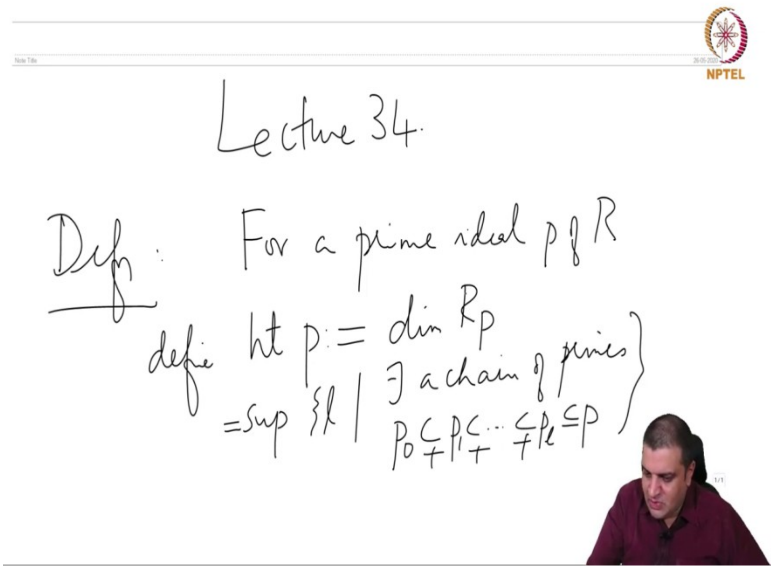
Computational Commutative Algebra
Prof. Manoj Kummini
Department of Mathematics
Chennai Mathematical Institute

Lecture – 34
Polynomial rings

(Refer Slide Time: 00:14)

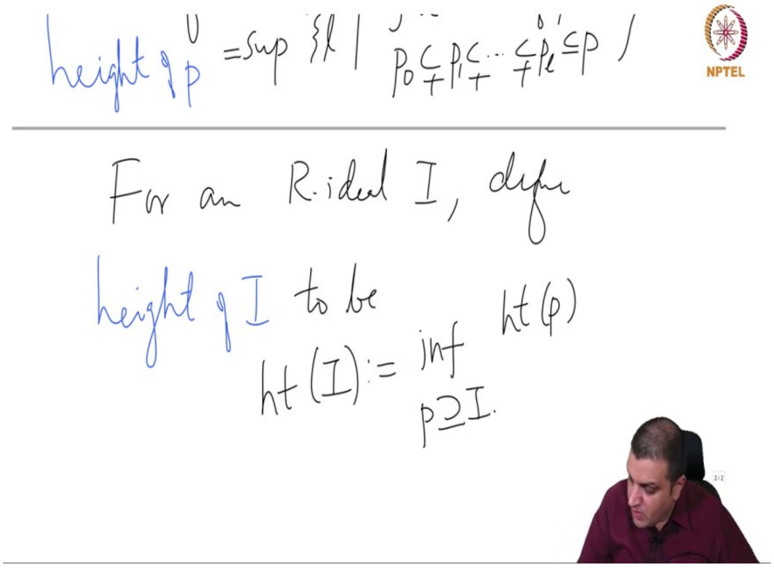


So, in this lecture, we continue our discussion about Integral Extensions, but before that we just need to define one notion of ideals called height. So, for a prime ideal p of R define height of p as $ht\, p = dim\, R_p$. This and what is this?

This is ¿{$l \vee \exists$ *a chain of primes* $p_0 \subsetneq p_1 \subsetneq \ldots \subsetneq p_l \subseteq p$. Here it could be equal and in fact, one should allow for equality at in the supremum.

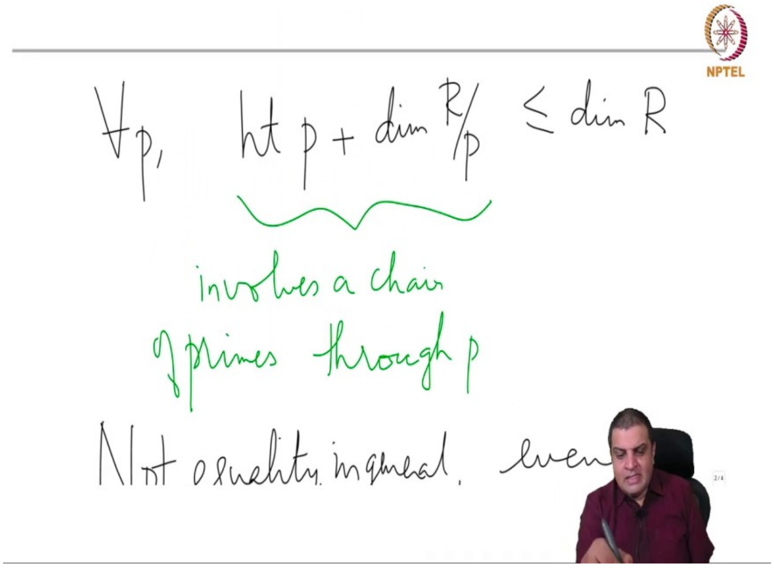
So, that is just because primes inside R_p are precisely correspond to primes with this property. So, these are strict inequalities and here one would allow the last one to be equal to p. Tthis is called height of a prime.

(Refer Slide Time: 01:37)



For an R ideal I define height $ht(I) := \inf_{p \supseteq I} ht(p)$. So, this is the height of an ideal.

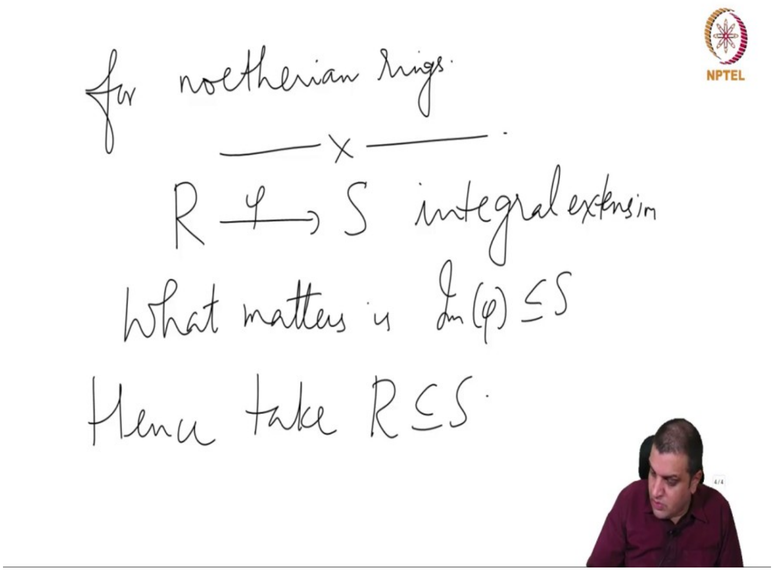
(Refer Slide Time: 02:56)



And, $\forall p, ht\, p + dim \frac{R}{P} \leq dim\, R$. So, why is that? Height of p measures length supremum of lengths of a chain that starts from somewhere below the minimal prime and goes up to p. $dim \frac{R}{P}$ means, well we will start from 0 of $\frac{R}{p}$ and because it is a domain, and then go up to a maximal ideal, but inside R it is same as starting from p and going to a maximal ideal.

So, on the left side this is I mean this involves a chain of primes through p; well, the right side does not take, it could take any chain ring. So, clearly any chain that gets counted in this calculation on the left side also gets counted on the while calculate in the right side. So, we get and this is a supremum so, we get this inequality and it is a fact that if this is not equality in general even for noetherian rings.

(Refer Slide Time: 04:31)



We will not discuss such examples, but it is well known, I mean it is known. There are books describing these examples. So, in fact this is not necessary an equality is one of the problems in trying to prove or that dimension of a polynomial (Refer Time: 04:59) n variables over a field is n. In fact, this is true I mean the equality is true in those rings, but one needs to prove that first or one just be one would just be going in circles.

So, just keep this ok, this is a notion that we need to use when we study ring extensions ideals in general. So, just question about terminology some books might refer to this height as co-dimension, but there is it is a slight misnomer. So, in the sense that it might give us the impression that we are talking about the difference of the dimensions, this is the dimension of some ambient space, this is a dimension of some subspace and we are talking about the difference when you use a word co-dimension.

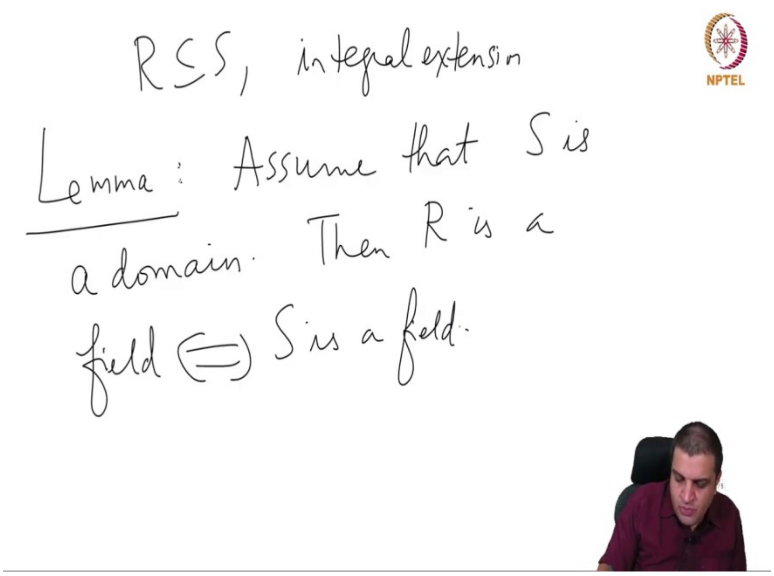
So, we will stick to you with the terminology as height. So, one should just keep this in mind when. So, some books do refer to this as co-dimension, but you know it can be because this it is not equality here they are not the difference between the dimensions is not the same as a height in general. So, that is just the just a definition.

So, now, what we want to understand is if you have an extension of rings R to S integral, well, just one point if you have an integral extension from R to S well we saw from earlier lectures, what R is not really relevant for the problem. What matters is so; let us call this ϕ. So, integral extension nothing to do with I mean in general I mean what integral.

So, what matters will be study things is it is ok. So, what matters is so this is the image of ϕ inside S. I mean whether an element $s \in S$ is integral over R or whether all of S is integral over R; is S finite over R; all those conditions are tested for this image, not for what the ring is.

You could add many many things in the ring and then make put all of them in the kernel it is not going to affect the nature of this map. It is not going to affect the nature of the questions that we study when we look at integral extensions. So, what matters is this. So, hence we will assume that $R \subseteq S$. So, this is what (Refer Time: 07:49).

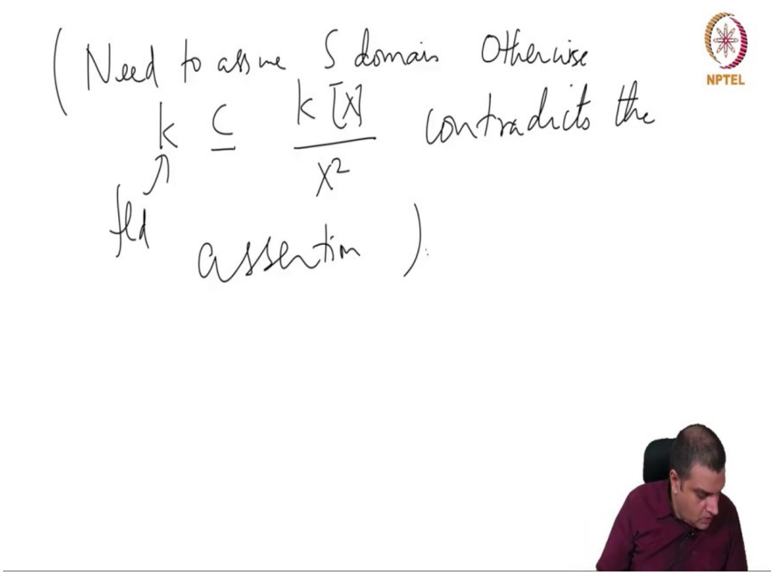
(Refer Slide Time: 07:52)



So, our setup is $R \subseteq S$ integral extension and we would like to understand this map little bit and better. So, here is a lemma and so, this is a topic in commutative algebra where we see certain techniques come I mean with the sense that localize go modulo this and this it is this is one where many arguments one does localize this first or go modulo this prime ideal then prove it and that sort of thing this is one phase where one (Refer Time: 08:37) for the first time quite a bit.

So, hence the following lemma is a sort of the base case for many arguments. Assume that S is a domain, then R is a field if and only if S is a field. So, where we assume that S is a domain.

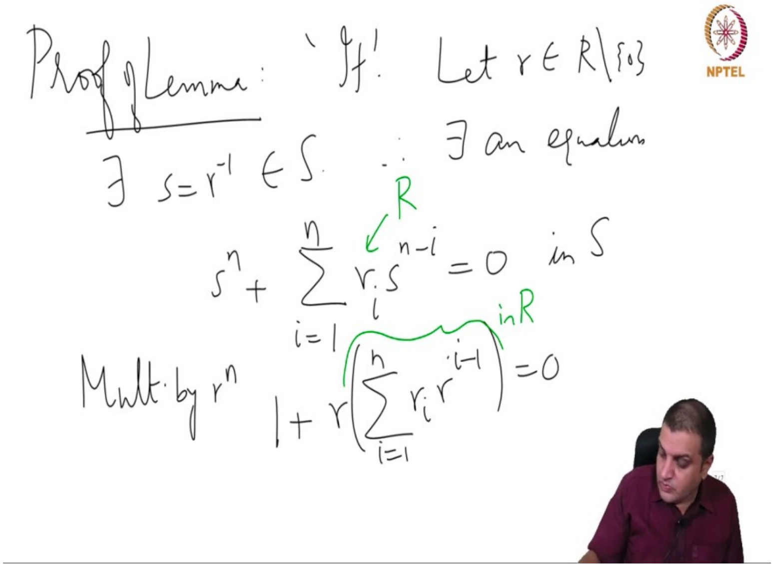
(Refer Slide Time: 09:12)



We need to assume this so, it just a remark need to assume S is a domain, otherwise k this is a field in sitting inside $\frac{k[x]}{(x^2)}$ contradicts the assertion x is integral over it but this is not a field. The reason is we did not assume it is a domain.

So, let us prove. proof of the lemma.

(Refer Slide Time: 10:02)

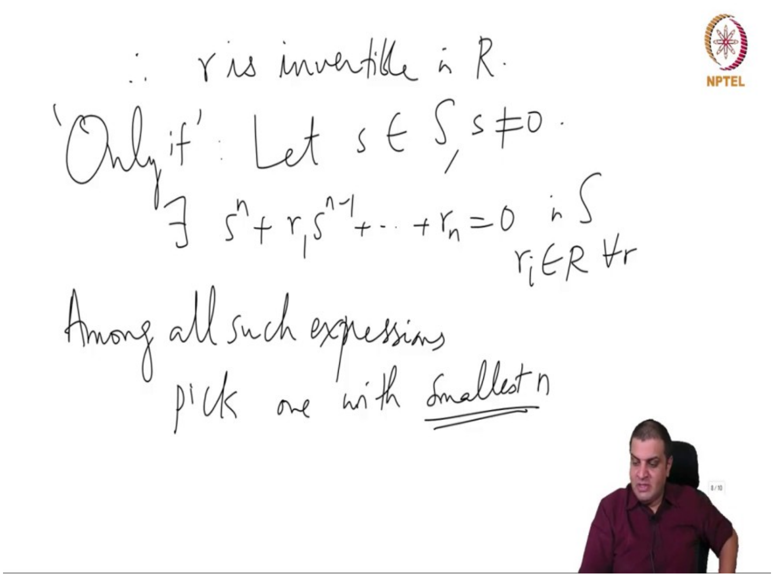


This is the if direction, if S is a domain then R is the domain; if S is a field R is a field. So, let $r \in R$¿$0\}$. There exist $s = r^{-1} \in S$ because S is a field. So, what does that mean? But, this is integral over. Therefore, there exist an equation $s^n + \sum_{i=1}^{n} r_i s^{n-i} = 0$ in S.

So, let us multiply throughout by r^n. $rs = 1$ 1 of S, but R is a sub ring. So, it is 1 of R also. So, this gives 1 multiplied by r^n. So, this term gives $1 + r\left(\sum_{i=1}^{n} r_i r^{i-n}\right) = 0$.

But, then this is an inverse because see $r_i \in R$. So, remember this is inside R that is the integral equation part and this is inside R. So, the sum is inside R. So, this is satisfied I mean this sum is inside R and this. So, 1 plus r times an element of R equals 0. So, therefore, r is invertible. So R is a field.

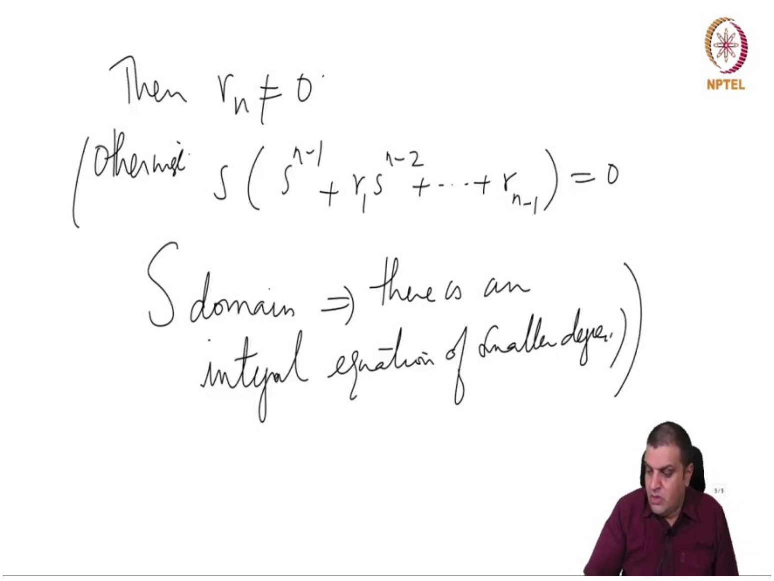
(Refer Slide Time: 12:28)



(Refer Time: 12:38). Now, only if. So, there exist some equation $s^n + r_1 s^{n-1} + \ldots + r_n = 0$ in S $r_i \in R$ for all i.

Now, among all such expressions pick one with smallest n; so, this is important. We pick the one in which the degree of this polynomial is the least that is possible.

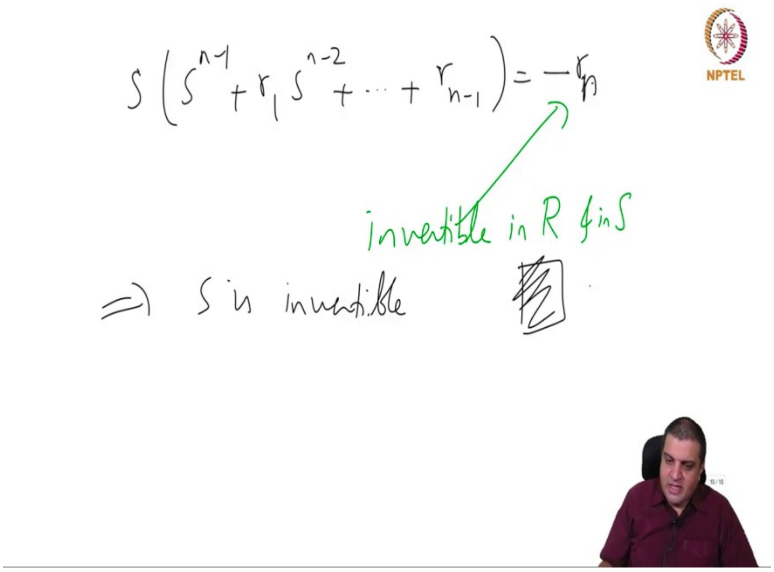
(Refer Slide Time: 13:40)



So, for that one then $r_n \neq 0$. Why? otherwise we can rewrite that as $s(s^{n-1} + r_1 s^{n-2} + \ldots + r_{n-1}) = 0$ but S is a domain and this is where it is used and exactly this is the step which would fail in the example I mentioned above. S domain implies that there is a there is an integral equation of smaller degree. So, this is where domain is used.

So, we can take r_n to this side.

(Refer Slide Time: 14:52)



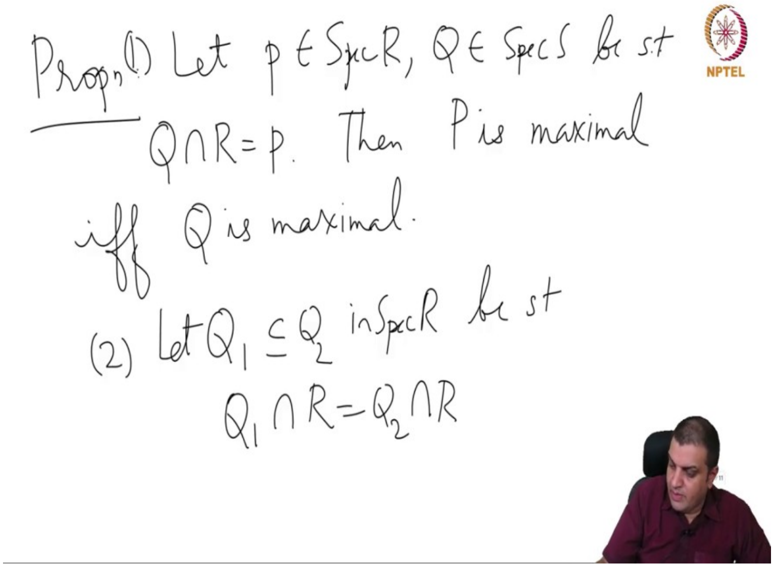
So, this similar way we write $s\left(s^{n-1}+r_1 s^{n-2}+\ldots+r_{n-1}\right)=-r_n$, but this is invertible. So, this is invertible. s times something is a unit invertible in S because if it has an inverse in R that will remain inverse in S also.

So, s times something is invertible, then s times something else would be 1. This implies now that s is invertible. So, we assume s is nonzero to say that this n is at least something. If $s=0$, then you are already done.

And, then this argument will not go through. So, be careful $s\neq 0$. In fact, that is the only correct statement. So, this is every nonzero element is invertible. So, this is the proof, that proof of that proposition the lemma.

So, now we prove a proposition which tells us substantial amount of given an integral extension of rings what does the map on spec do. This following proposition gives us it is a good amount of information about that which is not true for many arbitrary morphisms.

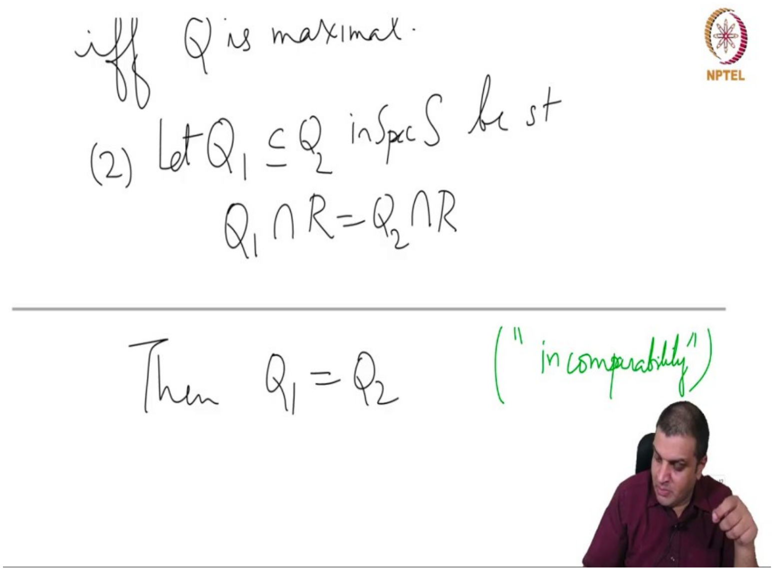
(Refer Slide Time: 17:03)



So, proposition: Let $P \in Spec\, R$ and $Q \in Spec\, S$ be such that $Q \cap R = P$ Q. So, this is the image of the map from Spec S to Spec R the image of Q under that map is P. So, this is point 1. Then P is maximal if and only if Q is maximal.

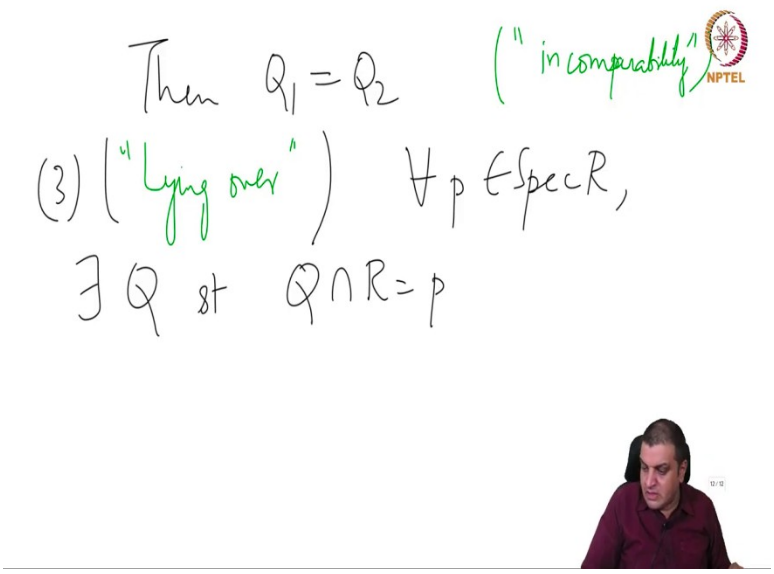
2, Suppose, we have $Q_1 \subseteq Q_2$ in Spec R be such that $Q_1 \cap R = Q_2 \cap R$ that is they contract to the same prime ideal. Then $Q_1 = Q_2$.

(Refer Slide Time: 18:33)



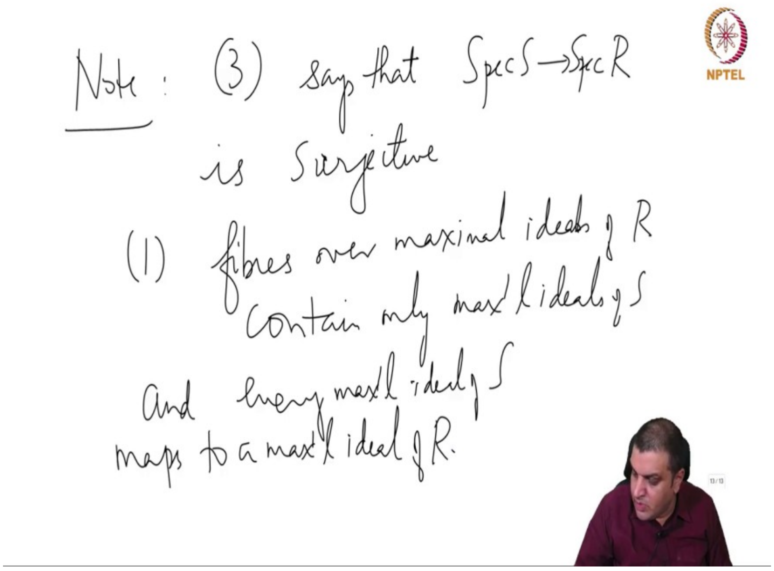
So, this is sometimes some textbooks might refer at least informally as incomparability. Two distinct Q's that contract to the same P below must be incomparable; if one is contain with the other they must be equal, that is what it say. There are two Q's in Spec S two primes in Spec S which map to the same point below with this property that they are comparable, then they must be equal. So, this is sometimes called in comparability.

(Refer Slide Time: 19:21)



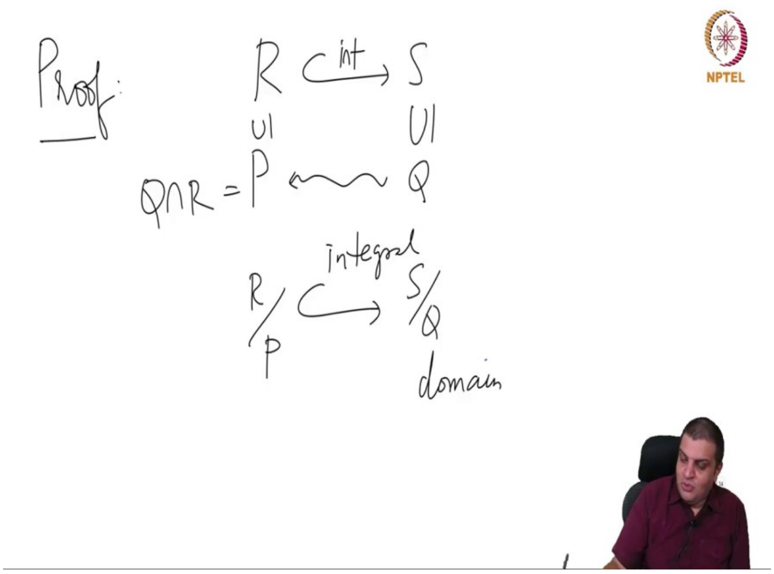
And, 3; this is sometimes called lying over, $\forall\, p \in Spec\, R$, $\exists\, Q$ such that $Q \cap R = P$. So, let us just briefly look at this proposition before we prove it. Proof is not very difficult.

(Refer Slide Time: 20:07)



You just note, 3 says that the map $Spec\, S \to Spec\, R$ is surjective. So, the finite surjective and 1 says that fibers over maximal ideals of R contains only maximal ideals of S and every maximal ideal of S maps to a maximal ideal of R. So, this one can think of this as a so, this one for example, would apply in the case of Noether normalization. So, this is one thing to so one nice property about these maps.

(Refer Slide Time: 21:41)

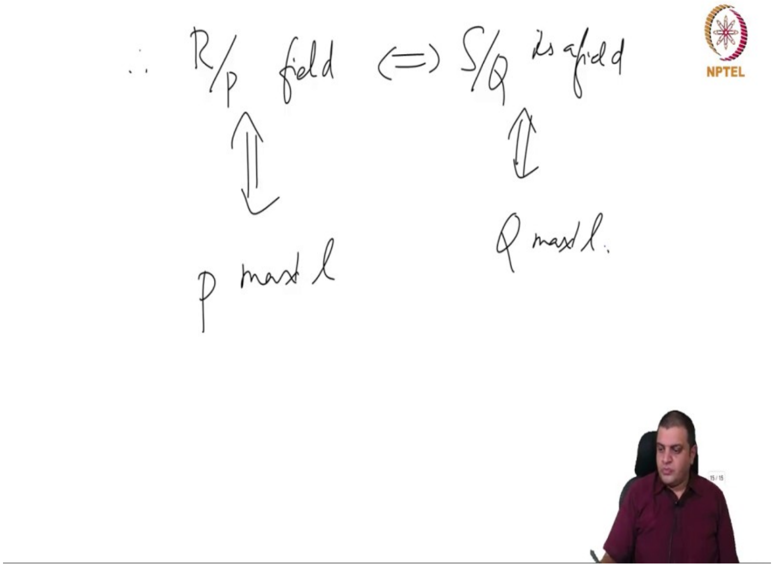


So, proof say said that proof is not very difficult. So, what are we given? We are given an integral extension of maps like this and we want to prove that. So, then we are given some Q inside here and a P inside here and this maps to I mean I am drawing the arrow not as a map of say Spec, but now think of it as a point inside Spec S.

So, this curly arrow here denotes the map in Spec S in the opposite direction Q goes to P, but remember Q is just a point in Spec S. So, with this thing we can look at $\frac{S}{Q}$ and $\frac{R}{P}$ that is because P is $Q \cap R$ this is again injective and it is integral. So, if you have an integral extension this is also integral.

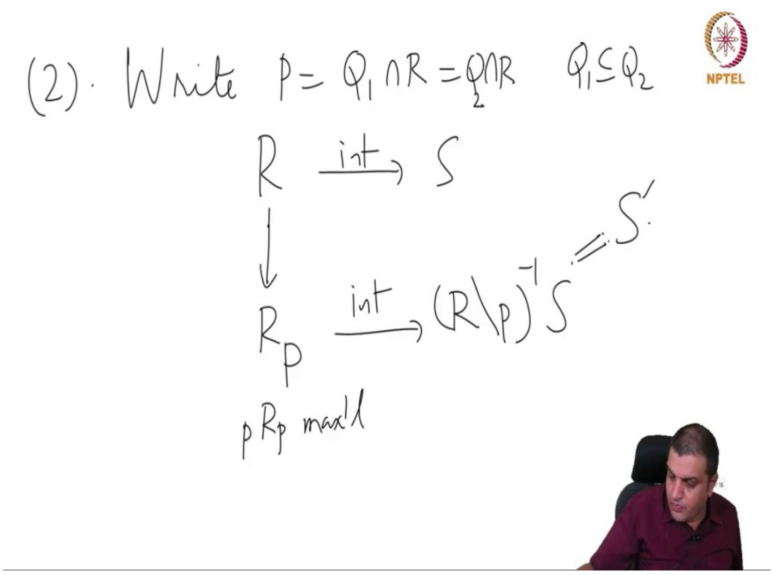
If you have an integral extension there and if you do this step it remains integral this also remains integral. Now, this is the domain and therefore, $\frac{R}{P}$ is a field if and only if $\frac{S}{Q}$ is a field.

(Refer Slide Time: 23:04)



This is equivalent to saying P is maximal and this is equivalent to say Q is maximal. So, this proves 1.

(Refer Slide Time: 23:26)

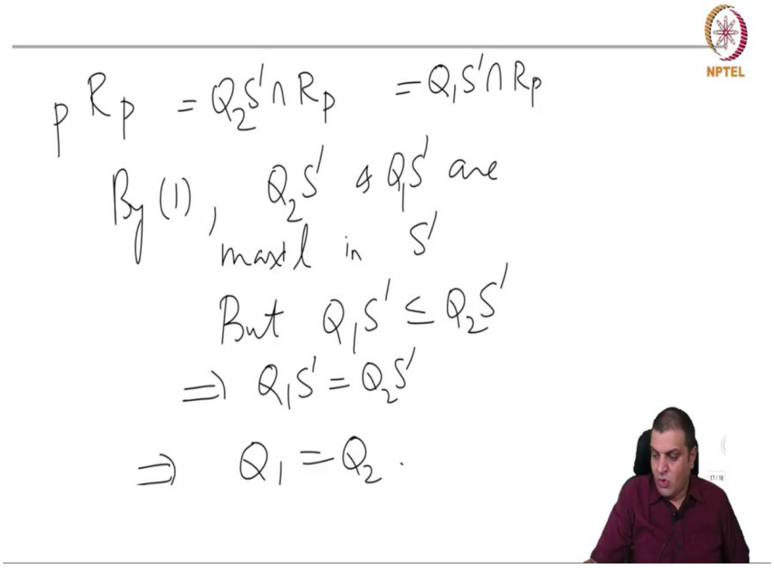


Proof of 2. So, let us write P. So, this is the statement where 2 prime ideals of S contract to the same prime ideal of R and $Q_1 \subseteq Q_2$ and we want to show that that is an equality. So, let us invert elements outside P.

So, we have $R \to S$ and while this gives a map from $R_P \to (R¿)^{-1} S$. So, this is I mean P extended to S need not be a prime ideal. It is complement in complement of the extension of P need not be multiplicatively closed.

What we are inverting is just those elements of R that are outside P. So, invert this. So, this is integral please check that this is also integral. So, $P R_P$ is a maximal ideal. So, notice that. So, let us call this ring $S^{'}$. This is local after inverting call it $S^{'}$.

(Refer Slide Time: 25:12)

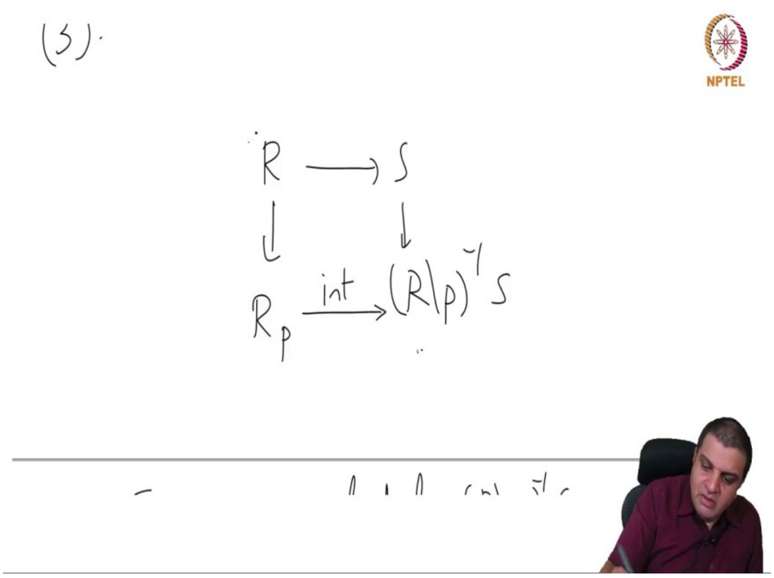


So $P R_P = Q_2 S^{'} \cap R_P = Q_1 S^{'} \cap R_P$. Therefore, it must be a maximal ideal of this that contracts this is maximal by 1, $Q_1 S^{'}$ and $Q_2 S^{'}$ are maximal in $S^{'}$.

But, but $Q_1 S^{'} \subseteq Q_2 S^{'}$, because both are maximal it implies that $Q_1 S^{'} = Q_2 S^{'}$, but using the correspondence between primes in a localization and this one says $Q_1 = Q_2$.

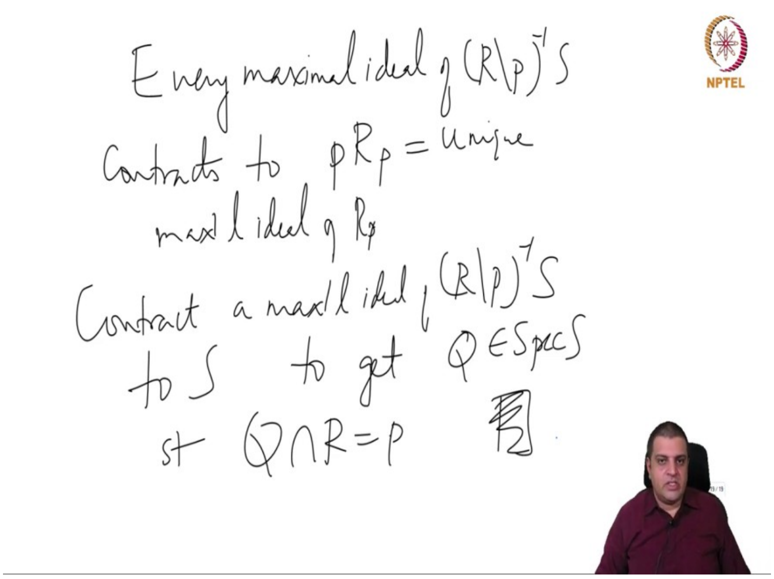
So, we localize here and then we are now looking over things that map on to the maximal ideal here therefore, there the primes here also must be maximal ideal in the fiber and, but then there cannot be a strict containment between them. And, this is preserved under localization. So, you can go back to the original case, ring.

(Refer Slide Time: 26:52)



So, 3. So, now again we have $R \rightarrow S$and then we have a map $R_P \rightarrow (R\text{¿})^{-1} S$. This is again integral and we observed that a maximal ideal here can only contract to a maximal ideal here.

(Refer Slide Time: 27:35)



So, every maximal ideal of $(R\text{¿})^{-1} S$ contracts to $P R_P$ which is the unique maximal ideal of R_P. So, let us go back to this diagram we have a maximal ideal here which contracts to $P R_P$ here. So, we start with a maximal ideal here we can contract from here from this corner to this corner in two different ways. You can contract like this or contract like this you would get the same result.

And so, contract like this we would get $P R_P$ and then P and contract like this we will get some Q and then to P. So, contract maximal ideal of $(R ¿)^{-1} S$ to S to get Q in Spec S such that Q contracted to R is P. As we mentioned earlier the last part says that the map is surjective.

So, we will stop here and we will continue in the next lecture with by start with proving what is called the going up theorem and which is a statement about chains of primes, how they behave in an integral extinction.