**Computational Commutative Algebra**
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**Lecture – 31**
**Integral Extensions**

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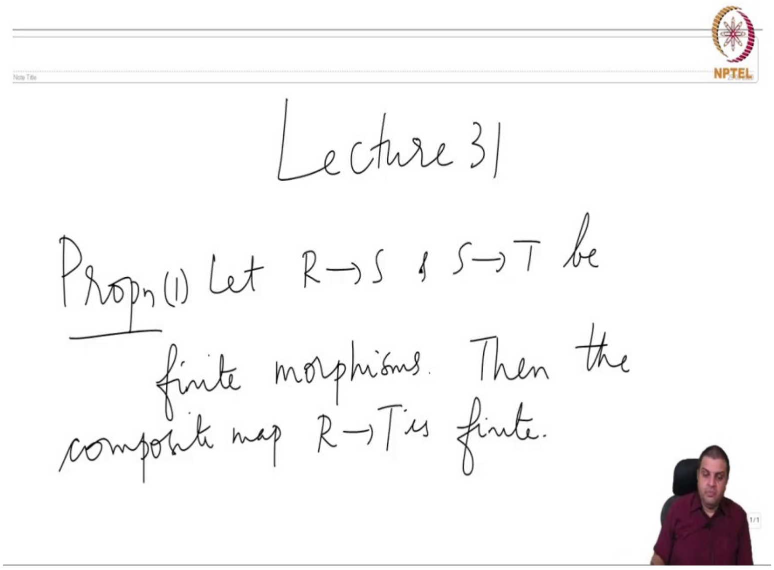


So, in this lecture we continue with discussing Integral Extensions and we first prove statement about transitivity of integral extensions. So, here is the proposition. So, it has some four parts. This is part 1, Let $R \to S$ and $S \to T$ be finite morphisms.

So S is a finitely generated R module and T is a finitely generated S module. Then the composite map $R \to S$ is finite. S is a finitely composite map $R \to T$ is finite. T is finitely generated R module through the composite map. So, this is the first statement.

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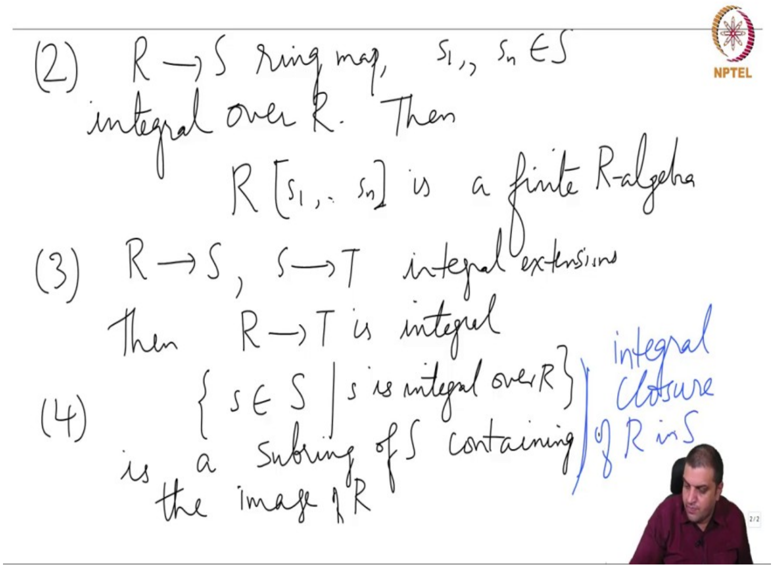


So, now, $R \rightarrow S$ is some ring map and $s_1, \ldots, s_n \in S$ integral over R. Then the subring generated these elements, see by this we mean the image of R inside S and these elements, but we will stop writing it once we get used to this thing is finite R algebra.

Now its finitely generated R module. Three, same as a first statement, but now for a integral; Let $R \rightarrow S$ and $S \rightarrow T$ integral extensions. Every element of S is integral over R and every element of T is integral over S, then the composite map $R \rightarrow T$ is integral. Four, the set $\{s \in S | s\, is \frac{integral}{R}\}$ is a subring of S containing image of R, in other words it is a R sub algebra of S.

So, you just look at the statements. First statement is that, if you have two finite morphisms in the composite, it is a finite morphism. Second statement is if you have a ring map and some finitely many integral elements, so the sub algebra generated by that finitely many integral elements is actually a finite R algebra, this is only finite type, a priori this just means finite type, but because these are integral elements it is actually finite.

And third is if you have two integral extensions, the composite is also an integral extension and final is the set of integral elements inside a ring, is actually a sub ring. So, this is called the integral closure of R in S. So, as so we will come back to that in a minute, but let us prove this.

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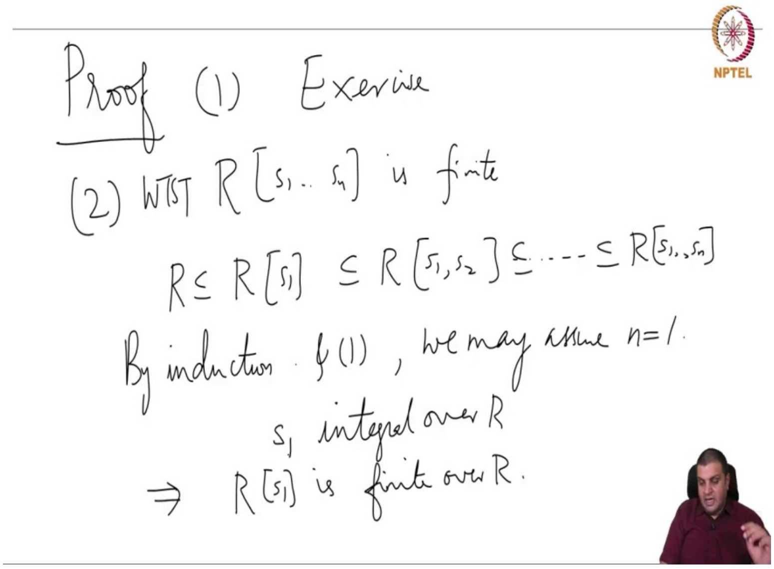


So, proof. 1 is an exercise because it is done; this is exactly how one would prove that if you have two finite extensions of fields, then the composite extension is a finite extension. So, this is the same proof just repeat it, I mean with appropriate changes of words and you will get this. So, first one is an exercise. So, for the second one, we want to show that, $R[s_1, s_2, \ldots, s_n]$ is finite.

So, this is what we want to show, but we can use 1 alright. So, we can start with R, then $\subseteq R[s_1] \subseteq R[s_1, s_2] \subseteq \ldots \subseteq R[s_1, s_2, \ldots, s_n]$. Suppose we prove that individual ones of finite. Then the composite is finite.

Just one more observation, $s_2$ is integral over R, hence this intermediate subring also. Similarly, any $s_i$ here is integral over the subring generated by R and the previous elements.

So, at each stage, it is just adjoining one integral element. So, if you prove that each extension here is a finite, then the composite is finite. So, by induction and 1 we may assume that, n equals 1. So, now let us go back to the previous proposition. So, we are just interested in; so, $s_1$ integral over R. Now, let us go back to the proposition from the previous lecture.

So, this now implies that $R[s_1]$ is finite over R. So, in the previous lecture we found various conditions for equivalent to an element of an extension ring being integral over the base ring R, one of which was that the subalgebra generated by that element is a finite algebra. So, this proves 2.

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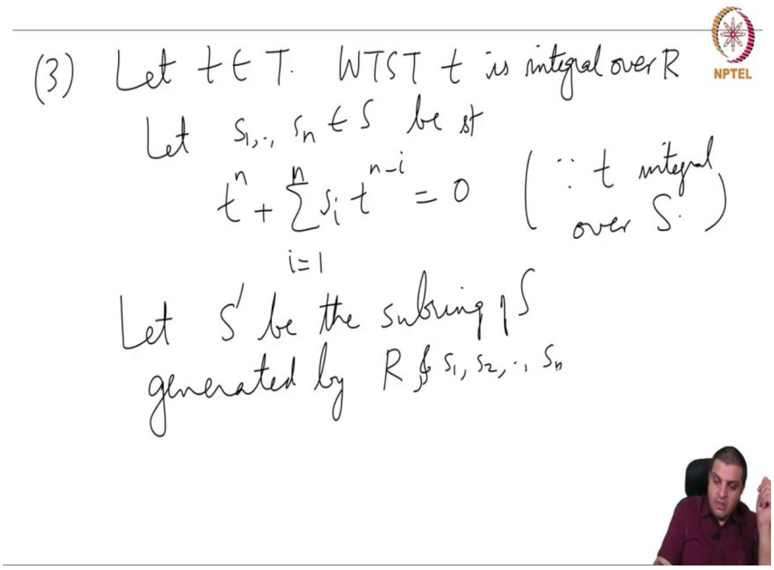


So, now 3; we need to prove that every element of T is integral over R. So, let $t \in T$ and we want to show that t is integral over R.

We know that t is integral over S. So, let $s_1, s_2, \ldots, s_n \in S$ be such that $t^n + \sum_{i=1}^{n} s_i t^{n-i} = 0$. This is because t is integral over S. Let $S'$ be the subring of S generated by R and $s_1, s_2, \ldots, s_n$. So, this is the subalgebra these are integral over R.

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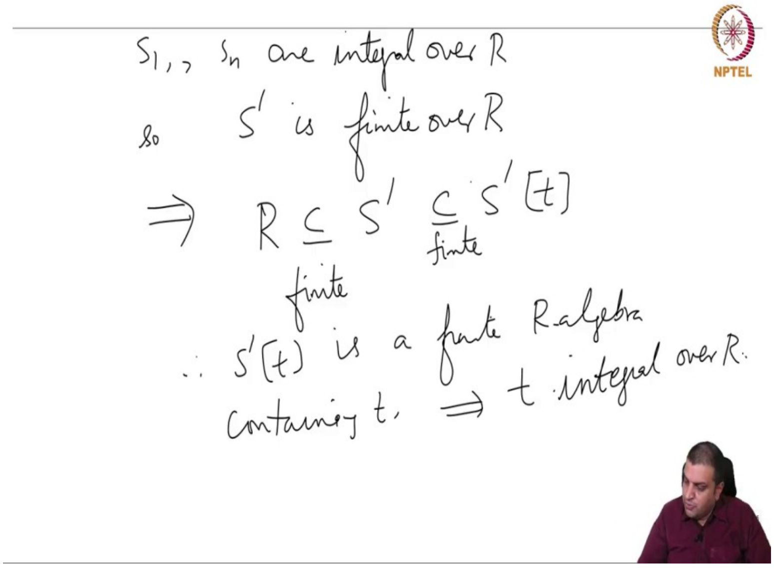


So, $s_1, \ldots, s_n$ are integral over R. So, $S'$ is finite over R. Now this element t here satisfies an integral equation over $S'$. So, this subalgebra generated by $S'$ and t is integral over R. So, this is finite over R. So, now, look at the extensions $R \subseteq S'$ this is finite.

And this is $S'[t]$, this is finite, that is because t is integral over $S'$, composite of two finite is finite. Therefore, this is finite. So, $S'[t]$ is a finite R algebra, containing t, which now implies that t is integral over R. So, this proves the thirds that integral extensions their the composite of two integral extensions is an integral extension.

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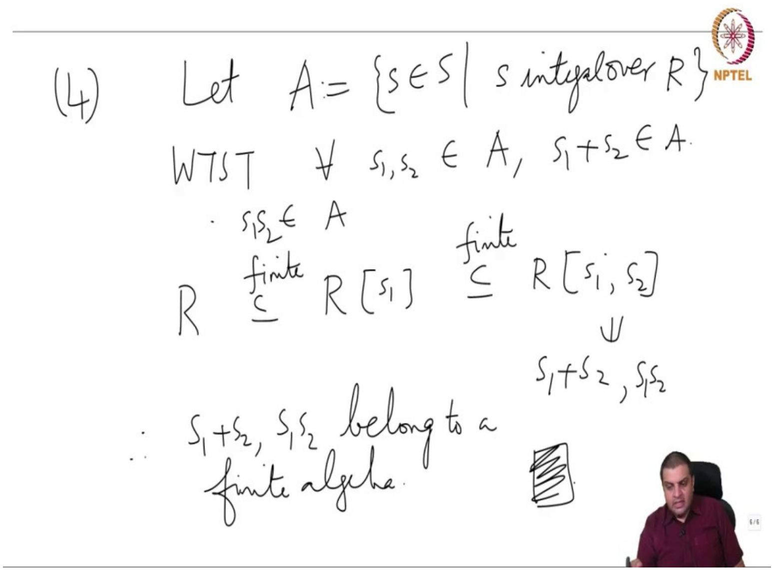

So, now for 4 we need to show that let $A=\{s\in S|s\frac{integral}{R}\}$.

So, we want to show that, $\forall s_1, s_2 \in A, s_1+s_2 \in A \wedge s_1 s_2 \in A$.

So, now let us look at these extensions. $R\subseteq R[s_1]\subseteq R[s_1, s_2]$, these $s_1$ and $s_2$ themselves are integral. So, this is finite, this is also finite and $s_1+s_2$ and $s_1 s_2$ are elements here. So, these are elements inside a finite algebra. So, therefore they are integral. $s_1+s_2$ and $s_1 s_2$ belong to a finite R algebra; so, they are integral.

So, this proves this proposition. So, I mean what it says is, composites of finite is finite. Composites of integral is integral and it also proves I mean makes this 2 is the others thing, that if you have a finitely generated algebra generated by integral elements; then it is actually a finite algebra. So, that is true and final statement is the integral closure of a ring in a larger ring is a ring.

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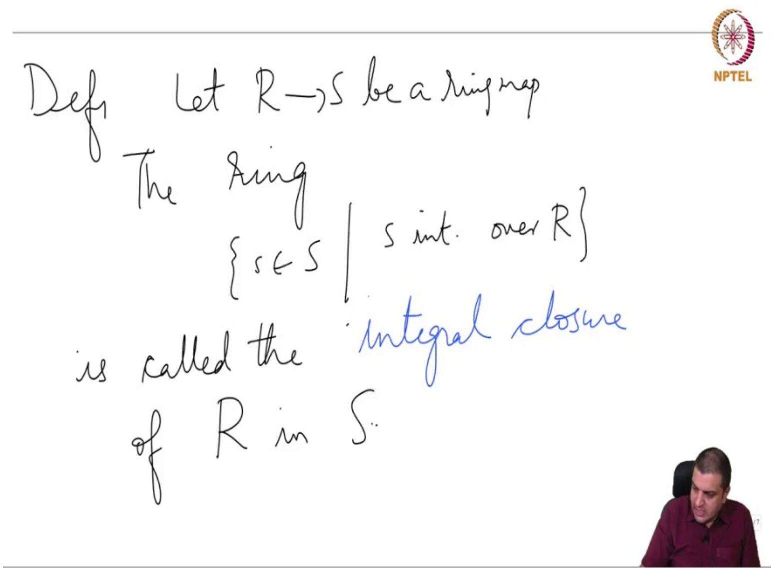


So, we will define integral closure definition. Let $R\to S$ be a ring map. The the ring which we just proved, $\{s\in S|s\frac{integral}{R}\}$ is called the integral closure of R in S.

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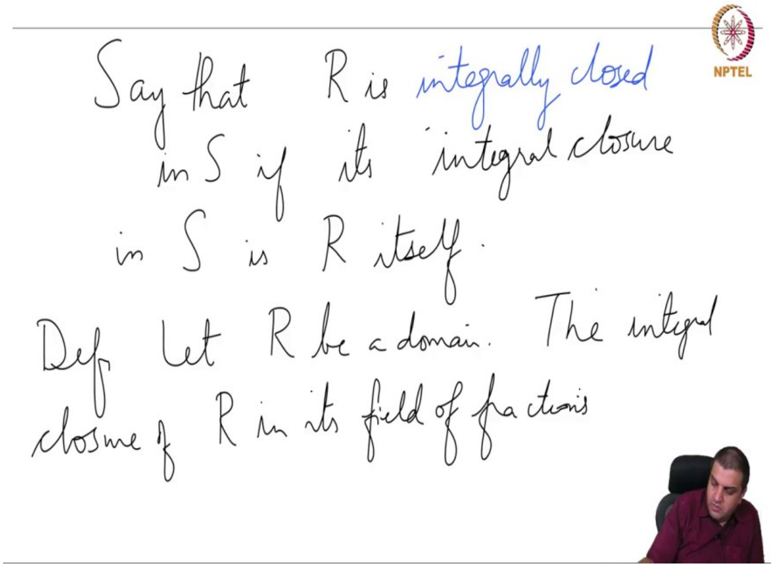


And we say that R is integrally closed in S, if its integral closure in R in S is R itself. Just one observation, when we say this is integral closure, it should at least in principle suggest that, if something is integral over this ring its already integral over R, the only then you can think of it as a closure. So, that when you apply that operation twice you do not get anything new.

So, which is indeed true like the case is indeed true, suppose that there is some t inside S, which is integral over this ring, but then we go back to the previous proposition to conclude that t is integral over R, which means t would have been in this set. So, it is indeed a closure operation. So, just one more definition, let R be a domain and K its field of fraction. The integral closure of R in its field of fractions is called the normalization of R.

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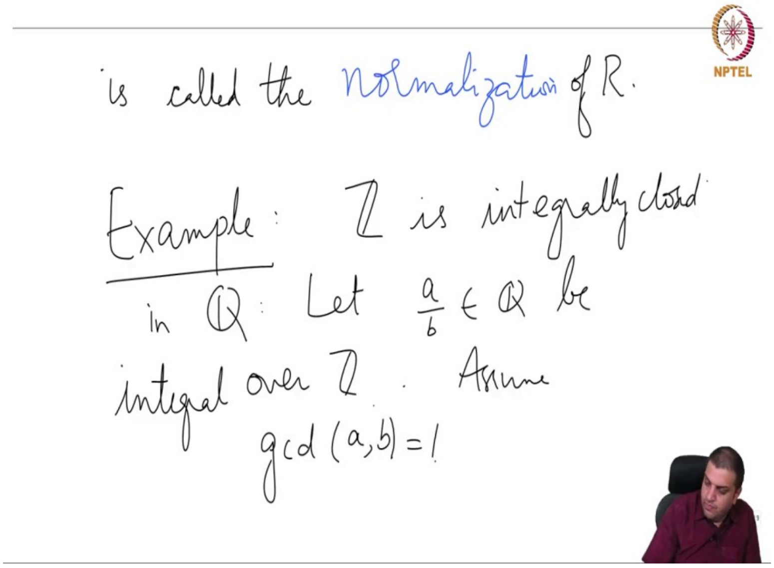


So, this is definition, we will not worry about normalization exactly at this point, but we will develop more about integral extensions, but we can nonetheless look at one example, $Z$ is integrally closed and the field of fractions of $Z$ is rationales and so it is integrally closed in $Q$. So, how do we do this? So, let $\frac{a}{b} \in Q$ be integral over $Z$ and assume without loss of generality that they are co-prime integers, that is $gcd(a,b)=1$.

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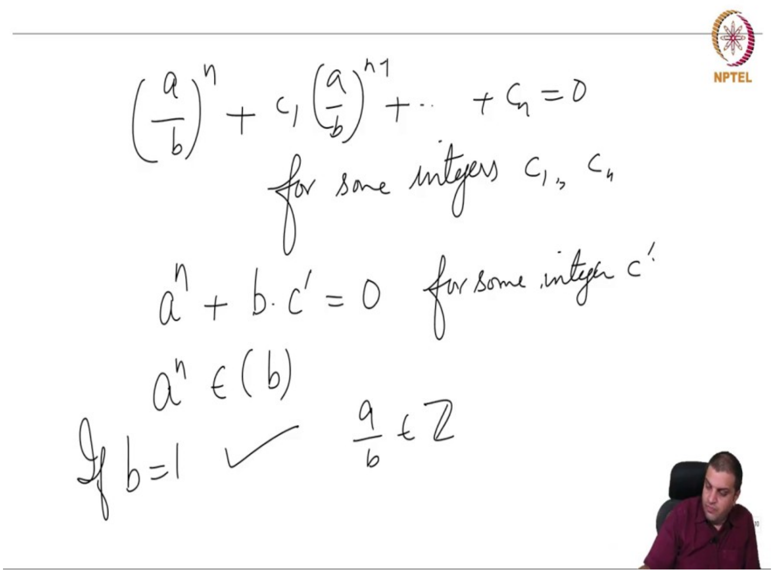

So, then we have $\left(\frac{a}{b}\right)^n + c_1\left(\frac{a}{b}\right)^{n-1} + \ldots + c_n = 0$ for some integers $c_1, \ldots, c_n$. This is what integral means, multiply by $b^n$. So, then we would get $a^n + bc' = 0$ for some integer $c'$. There is only $n-1$copies of b here. So, when you multiply everywhere we would get a $b$ in all these terms.

Now, what does this say? This says that $a^n \in (b)$. Now, if b is 1, there is no issue because this is true and if b is 1, then $\frac{a}{b} \in Z$.

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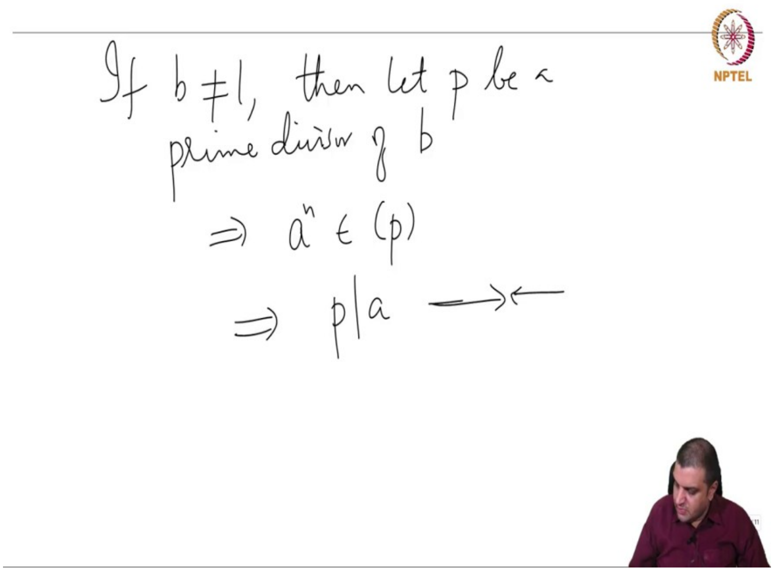


If $b \neq 1$, then let $p$ be a prime number dividing b, then this one says that $a^n \in (p)$ because these are inside b and $(p)$ is maximal ideal containing $(b)$.

So which now implies that $p$ divides $a$, but then this contradicts the hypothesis that a and b were chosen to be relatively prime. So, you can essentially rehash this argument to prove that every UFD is integrally closed in its field of fractions, which you will do in their exercises. So, let us look at another example.

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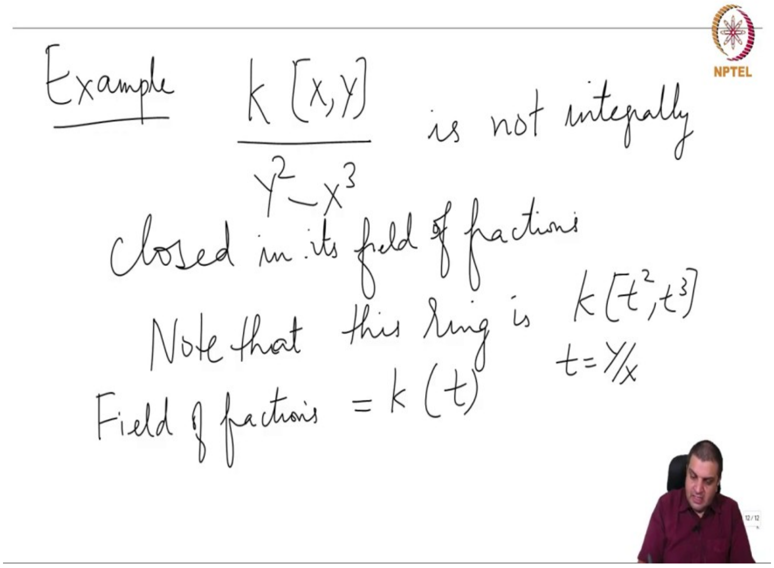


So, the same argument will prove that polynomial rings are integrally closed in their respective field of fractions. But here let us consider slightly different.

$\frac{k[x,y]}{y^2-x^3}$ is not integrally closed in its field of fractions and the easiest way to understand that is to rewrite this as somewhere we have seen earlier. So, note that this ring is $k[t^2,t^3]$. So, the field of fractions contains t; fractions is $k(t)$, rational function field in one variable, because notice that $t=\frac{y}{x}$.

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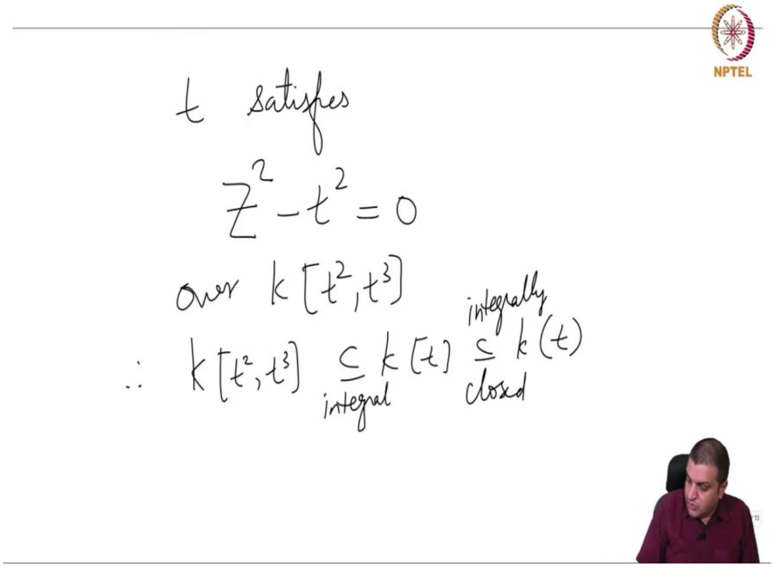


So, that is inside here and t is there in the field of fractions and t is integral over this ring. Let me just use a different variable now. $Z^2 - t^2 = 0$ over $k[t^2, t^3]$. This $t^2$ is from here, this $Z^2$ is a variable and that is where we substitute t.

So, therefore $k\left[t^2, t^3\right] \subseteq k[t] \subseteq k(t)$. This is an integral extension or a finite extension and this is integrally closed here, that is the essentially that, that UFD's are integrally closed, one can just redo the argument for $Z$ in this case.

So, we have this is the field of fractions. Between this in the field of fraction, there is this ring which is integral and that is integrally closed in its field of fractions. So, this is the picture for this one.

So, now, we come to one of the results that I informally mentioned earlier, which is that given any finite type algebra over a field, there is a polynomial subring over which the algebra is finite.

We only mentioned about maps between Spec, but this is a slightly stronger this is a stronger statement and this we will prove not immediately, but right now we will use it to derive to draw applications one of which is I mean the main part of which is to prove the Nullstellensatz.

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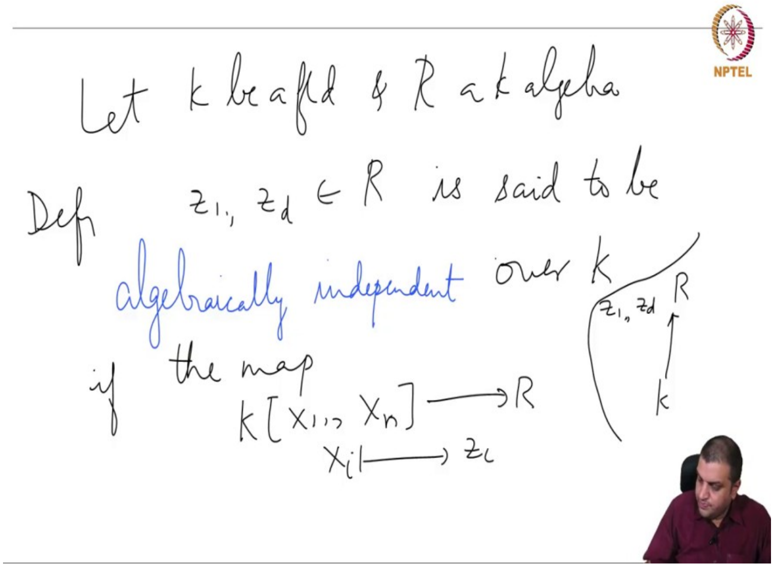


Before we state the theorem we need a definition. So, let k be a field and R a k algebra. So, definition;$z_1, \ldots, z_d \in R$. So, the definition we do not need to assume that its finite type, but we are taking finitely many elements inside R, is said to be algebraically independent over k. So, let us just say what we are saying. So, $k \rightarrow R$ is a ring map and here we have taken some element $z_1, \ldots, z_d$.

So, we have seen this earlier that if you have a ring map like this, then it extends to a map uniquely from a polynomial ring over k mapping the variables to this and scalars mapping accordingly. So, the map from $k\left[X_1, \ldots, X_n\right] \rightarrow R$, $X_i$ going to $z_i$. So, these are variables is injective.

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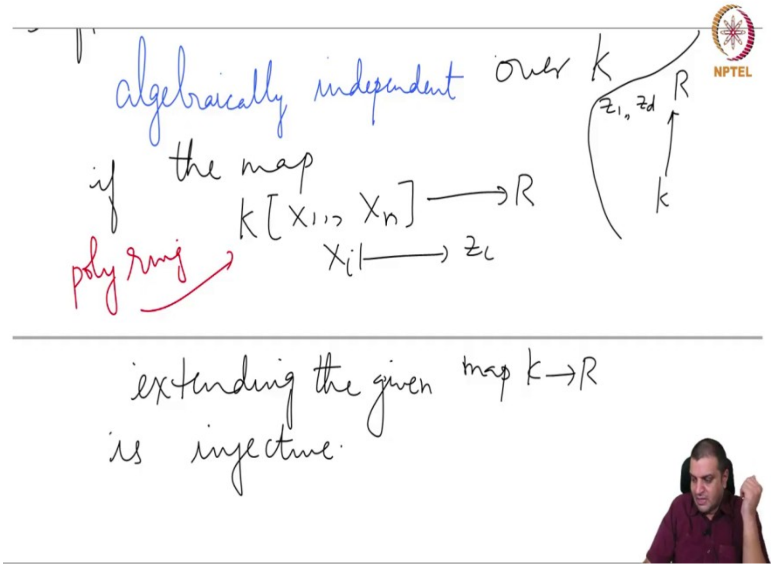


So, the map extending the given map k to R. One of the first few lectures, we have given a map from k to R and a bunch of finitely many elements. So, that is the; so, maybe I should just say this is a polynomial ring. So, that extends to a map form the polynomial ring to the same R itself extending the given map.

So, this is injective. In other words the algebra $k[z_1,\ldots,z_d]$ inside R is a polynomial subalgebra. So, we say that this is a algebraically independent. They behave exactly like variables.

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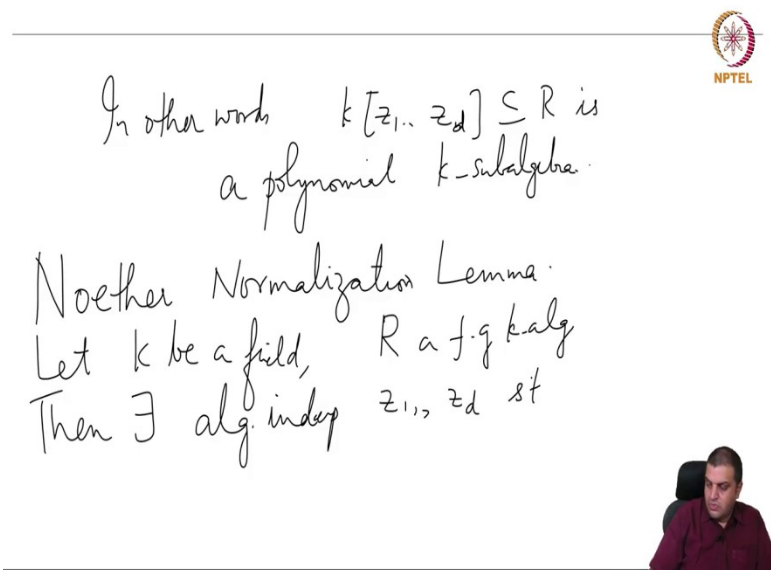

So, in other words $k[z_1,\ldots,z_d]\subseteq R$ is a polynomial k subalgebra. There are no algebraic relations among them, that is what algebraically independent. So, here is what is the theorem called Noether normalization.

The same Noether as Noetherian, Emmy Noether Lemma. Let k be a field, R a finitely generated k algebra. Then there exist algebraically independent $z_1,\ldots,z_d$ such that R is finite over the subalgebra $k[z_1,\ldots,z_d]$ which is a polynomial algebra.

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So, this is the theorem called Noether Normalization Lemma. This has lots of applications in various contexts studying of rings that are finite type of over a field. We will not prove it right now. We will first draw some corollaries from this. First draw one corollary from which we will prove the one version of the Nullstellensatz.