Computational Commutative Algebra
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Lecture – 30
Morphisms – Part 2

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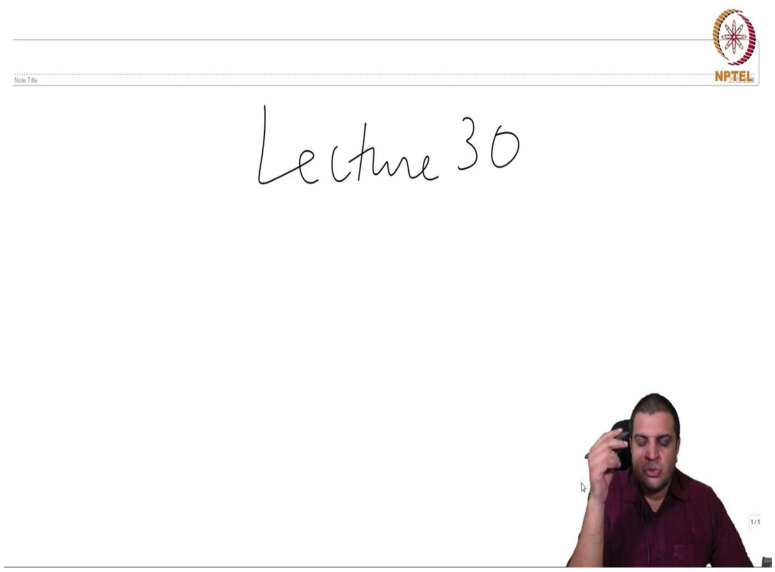

This is lecture 30. So, in this we continue the discussion about examples of Morphisms and fibers of that to come to the as a starting point for the next question to a next topic to understand.

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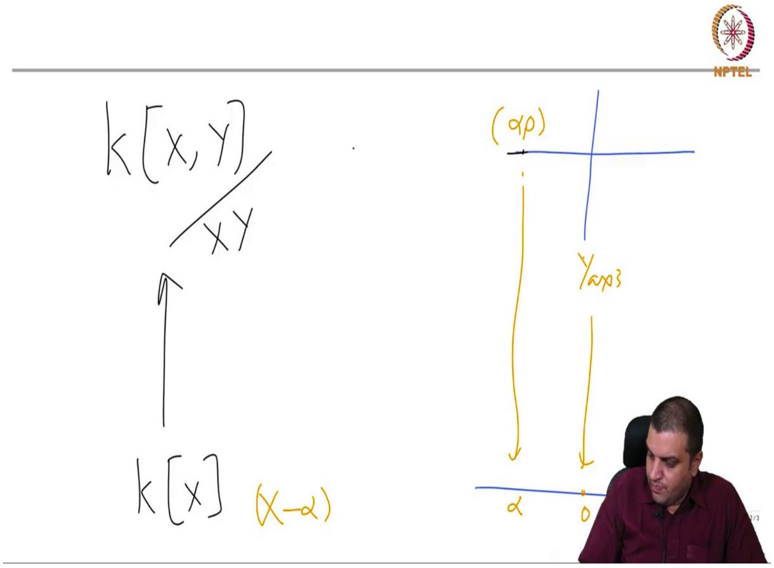

So, back similar example, because I want projection to the X axis, let me orient the picture this way, the arrow as this way. $k[X] \to \frac{k[x,y]}{(yx)}$

So, this is the picture and the variety of this would be so,every point where either x vanishes or x vanishes, which is just the union of the axis, and here is just the line X. So, now, let us take some α here. So, we get $X-\alpha$ So, let us work that out.

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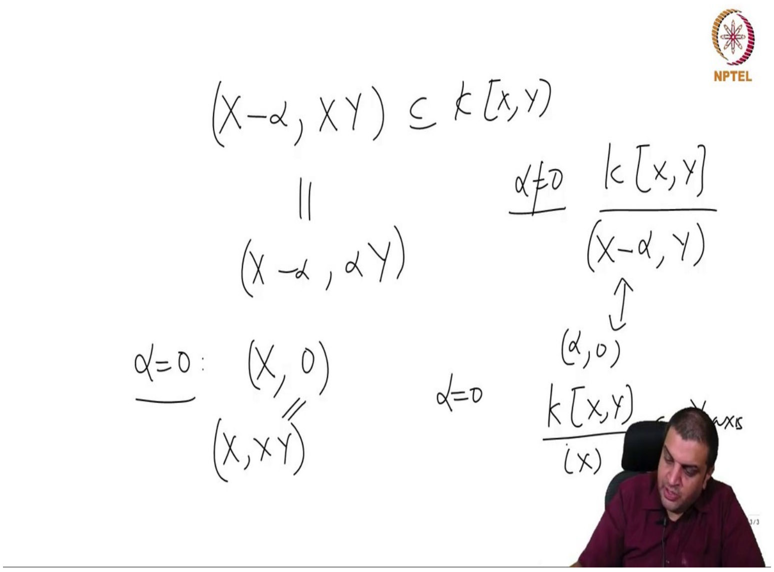

So, $(x-\alpha, xy)$. This is the ideal inside $k[x, y]$ that we need to worry about. But, this is the same as same as $(x-\alpha, \alpha y)$, because we are just substituting x as α, just one observation, if alpha is 0, it is subsumed in this case but let us just make sure that we understand.

If, α is non zero then it is fine even. if α is 0., then we just get $(x, xy)=(x)$ So, even if x is 0 then it is still fine.

So, this is the thing. However, the behavior is quite different in the two cases, depending on alpha 0 and alpha is non zero. What the fiber is very different. So, let us look, let us work these both out let us just work for $\alpha = 1$ and $\alpha = 0$. The fiber is $\frac{k[x,y]}{(x-1, y)}$.

So, this corresponds to the point (1 , 0). Because, if α is non zero, although the I mean in both cases the description of the ideal is as a formal thing it is correct, there is nothing wrong, this ideal is depends on what that alpha is.

If, we take. The ideal generated by αY and Y are the same. So, we can just take if α is 0, then we would just get $\frac{k[x,y]}{(x)}$

, but this is the Y-axis right every point on the Y-axis satisfies this equation. So, which is what the fiber over some point α , , here is $(\alpha,0)$ that is what goes to this point.

However, to the point 0 here the entire Y axis maps and that is the. So, there is a difference in earlier case all the maps were finite, may be the fibers were finite, but here that is not the case. Although, I mean if you look at in some sense there is not much difference, which is your polynomial ring in two variables, we went to modulo 1 equation and then we are just looking at the map from one of those variables to this.

So, in that sense there is not much difference between this and the previous problem, but there is which is what we just saw. So, now, let us look at another let us modify this with slightly different fashion.

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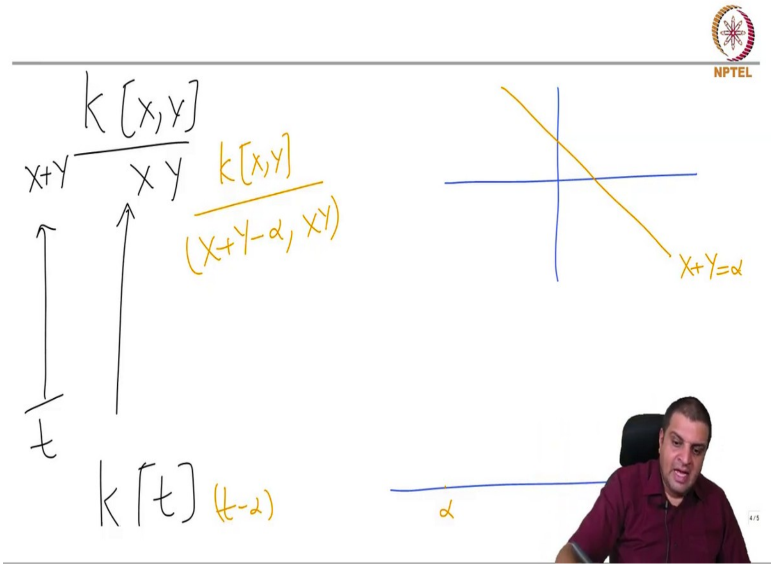

So, now we consider a k[t] and here is just $\frac{k[x,y]}{(y\,x)}$

.And, this map , which is now $t \to x + y$.

So, in other words when t when we say t to some value α , we are setting x + y to some value α . So, that would be an entire line ok. Some line I mean if you draw in a real picture it will be some line with the so, this is the x, y gives that t is just this line. And, if you take some α here, it is not just the point t is not same as X, t corresponds to some line here which so, let us do that so some α here.

So, here is $t-\alpha$. Here it means, $\frac{k[x,y]}{(x+y-\alpha, xy)}$. Now, irrespective of the value of α this has exactly this has one or two solutions. So, let us do that.

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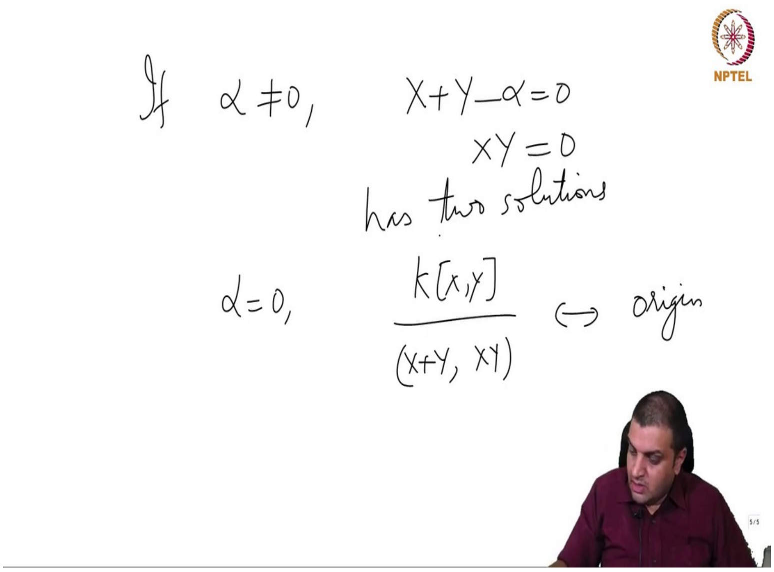

So, if α is not equal to 0, the equations $x+y-\alpha=0$ and $xy=0$ has two solutions.

You can work that out. And, if $\alpha=0$ we get $\frac{k[x,y]}{(x+y, xy)}$ which is really if both these functions vanish it can only vanish at the origin right. X and Y have to vanish so, either x is 0 or y is 0, but as soon as x is 0 this forces y also to be 0. So, this is has only so, the max spec of this corresponds to one point which is just the origin.

But, as we saw earlier it has some additional structure to it not just 1 point, but whatever it is when we go back to this picture, the fiber over alpha is this not a point, but it is a line like this $x+y=\alpha$

ok. So, it intersects the given variety in only infinitely many points.

So, the fiber of this map has the fibers of this map are finite while in the previous example, there are fibers which are not finite. So, we will come back we visit this point after we discuss a few we prove some results I mean we set this up and prove some results.

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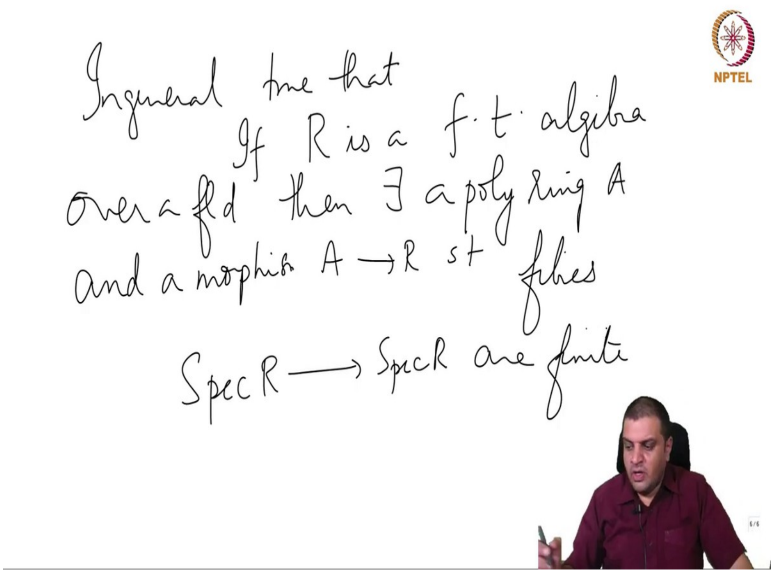

So, it is in general true, in general it is true that we will prove this in the next lecture or the one after that, a finite type algebra meaning, finitely generated algebra over a field, then there exists a polynomial ring A such that the fibers of this map.

Let us say, algebraically closed field for this. A morphism $A \to R$ such that $Spec(R) \to Spec(A)$ the fibers of this map fibers are finite. So, that is the picture that we had here. So, here is some finitely generated algebra R, there is a polynomial ring A with this as finite. So, this is in general true it is.

But, it is not always true, that you one could have sort of bizarre behavior about fibers, but here we some somehow find nice fibers. So, we need to develop enough to understand and prove this theorem. We will prove something stronger than this and this is what we want.

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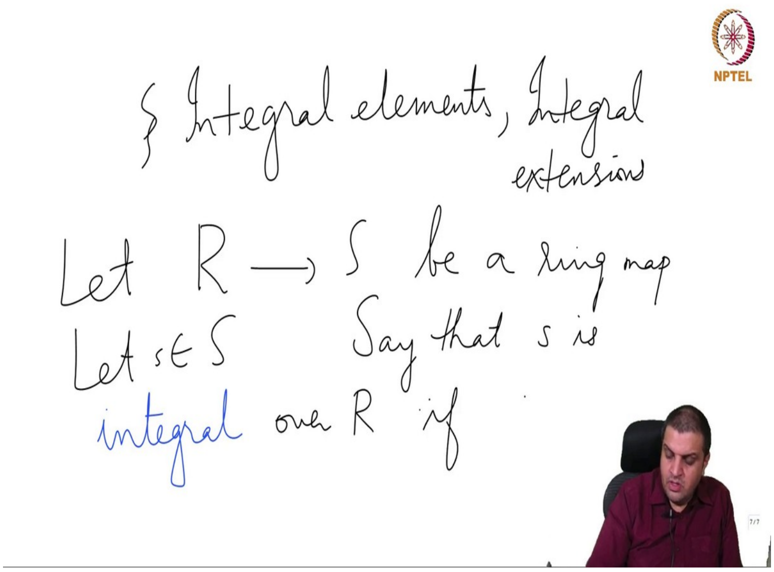

So, now we start a section called integral elements, integral extensions. So, this is what? So, let R to S be a ring map . So, given a map $R \to S$, We say that so, let $s \in S$ say that s is integral over R if.

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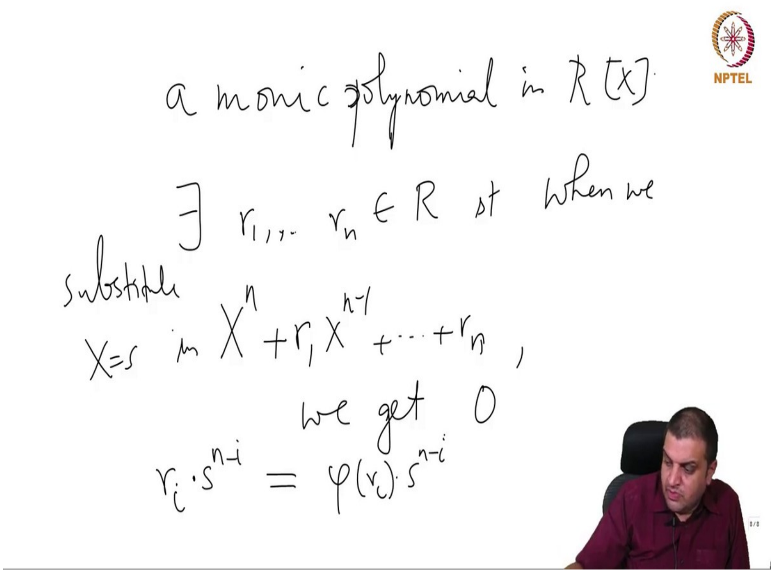

s satisfies a monic polynomial in R[X]. So, just so, what does that say, it says that. So, there exist $r_1, \ldots, r_n \in R$, such that when we substitute X=s in $X^n + r_1 X^{n-1} + \ldots + r_n$, when you substitute s for this one right, we get 0.

So, just 1 point when we say r_1 . So, when we say so, here we have to evaluate things of the form $r_i s s^{n-i}$. This remember is just $\varphi(r_i) s s^{n-i}$.

So, whenever we are not seeing sorry this is phi we are not saying this is injective, and whenever we multiply two elements inside here the action is always through this ring of morphism.

I mean that is a standard thing we often do not write this we just say $r_i s s^{n-i}$ that is what this means.

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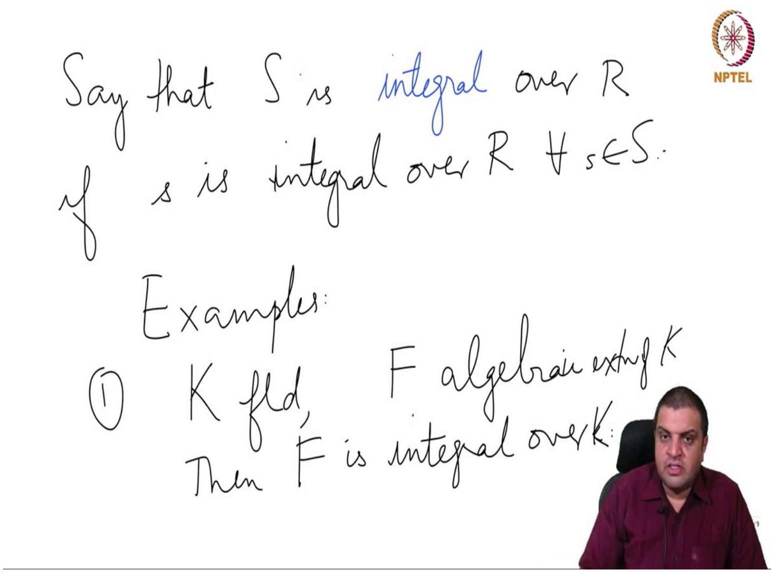

Say that, S is integral is integral over R if $s \in S$ is integral over R for every $s \in S$. So, let us look at some examples.

First one suppose K is a field and F an algebraic extension of K, then F is every element of F is integral over K. This is nothing different from just algebraic in this situation integral means algebraic there is nothing different.

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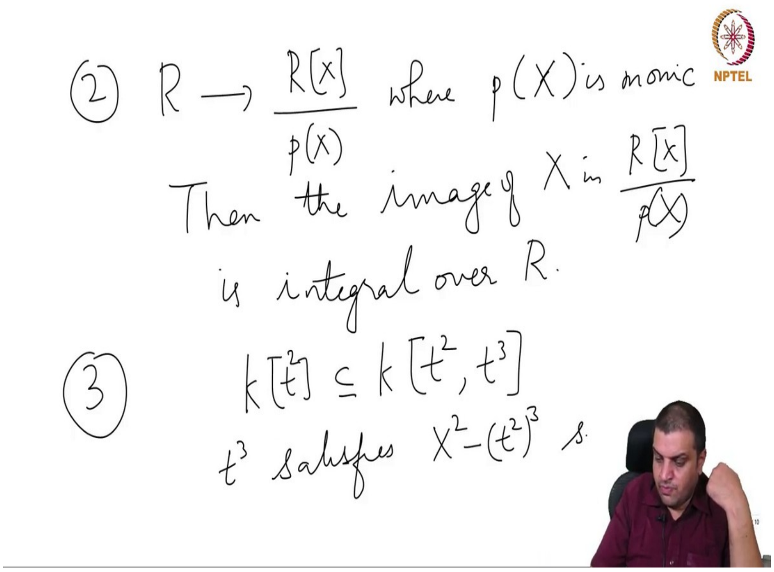

Example 2; we could just think of this in a formal setup, that we could just adjoint a variable X and then just kill polynomial like this. And, the new ring will the image of X will automatically be integral.

So, if you take $R \to \frac{R[X]}{p(X)}$ where, p(X) is monic . Then, the image of X in $\frac{R[X]}{p(X)}$ the residue class is integral over R. That is the another example, it is sort of formally constructed, but .

3) we another ring that we are we have seen and we have become familiar with in is this $k[t^2, t^3]$.

And, so, inside this look at the sub ring $k[t^2] \subset k[t^2, t^3]$. Now, t^3 so, this is the s that we are looking at t^3 satisfies $X^2 - (t^2)^3$, t^2 from the sub ring R. This is from R this is t^3 is from S.

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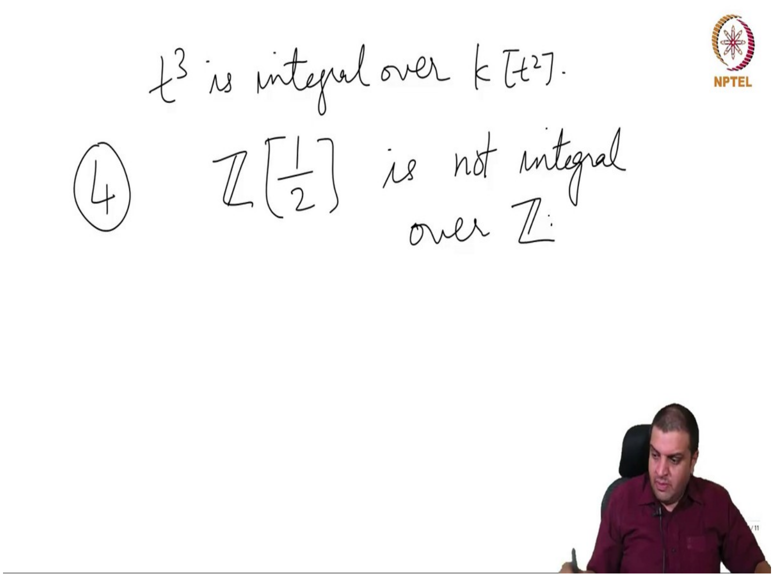

So, t^3 is integral over $k[t^2]$. In that ring and finally, a non example. So, this is a an exercise if you take Z and you take some invert something let us say we just invert 2, $Z[\frac{1}{2}]$. So, this is localizing. Inverting powers of 2 is so, in so, $Z[\frac{1}{2}]$ is not integral over Z. That is if you try to write half as to satisfy an integral equation we will fail. So, do this as an exercise. So, it is not integral over S.

So, we would like to understand integral extensions.

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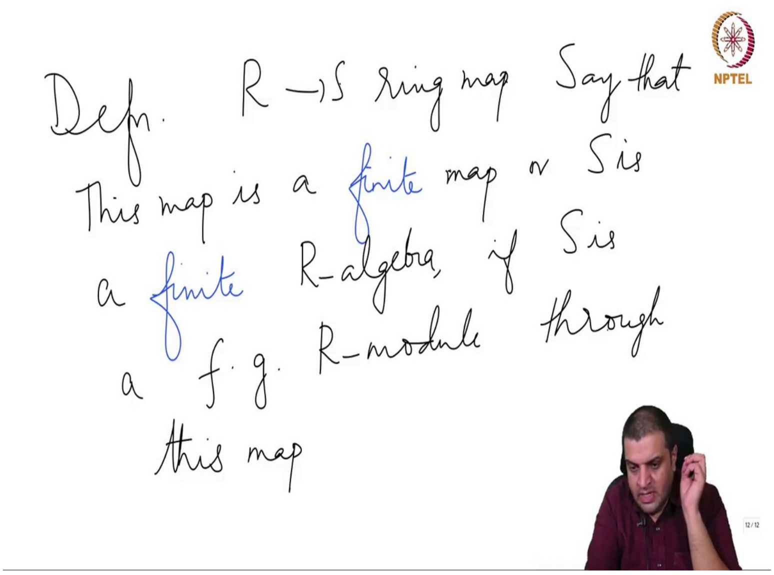

So, definition $R \to S$ ring map, say that this map is a finite map or S is a finite R-algebra. If S is a finitely generated R-module through this map, through this map.

So, then the terminology is a little confusing, we talk about finite R-algebras and finite type R-algebras, finite type is as an algebra it is finitely generated, finite is as a module it is finitely generated. So, and any algebra has a module structure through those map so.

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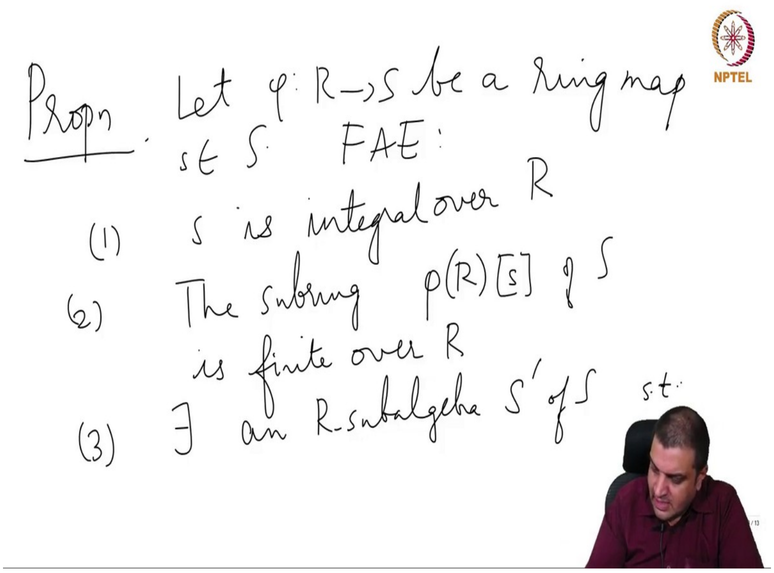

So, now here is a proposition. Let $\varphi : R \to S$ be a ring map right $s \in S$, then the following are equivalent

1) s is integral over R.

2) the sub ring which is generated by the image of $\varphi(R)[s]$ of S. the subring is finite over R. And,

30 there exist an R-subalgebra S' of S. So, it is a subset and it is a sub ring, but the map from $R \to S$ goes through this S'. So, it contains the image of phi. So, it is that is what we mean by sub algebra?

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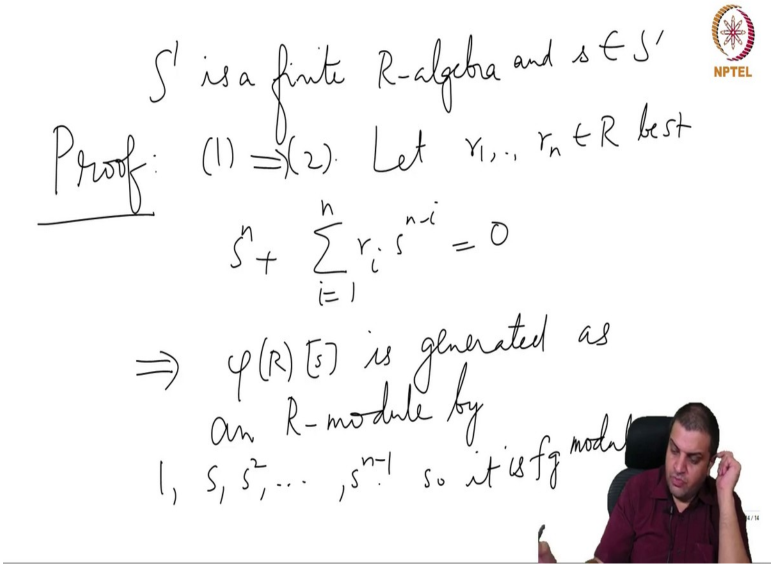

Such that S' is a finite R-algebra. So, let us just revisit the thing. So, φ is a map and we take an element. First statement that element is integral over R, the second statement is the sub this is so, this is an example of a sub algebra. So, it is a sub ring of s containing the image of φ and some other things.

So, $\varphi(R)$ this sub ring is finite over R and s is inside S'. That S' is contained inside some finite subalgebra, not necessarily just that. So, that those three statements and the claim is that these two things are equivalent to S being integral proof 1 implies 2.

So, if you consider the elements so, let $r_{1,\ldots,}r_n$ be in R such that, $s^n+\sum_{i=1}^{n} r_i s^{n-i}=0$,, because it is a monic polynomial. This implies now that $\varphi(R)[s]$ is generated as an R-module by so think of it as a polynomial ring. If, you have a polynomial ring let us say k[x], it is generated as a module; as a module by the powers of the variable.

So, here it will be generated by $1, s^{2,\ldots}$. , but as soon as we see an s^n we can use this expression to bring that degree down. So, this is generated by up to s^{n-1}. So, let me just repeat this as it is; as it is written here. So, this just says how the r_i act?

So, for now we can ignore this. When we adjoint an element like this, we have to worry about all it is powers $1, s^{2,\ldots}$. But, So, it will be a polynomial expression in the s, s will be like a variable I mean it will be a polynomial in which s takes the value of that variable takes the value of s and coefficients are from this ring. So, that is what we are considering.

And, but as soon as we see an s^n we can use this expression to bring the degree back, degree down, using elements of R. So, this is generated by this. So, it is finitely generated module. So, remember this finite algebra means finitely generated module.

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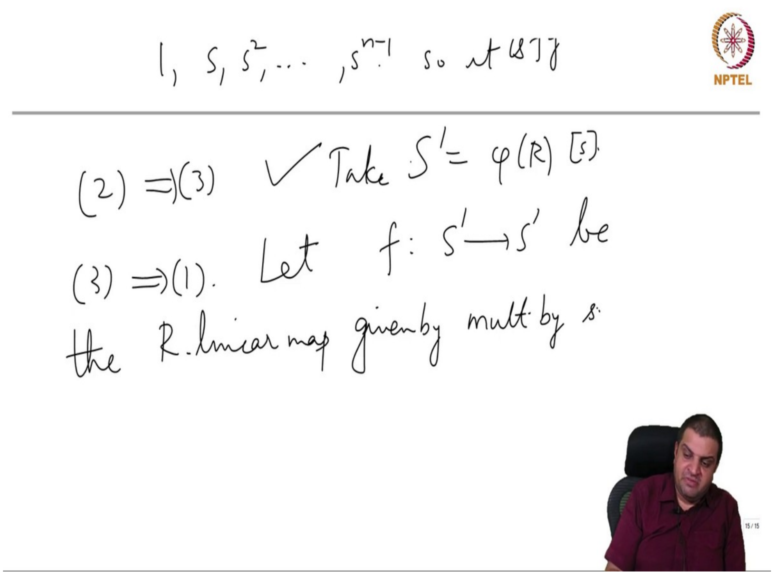

So, it is 2 implies 3 is immediate, because here it says one specific subalgebra is finite. Here it is just asserting some algebra is finite.

So, 2 implies 3 is immediate and we so, take $S' = \varphi(R)[s]$. So, take and 3 implies 1. So, let we will use a determinant trick. So, let $f : S' \to S'$ the R-linear map, given by multiplication by s. So, $1 \to s$ and then that is not we have to specify it for every element of S', but multiplication by s.

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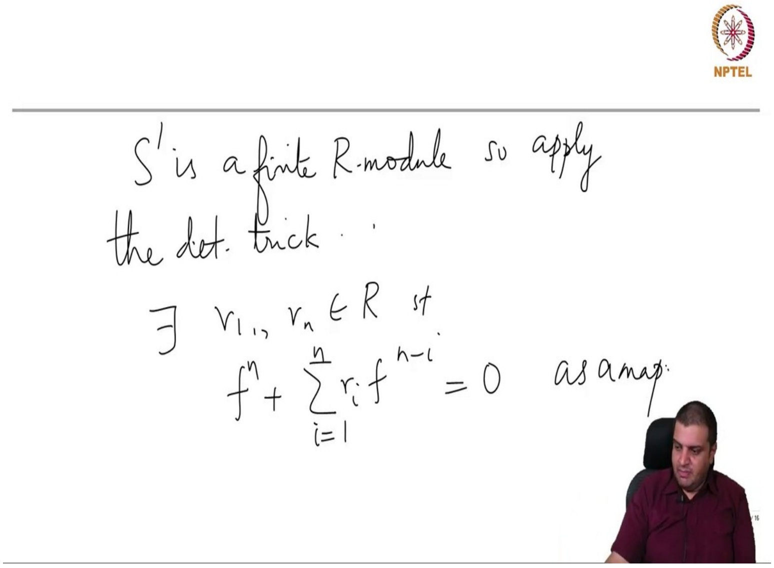

So, this is a finite module. So, we can apply the determinant trick which we learned a few lectures ago. So, what do we need? We need f of this to be inside some ideal times S and then there is some statement, but here we can take ideal to be R itself.

So, what does the determinant trick give? There exist an n such that n depends on the number of generators for S' , $r_{1,} \ldots , r_n \in R$.

So, if it were inside some ideal we would get some extra conditions on these things, but here we are just taking the unit idea. Such that $f^n + \sum_{i=1}^{n} r_i f^{n-i} = 0$ as an R-linear map.

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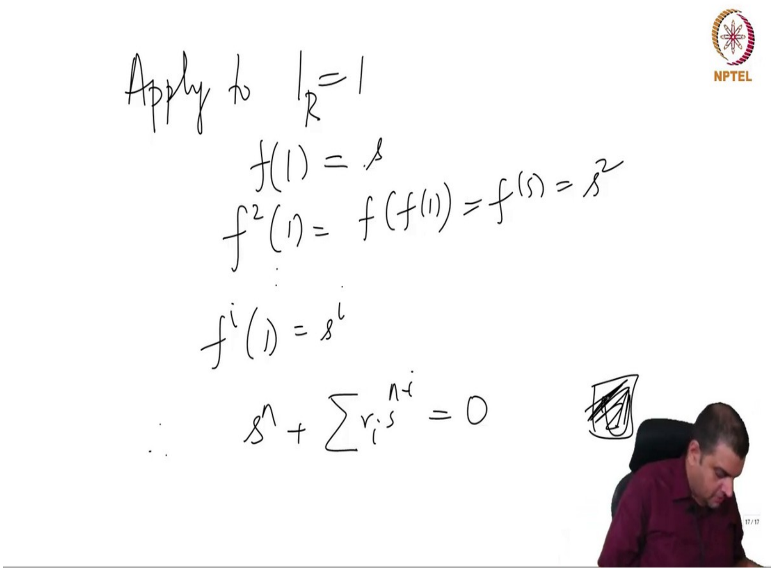

But apply this to 1_R. So, f(1) = s, $f^2(1)=f(1)^2=f(s)=s^2$. So, $f^i(1)=s^i$.

And, now apply this whole function to s and then we would just get therefore,

$$s^n+\sum_{i=1}^{n} r_i s^{n-i}=0$$

, this is the. So, we will stop this lecture here. Next is again a property about transitive nature of integral extensions and finite extensions. So, we will prove that proposition in the next lecture and then, we will then, discuss what is called noether normalization lemma.

And, then we will get we will first use an noether normalization lemma to prove the version of (Refer Time: 29:19) as we are interested in, after that we will prove the noether normalization lemma. Not just in one lecture, but in the next few lectures.