneq \ldots \subsetneq Q_l$. So, this chain goes up as a chain like this. So, that is why it is called a going up theorem.

So, here is a chain of primes, there is a prime which is in the over in the fiber of over this smallest one. Then this chain can be lifted to in Spec S. So, that is what this theorem says.

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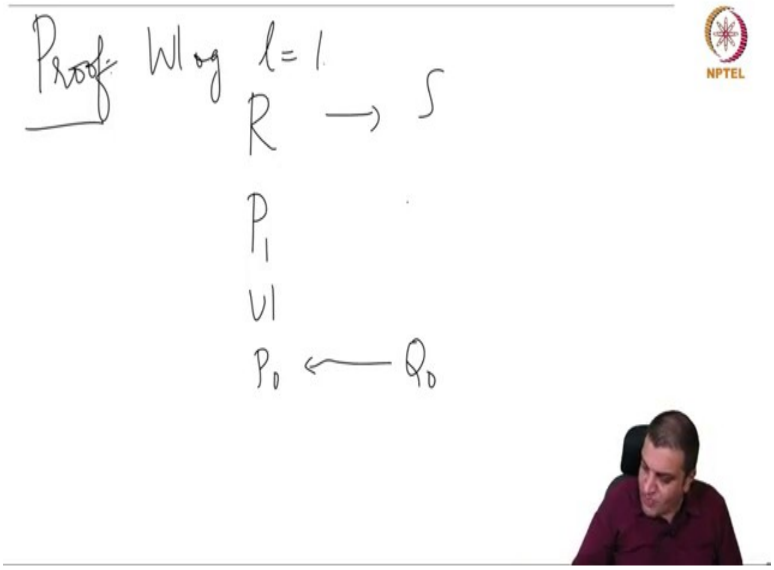


So, let us prove this. So, if we have to prove this we can just prove for $l=1$, because if you can extend this, then we will get a $Q_1$. Then between $P_1$ and $P_2$ apply the same theorem to get $Q_2$, because $Q_1$ has already been obtained. So, now, $Q_2$ has been obtained.

So, now, apply that for to $P_2$ and $P_3$ and so on. So, one would get a chain of primes this way inside Spec S. So, without loss of generality $l=1$. So, this rest can be done by induction. So, this is the picture.

So, let me just repeat the picture that we have here, this is the integral extension. Then there is a $P_1 \supseteq P_0$ here and the $Q_0$ mapping to this $P_0$ contracting to that $P_0$ and we would like to construct something here. So, go modulo $P_0$.

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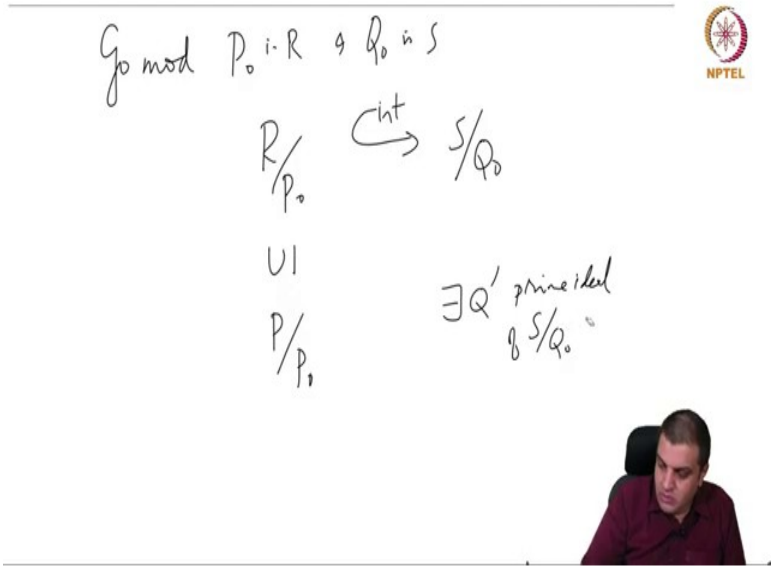


So, then we have $\dfrac{R}{P_0}$ inside here we get $\dfrac{P}{P_0}$ and then we have $\dfrac{S}{Q_0}$. So, we just get, so this is what this is integral extension, that is the point and in this situation there must be something here. So, there exists a $Q^{'}$ prime ideal of $\dfrac{S}{Q_0}$.

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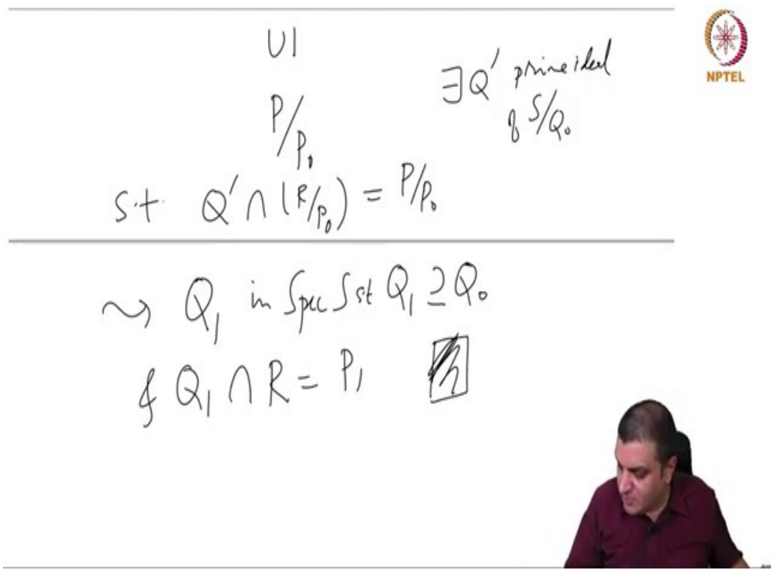

There exist a prime $Q^{'}$, such that $Q^{'} \cap \frac{R}{P_0} = \frac{P}{P_0}$, but then any prime $Q^{'}$ is of this form. So, this one give will give a $Q_1$ in Spec S. So, here it is a prime ideal of $\frac{S}{Q_0}$, $Q_1 \supseteq Q_0$ and $Q_1 \cap R = P_1$.

So, such a $Q^{'}$ will give a $Q_1$ its pre-image would have this property. So, this is the going up theorem. So, now, let us try to understand these things in terms of dimensions.

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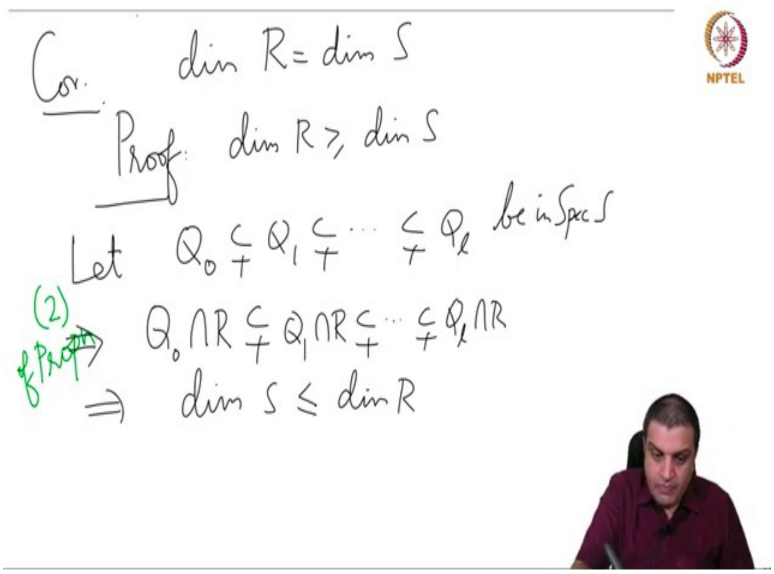


So, corollary, $dim\,R = dim\,S$, again same situation R to S is integral. So, proof. So, we have to prove that supremum of lengths of chains of primes in R is equal to supremum of length of chains of primes in S.

So, we will prove that for each chain here, there is a corresponding chain of a same length on this side and then the other direction and then when we take supremum this would be fine and we will prove the two inequalities this way.

So, let us just first prove that $dim\,R \geq dim\,S$. So, let $Q_0 \subsetneq Q_1 \subsetneq \ldots \subsetneq Q_l$ be in Spec S. So, supremum of such $l$ is what would give us this dimension. This implies that by the incomparability theorem which is part 2 of the earlier proposition, $Q_0 \cap R \subsetneq Q_1 \cap R \subsetneq \ldots \subsetneq Q_l \cap R$ which means any chain that we pick in Spec S, there is a chain of at least that length in Spec R.

So, now this implies that $dim\,S \leq dim\,R$. You can pick any chain here, you we get a chain of equal length here. So, the supremum of all such $l$ will be less than or equal to supremum of lengths of chains of prime ideals inside a Spec R. So, this is this direction. So, this was 2 of the propositions. The statement what we called incomparability.

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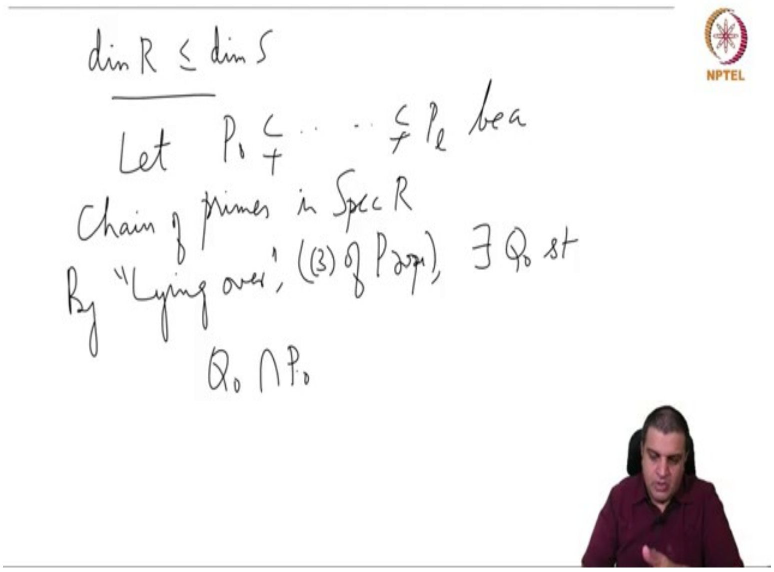


Now for the other direction. So, let $P_0 \subsetneq P_1 \subsetneq \ldots \subsetneq P_l$ be a chain of primes in Spec R, by lying over, this is part 3 of the proposition, there exist a $Q_0$ such that $Q_0 \cap R = P_0$. So, going up does not assert the existence of $Q_0$, it assumes for every $Q_0$ mapping to $P_0$, there is a chain that begins from $Q_0$, it is lying over which say which said there is such a $Q_0$.

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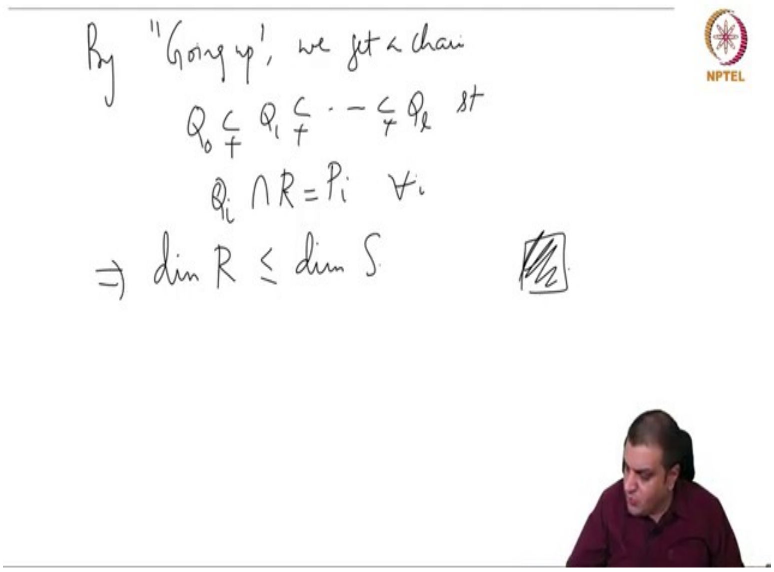


By going up, we get a chain $Q_0 \subsetneq Q_1 \subsetneq \ldots \subsetneq Q_l$ such that $Q_i \cap R = P_i$ for all i. So, now this is the same argument given any chain inside Spec R, there is a chain of at least that length in Spec S. So, this now implies that $dim\, R \leq dim\, S$.

So, therefore, this is equal. So far the only concrete integral extension that we saw on sort of sufficiently general was Noether normalization. So, this one now says that, given a given a finitely generated algebra or a field, there is a map from a polynomial ring finite map.

So, if you look at the map in Spec it is surjective and these two rings are the same dimension and hence we should know a way to find the dimension of a polynomial ring, but as I mentioned earlier it would take us a little bit work to prove that statement, that it is dimension of a polynomial ring over a field equals the number of variables, but we will prove that after we develop a little bit about the dimension theory of Noetherian rings.

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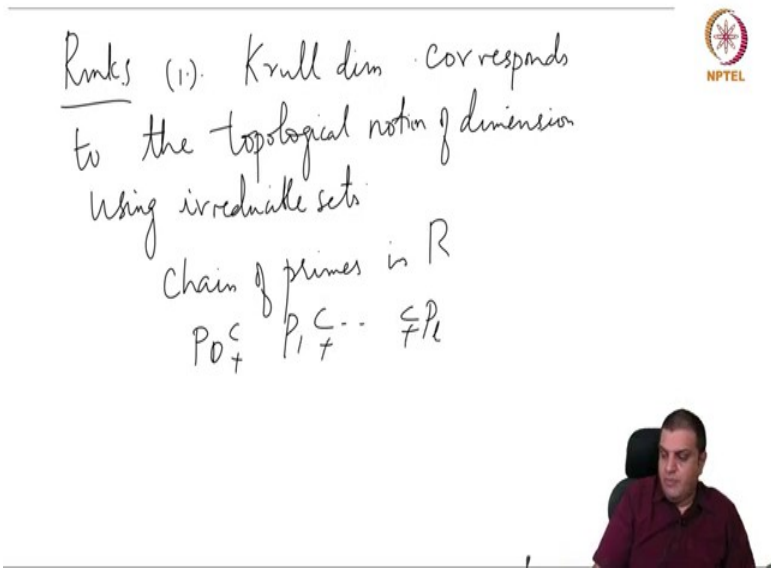


So, just a few remarks. Before we go on to the next, because we have to slightly digress develop various things to prove to study dimension more generally.

So, let me just make two remarks. One, so Krull dimension of Krull dimension corresponds to the notion of dimension for topological spaces to the topological notion of dimension using irreducible sub sets. In other words, a chain of primes $P_0 \subsetneq P_1 \subsetneq \ldots \subsetneq P_l$ in R corresponds to

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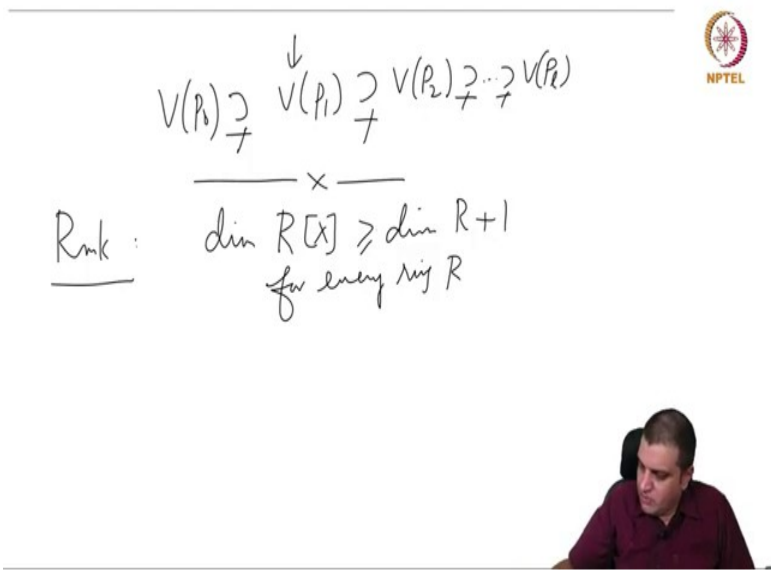

$V(P_0) \supsetneq V(P_1) \supsetneq \ldots \supsetneq V(P_l)$, and this is going to be different because for example, $P_1$ is inside here, but not inside here, other way around, and if we are working if you are thinking about finitely generated algebras over algebraically closed fields, then one can also write $Z(P_1)$ strictly containing $Z(P_2)$ strictly containing $Z(P_3)$ and so on. And these are the irreducible subsets in those topological spaces.

So, giving a chain of prime is the same thing as giving a chain of irreducible subsets in a topological space and this is. So, Krull dimension is really the topological dimension for Zariski topology. So, that is the first statement and the second is a small observation about polynomial rings and we will have to come back to these this later we will have to we will revisit the problem about dimensional polynomial rings a little later.

So, just a remark. Dimension of $R[X]$ at least $dim R+1$ , X is a variable for every ring R. This we will prove, it is not very difficult. The point is that if R is an Noetherian, then this is actually equality, but in order to prove that we need to understand more about the dimension of Noetherian rings. So, let us try to give a proof of this.

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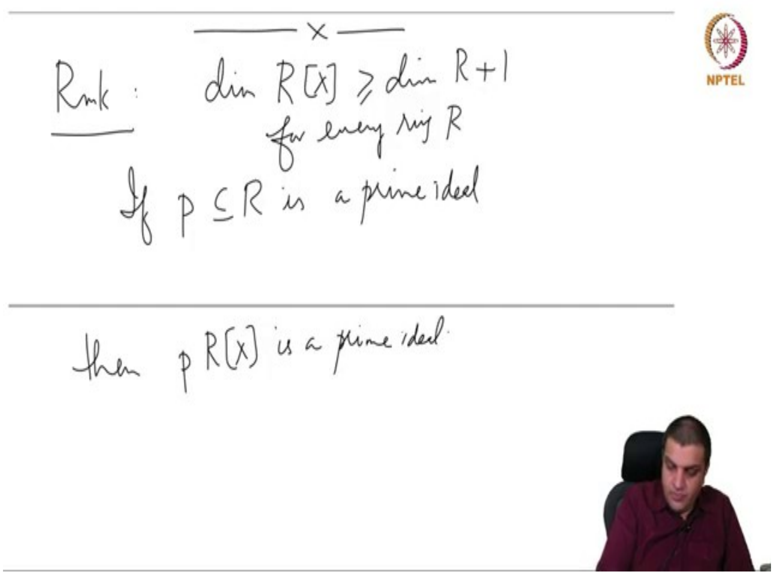


So, if P is a prime ideal of R, then its extension is also a prime ideal .

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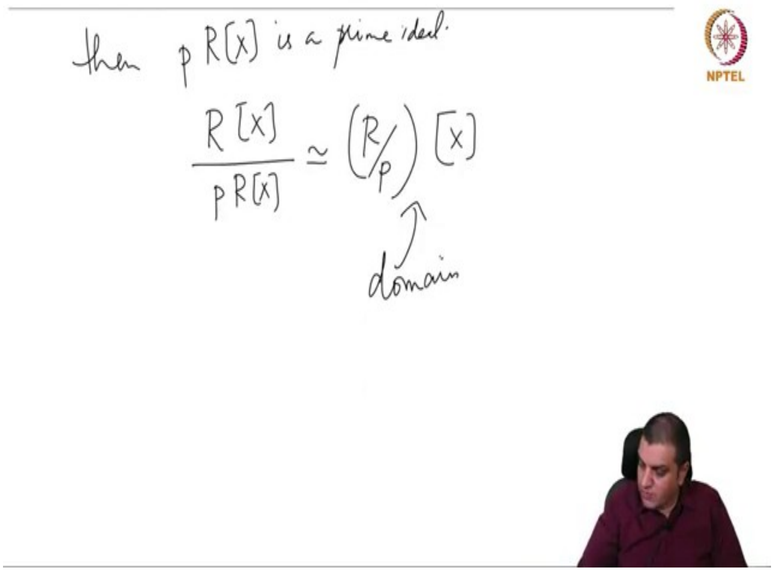

So, in other words, it is may the sense that, it is not very difficult to check that $\frac{R[X]}{PR[X]} \cong \left(\frac{R}{P}\right)[X]$, and this is a domain. If you join a variable to a polynomial rings over domains are domains. So, therefore, this is a prime ideal.

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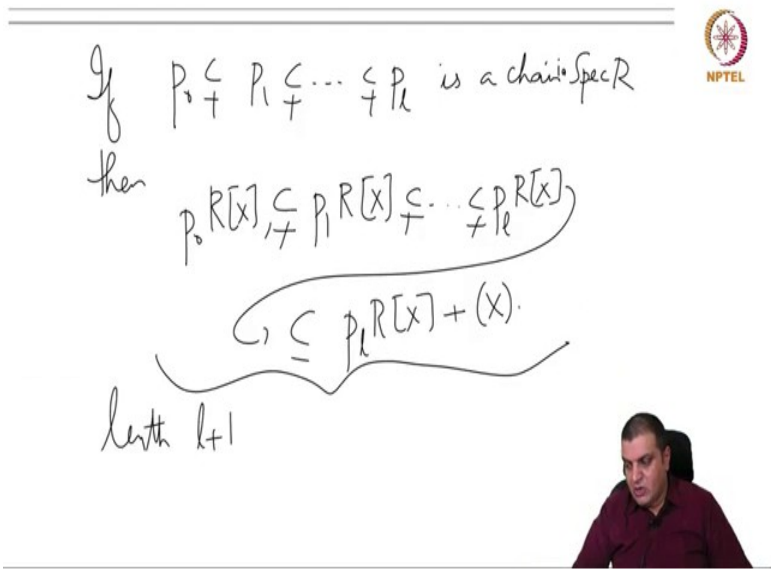

If we take a chain of distinct primes in Spec R, then $P_0 R[X] \subsetneq P_1 R[X] \subsetneq \ldots \subsetneq P_l R[X]$, we can add one more element and make it one larger chain which is. So, we continue it here, you take $P_l R[X] + (X)$.

So, one needs to check that the ideal generated by X is not in this extension, and also that this is a prime ideal. So, let us quickly check that thing.

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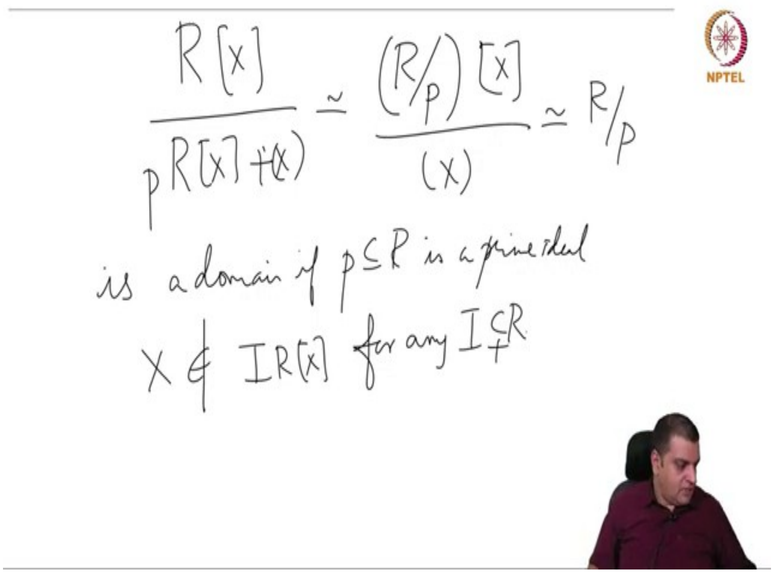


So, $\frac{R[X]}{PR[X]+(X)} \cong \frac{\left(\frac{R}{P}\right)[X]}{(X)} \cong \frac{R}{P}$. Well we can first kill this and then X. This is true for any ideal in particular it is true for prime ideal. So, is a domain if P is inside R is a prime ideal. So one needs to check that $X \notin IR[X]$ for any proper ideal and so, that would tell us that this is really a new prime ideal and here is the chain of length $l$, here is the chain of length $l+1$.

So, therefore, dimension goes up at least by 1 and for Noetherian local rings the dimension is goes exactly by 1. So, this is Noetherian rings. So, this we will prove after we develop some notion about of dimension. So, the next is therefore, what one has to do is to understand is to understand I mean is to prove or study the dimension theory of Noetherian local rings.

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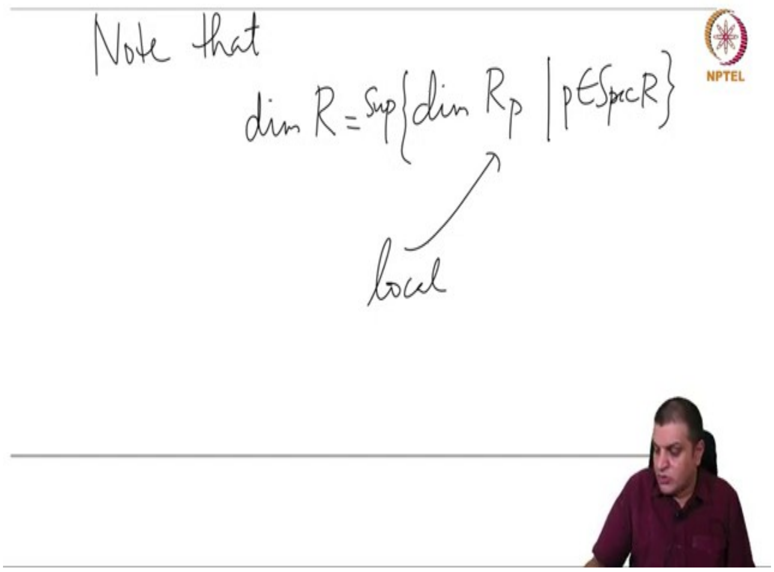


But if you, so notice that, note that dimension of R is, so this is going to this is as a supremum of chains of primes; so, wherever a chain ends at the ok, that that is going to contribute to the height of that prime. So, this is really $dim\, R \; =^{¿\, dim\, R_P \vee\, P \in Spec\, R\}\, ¿}$ ¿. So, we would like to therefore; so, in some sense, it is enough to understand a theory of dimension of Noetherian local rings. So, let us assume this is a local ring. So, we need to only worry about.

So, we are interested in Noetherian local rings and therefore, this tells us that it is enough to know for Noetherian local rings. So, the dimension theory of Noetherian local rings is done was developed by Krull.

So, the main theorem of it proves that, dimension can be determined by as two other numbers, dimension is equal to some number S and some numbers $\delta$, we will not call them S and $\delta$, but we will probably label them late once you get to the theorem and these one of these numbers is obtained by taking filtration of a module.

So, first of all of this instead of worrying about dimension of Noetherian local ring one can just worry about dimension of a finitely generated module over Noetherian local ring, conceptually there I mean proof wise there is no any extra difficulty and some of the arguments are easier to manipulate because one keeps in mind that one is working with modules and not just quotient rings.

So, in order to do this one has to do some filtrations of modules and their quotients and one needs to understand a little bit about Artinian rings and modules for that.

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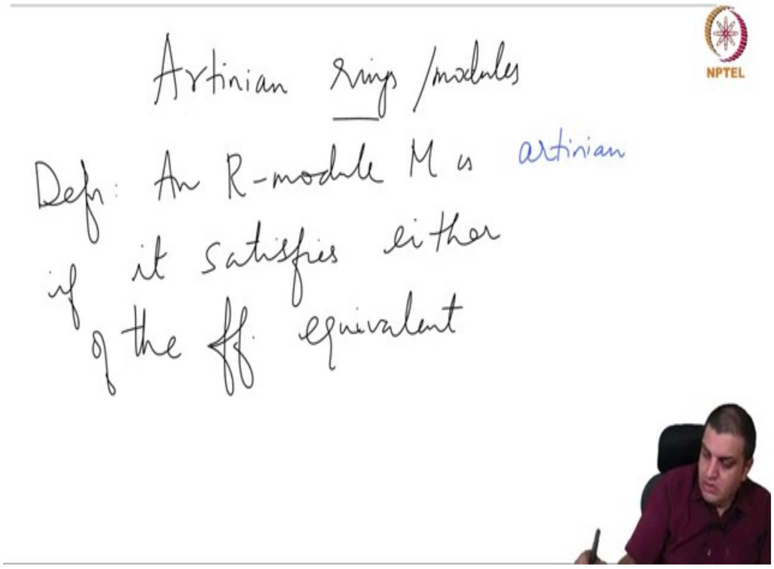


So, let us define that. So, this is Artinian rings and modules, mostly rings. We will just define what an Artinian module is that is all. Definition: An R module R is any ring, commutative ring now, R module M is Artinian, if it satisfies either of the following equivalent conditions.

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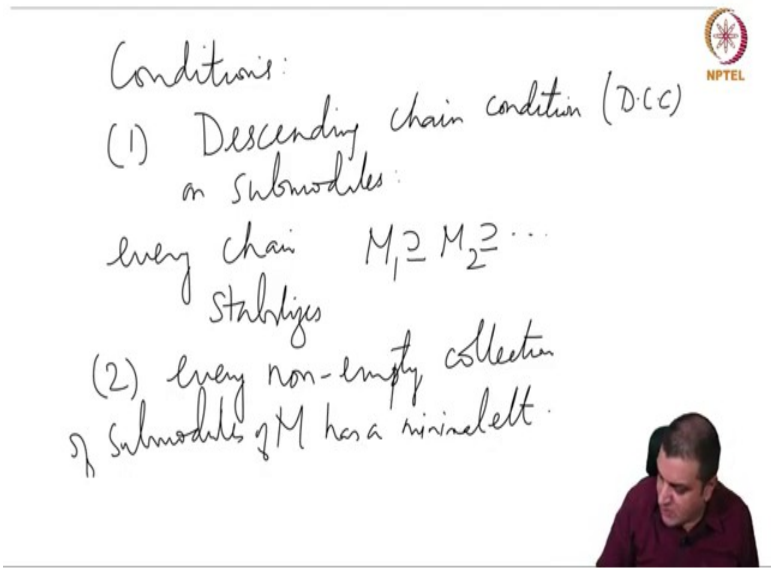

One, what is called descending chain condition on sub modules. So, this is abbreviated as DCC. What does this say? Every chain $M_1$ descending. So, it becomes gets more progressively smaller stabilizes. So, there is some $M_n$ such that there after it is equal to $M_n$.

So, analogous to the Noetherian condition and two every non-empty collection of submodules of M has a minimal element. So, the proof is exactly the same. So these two conditions are equivalent. So, M has to satisfy only one of them, hence it also satisfy the other and the proof that these are equivalent is exactly the proof I mean it is just reversing the containment and maximal with minimal in the proof for ACC from Noetherian modules. So, I will not prove it in this lecture. So, (Refer Time: 26:10).

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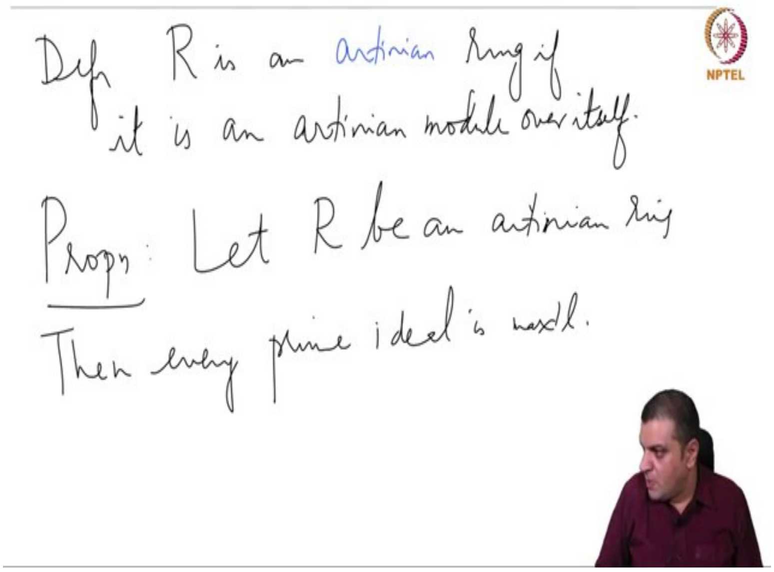


So, this is what is called Artinian and definition, R is an Artinian ring. If it is an Artinian module is over itself.

So, we will end this lecture with a strange observation about this kind of rings, well it is not strange, it is just we have not seen this definition so far, so we have not seen that such things could happen. Let R be an Artinian ring. Then every prime ideal is maximal.

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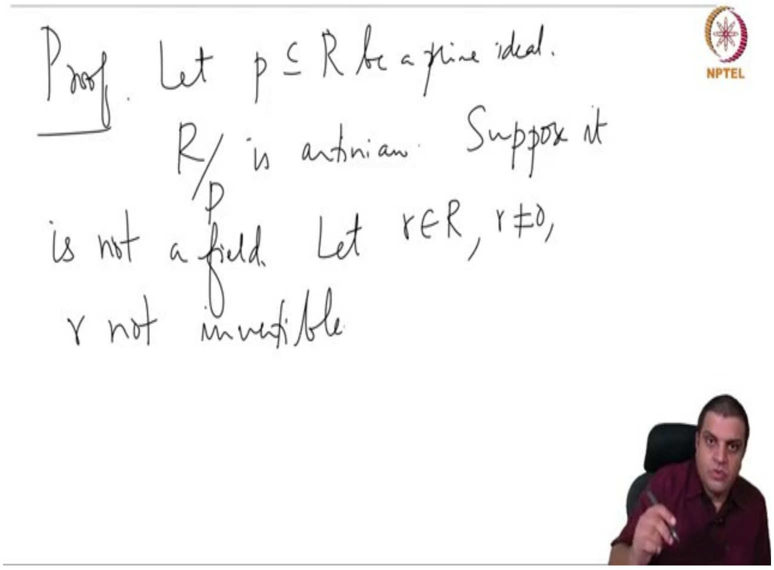


So, proof. So, let $P \subseteq R$ be a prime ideal. Then well, let us consider $\frac{R}{P}$. So, if R is Artinian, then a descending chain condition will also hold for $\frac{R}{P}$, so here it would mean descending chain of ideals.

So, descending chain of ideals in $\frac{R}{P}$ lift to a descending chain of ideals in R and if that stabilizes, it will stabilize inside $\frac{R}{P}$ also or any quotient of other matter. So $\frac{R}{P}$ is Artinian. Suppose it is not a field. Let $r \in R$ and $r \neq 0$, r not invertible.

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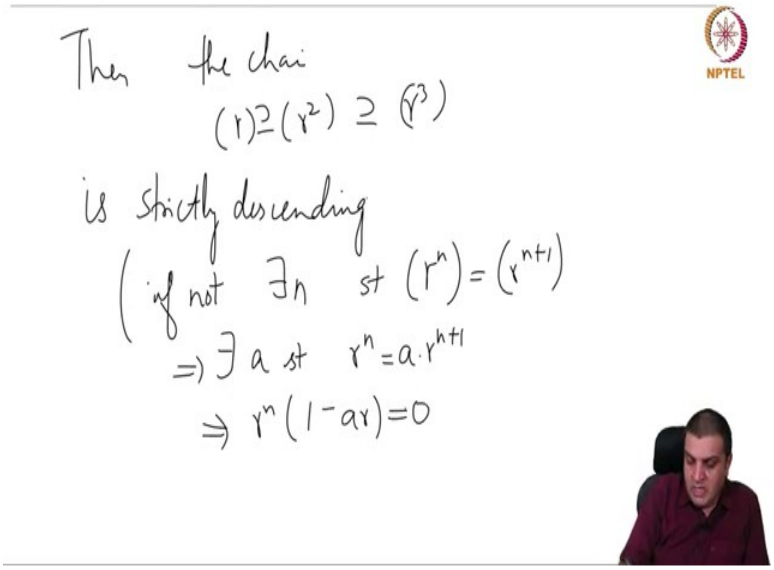


Then, the chain $(r)\supseteq(r^2)\supseteq(r^3)\supseteq\ldots$ . So, it is strictly descending. That is, if not there exist some r there exist n such that $(r^n)=(r^{n+1})$. What does that mean? It means that there exist some $a\in R$ such that $r^n=a\,r^{n+1}$. And now this implies that $r^n(1-ar)=0$ but this is a domain.

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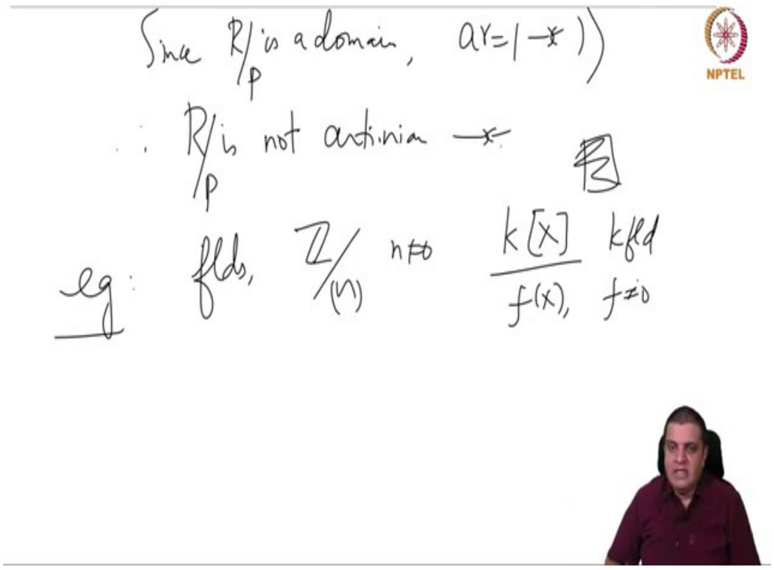


We are working with $\frac{R}{P}$ is a domain. What did we say? So, it is going to be $\frac{R}{P}$. Since, $\frac{R}{P}$ is a domain we will get that $1-ar=0$ or in other words we get $ar=1$, that is r is invertible, but that we assume it was not invertible this is the contradiction.

So, this contradicts, this if not part. So, it is strictly descending therefore, R is not Artinian, $\frac{R}{P}$ is not Artinian, this is a contradiction. This contradicts the assumption that $\frac{R}{P}$ is this.

So, just take some quick look at examples; fields, $\frac{Z}{(n)}$ where n is non-zero, same argument $\frac{k[X]}{f(X)}$, k field $f \neq 0$. So, these are all Artinian rings. $Z$ is not Artinian for the same argument as what we just showed in the proof PID's. polynomial rings, they are in larger number of variables; they are not Artinian. So, this is the end of this lecture. In the next lecture we will continue studying the structure of Artinian rings.