Computational Commutative Algebra
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Lecture – 29
Morphisms – Part 1

Welcome, this is Lecture 29. It is titled Morphisms very generically. So, what we want to understand in this lecture is so now, what does what do ring maps mean on Spec and how do we visualize them; what do they I mean using Nullstellensatz, what do they translate into geometry. And the reason, so, we will use this as a motivation to talk about finite and integral morphisms and then, proceed from there.

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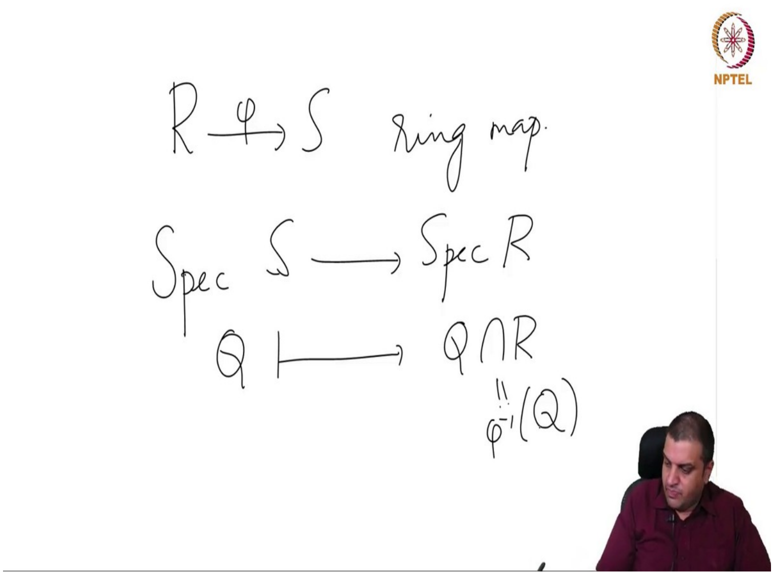

So, this is the setup that we have; $\varphi: R \to S$ is a ring map homomorphisms of rings,. This gives a map from Spec S $\to$ Spec R in which some prime ideal here goes to its contraction. So, if you label this map phi, remember this is just a handy notation for $\varphi^{-1}(Q)$.

I mean the notation although the notation is that of an intersection, it is not implied that R is a subring of S. It is just a morphism, ok. So, this is what we have and what exactly is this doing; that is what we would like to understand.

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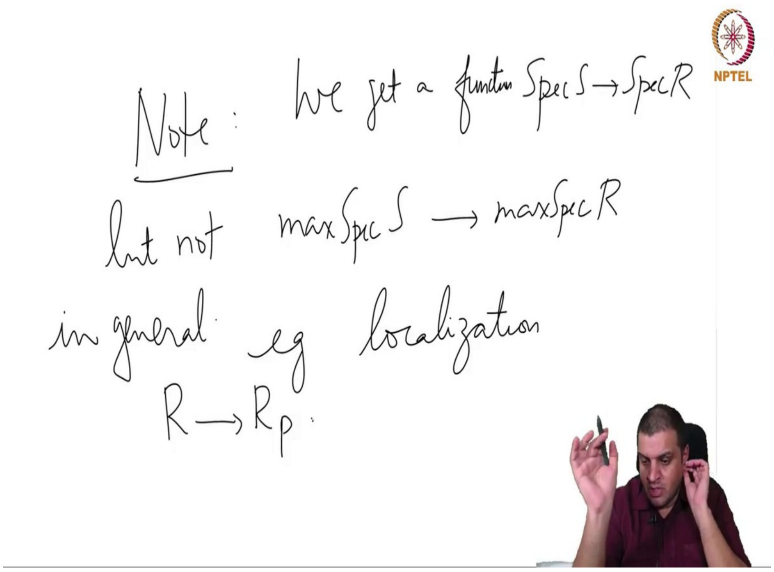

So, just one warning or note, we get a map from $Spec(S) \rightarrow Spec(R)$, we right now it is just a function. Actually, using in Zariski topology, it is even a continuous function. $Spec(S) \rightarrow Spec(R)$, but not from the $maxSpec(S) \rightarrow maxSpec(R)$ in general. for example, think about localization ; if you take the map $R \rightarrow R_p$ and p is not a maximal ideal. So, the image of p inside here would be a maximal ideal, but its inverse image here is not a maximal ideal So, in general, we do not get a map from $maxSpec(S) \rightarrow maxSpec(R)$.

But it so happens and we do not need it; we do not need the result in our course, so we will not prove it that if R and S are finite type algebras over a same field and we have a map like this given by polynomials, then maximal ideals will contract to maximal ideals.

However, that requires a proof and it is a statement which is sort of very special to polynomial rings over fields; so, or some similar rings. So, we will not attempt it, proving it and we will not explicitly need it either. But let us try to understand this using some pictures nonetheless.

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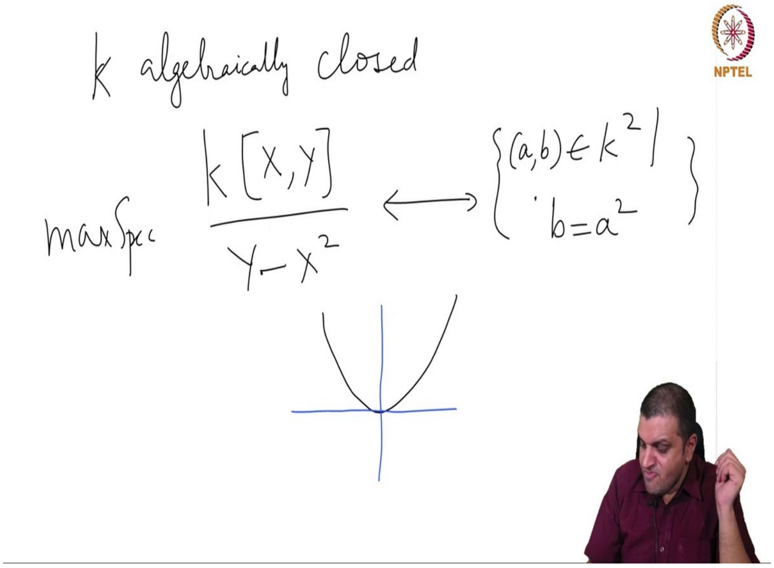

So, when we think about pictures, we will think of as k is algebraically closed. Although, we will draw pictures here thinking that it is R, but that is just for us to. Now, we can ask, so what maximal ideals are? let us say we take a simple enough equation $\frac{k[x,y]}{(y-x^2)}$.

So, what is the maxSpec of this? Under Nullstellensatz, this is all the points in k^2 at which this function vanishes or this ideal vanishes and that just means such that $b=a^2$ or we can think of it as a parabola.

So, again when we draw parabola, we drawing it in R, but that is only for us to sort of keep a picture in mind. I mean it is not this is over some algebraically closed field which cannot be inside R So, we get a parabola. This is just the axis. So, the line the curve is just the black one and the blue thing is just the axis, you can think of it as x-axis and y-axis.

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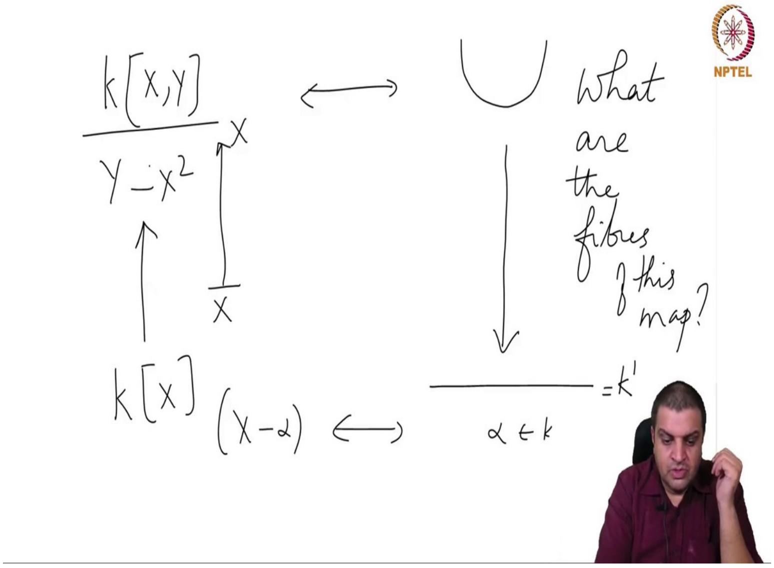

So, now, let us look at a map of rings. The above ring ring map from $k[X] \to \frac{k[x,y]}{(y-x^2)}$ and $X \to x$. So, we can think of k[x] as some subring of k[x,y]. So, k[x][y] and then, we say I mean kill $y-x^2$. So, this is just (Refer Time: 06:13). This we said corresponds to this parabola and this corresponds to the an axis like this. We can think of it as k itself because maximal.

So, when we draw pictures, we are thinking about the maximal ideals and the maximal ideal here is of the form $(X-\alpha)$;and that corresponds to the point $\alpha \in k^1$. So, we can think of this ring as corresponding to this line, the way we all of this ring.

Now, , in this case it just so happens that if you have a maximal ideal here, when you take its inverse image, it is a maximal ideal. So, the map here is in this direction. And, we would like to understand; what are the fibres of this map. I mean what sort of a map is this. In this case, at least intuitively one can think if you have a point α here, its fibre would be the point α, α^2. Because it has to satisfy $y-x^2$.

So, the first one would be α and the second one would be α^2. So, one can sort of imagine that and that is indeed the case and so, we should now try to formulate this idea of finding fibres from the description of the ring map. And we will come back to this picture. So, recall that we have a ring map $R \to S$.

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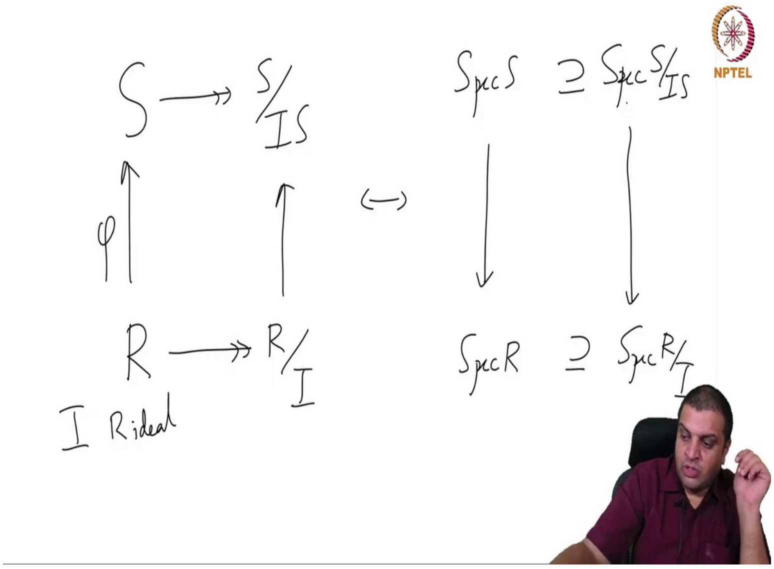

So, now if I is an R ideal, there is a surjective map $R \to \frac{R}{I}$ and there is a surjective map from $\frac{R}{I} \to \frac{S}{IS}$. The geometric side of the picture is we have $Spec(S) \to Spec(R)$.

the $Spec\left(\frac{S}{IS}\right)$ of this ring is same as one to one correspondence with primes containing I, so we can think of this as a closed subset of Spec (S). $Spec\left(\frac{S}{IS}\right)$ sits inside as a closed subset of Spec(S) and here, is $Spec\left(\frac{R}{I}\right)$ and then, we get a map like this.

So, a picture like this corresponds to a picture like this on the geometric side. Now, we would like to understand the fibres of this map which is what the picture there was. This is a quotient of a polynomial ring in two variables. So, we can think of this as S to S mod J something, it is not IS, but , it is something, and here we can think of this as, so this we can just think of as R, there is no I.

So, I agree that this is not the same as this picture; but, it is pictures like this that we would like to understand. Essentially, to figure out what are the; what are the fibres of this map.

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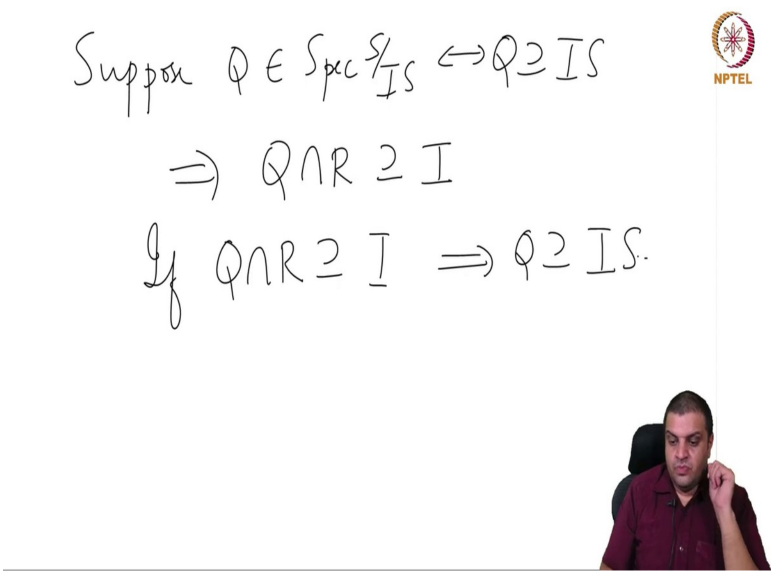

So, suppose $Q \in Spec(\frac{S}{IS})$ which from which we just say Q contains I S. So, abuse of notation here, we are discussing a prime ideal of the quotient, but here I am looking at this preimage and just calling it again Q. I mean instead there will be situations where one should not do this; but this is not one of them. which now implies that $I \subset Q \cap R$.

So, we were given only this arrow, we did not have this arrow. We just had these two closed subsets. By drawing this arrow, I had implicitly asserted that image of this set inside here, it lands inside this closed set which I have not proved and now, I proved it that if Q is an ideal containing I S, its image will contain I. So, in other words, a point here will map here. So, till then this arrow I had not proved that is exist, but there is such an arrow.

So, now conversely ok, if $I \subset Q \cap R$, then, $IS \subset Q$. So, that is the other direction which again is not explicitly stated here. Suppose, I take a closed subset here of Spec(R) which is $Spec(\frac{R}{I})$, what is its inverse image? Well, all the primes in the inverse image, I mean inverse image through this map will actually be primes here,. So, therefore, we get this diagram.

I had not checked that, this is a meaningful diagram, when I wrote it. Now, we have checked it. So, now, we would like to understand what is called fibre over a point in Spec(R). We do not need to worry. So, this explanation with I set was just to justify that this diagram with four arrows is correct. Now, we would like to understand the fibres of this map; $Spec(S) \to Spec(R),$.

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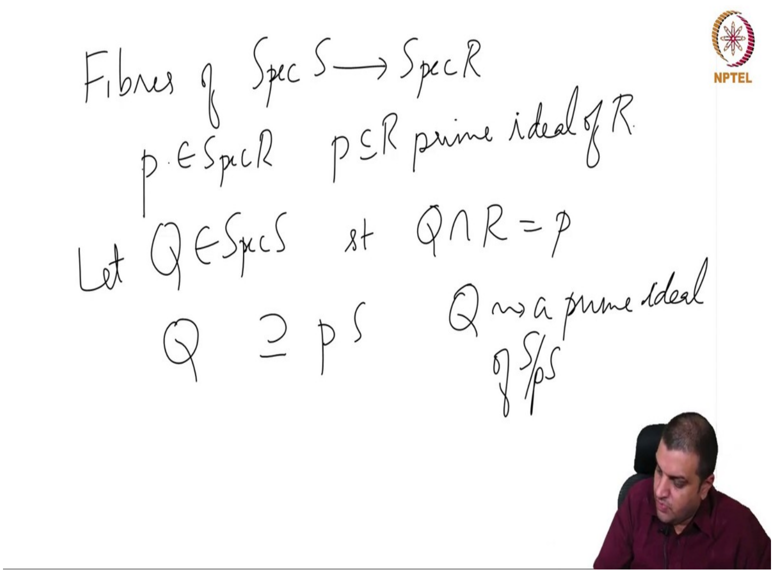

So, fibres of $Spec(S) \to Spec(R)$. So, what are points here? A point here is a prime ideal $pins\,Spec(R)$ that means, p is a prime ideal of R . Now, let Q be a prime ideal of S such that $Q \cap R = p$. So, what can it say? So, then Q will contain pS; is in other words, Q gives a prime ideal of $\frac{S}{pS}$. So, that is one observation about such Q.

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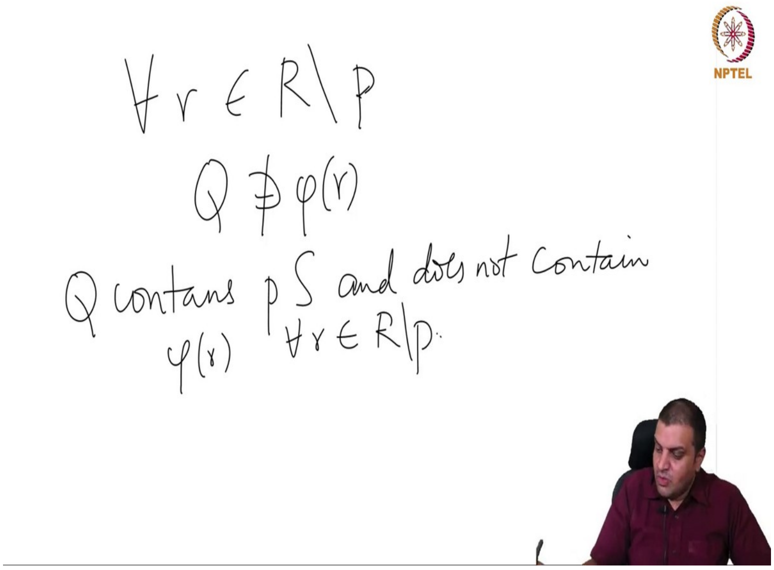

Another observation is for all $r \in R \setminus p$; Q cannot contain $\varphi(r)$. Remember phi is a map from R to S because if Q contains $\varphi(r)$, then $r \in Q \cap R$, but $r \notin p$. So, the Q cannot contain this. This is true for all r. So, let us put these things together. Q contains pS and does not contain $\varphi(r)$ for all $r \in R \setminus p$. When can this happen?

We can think of $\frac{S}{(PS)}$; so first of all Q has to correspond to a prime like this and. So, notice that there is a map. So, $\frac{S}{(PS)}$ is an S-module, this is an also an R-module.

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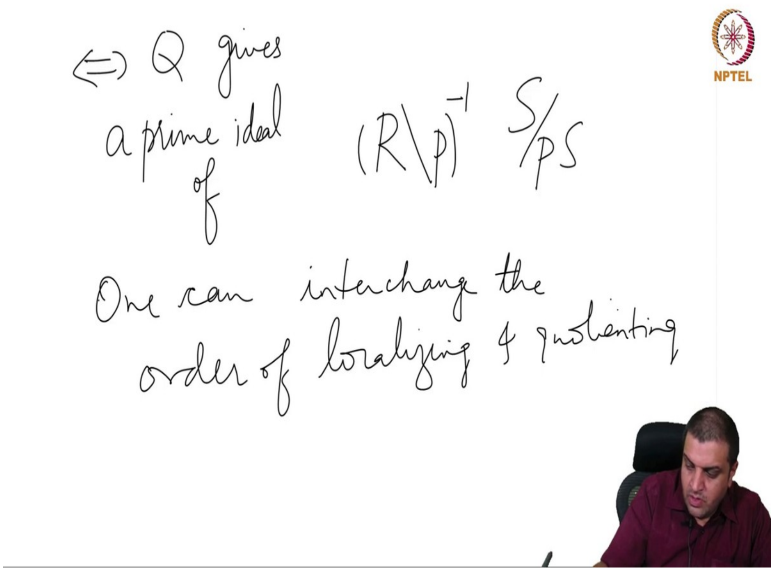

So, we can invert elements of this $R \setminus p$ inside that, this is a multiplicatively closed set inside R, we think of this as an R-module and then we invert this set. So, this will also be a ring, these things can be checked, ok. So, what it is saying is that Q has this behaviour if and only if Q gives a prime ideal of this ring. Again, when we say $R \setminus p$ inverted, we mean their images inside S inverted.

So, that is how R acts on S, so this is what the about this fibre over p looks like. Every prime Q with this property, this is what it looks like. So, typically when we say the fibre over p which ring theoretically, we just mean this ring. So, this will be a ring.

So, we just mean this ring. There are one can also think of it as we can invert elements, so going modulo some submodule and localizing here commute with each other, so we could just first take S invert all these elements; then, take pS and invert all these elements and then, go modulo this. but this is the ring that we have; we will this is the ring that is would be considered ring theoretically as the fibre over the prime p. So, we can think of it in slightly just one remark. so, one can reverse the order of localizing and inverting and going modulo. So, one can interchange the order of localizing and quotienting and can also describe this as.

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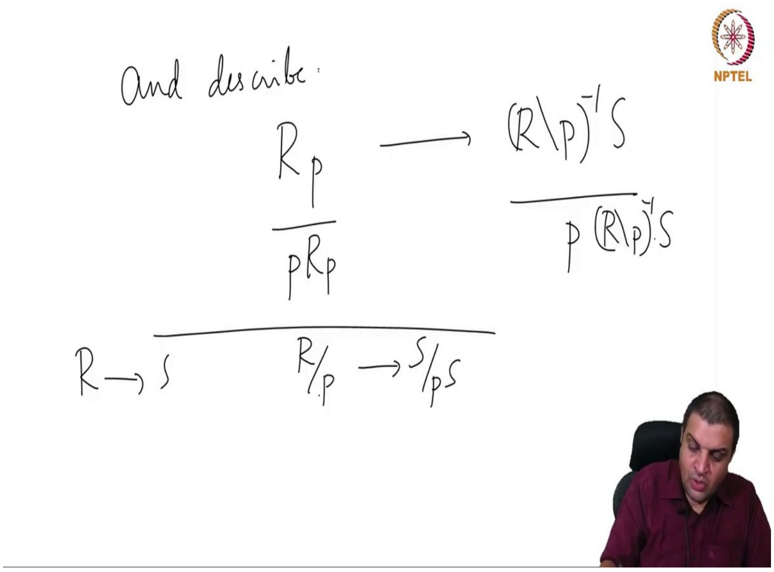

So, we can and describe this as follows. We take $R_p \to R \setminus p^{-1}$ then, we kill pR_p .

$$\frac{R_p}{pR_p} \to \frac{R \setminus p^{-1} S}{p R \setminus p^{-1} S}$$

This is the same thing as $\frac{R}{p}$ localized in the complement of this and here, we would just kill the extension of this ideal. So, we take that map and then, go modulo of this and then, here we would just kill the extension of p into this ring. So, this is also another way to think about it and yeah.

So, this is how fibres are described , I mean given a ring homomorphism what do the fibres of the corresponding map and Spec look like. So, just one more observation that we would like to make here, here we have first. So, what is the operation? So, this is say so we have $R \to S$, then we take $\frac{R}{p} \to \frac{S}{pS}$ and then, we just invert all the non-zero elements here. This is the domain; $\frac{R}{p}$ is a domain .

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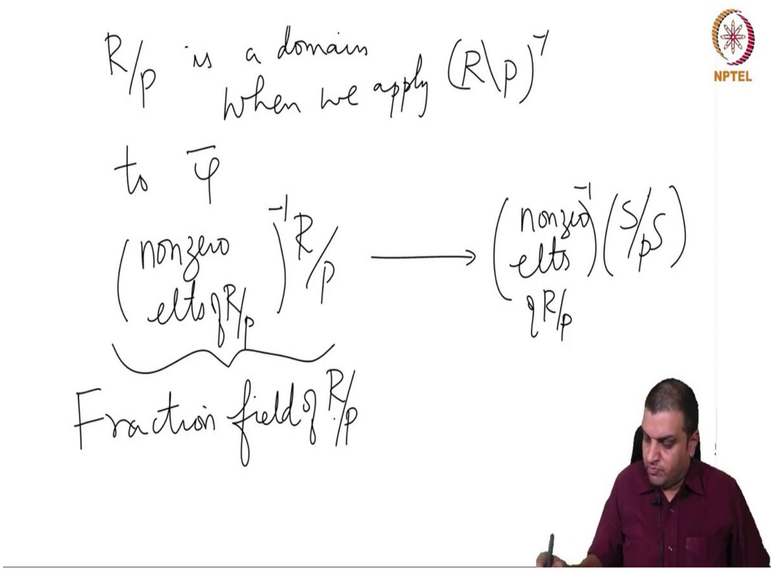

So, then what we are when we invert, so when we apply $R \setminus p^{-1}$; so. Let us call this map φ ; what are we doing? We are we do the we take $\frac{R}{p}$. So, image of $R \setminus p$ inside $\frac{R}{p}$ is a set of non-zero elements. So, here we invert non-zero elements of $\frac{R}{p}$. This is a domain, so the set of non-zero elements is a multiplicatively closed set and then, we do the same thing here, non-zero elements to this ring $\frac{S}{pS}$.

So, this is a more concrete way of saying what is going on in that thing and what is this? This is the fraction field of $\frac{R}{p}$. $\frac{R}{p}$ is a domain; so, this is just a fraction field and this is the map that you would get.

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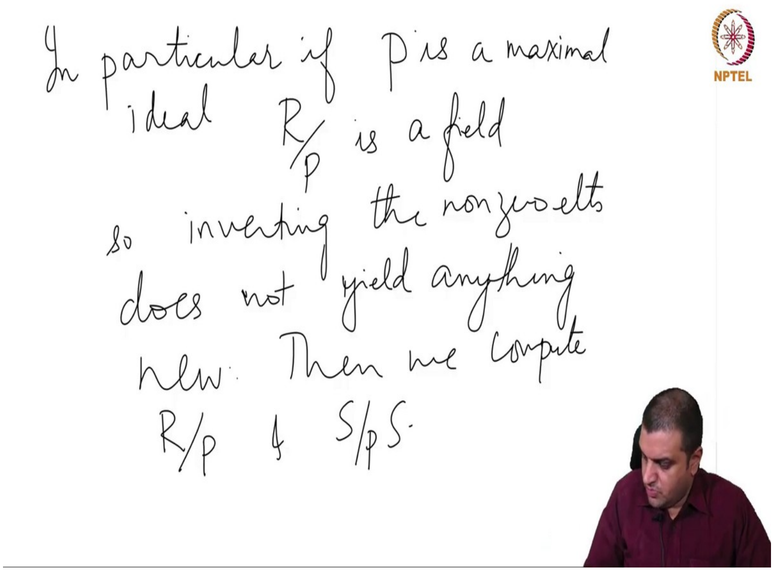

So, in particular, if p is a maximal ideal, we do not need to do this invert elements of R non-zero elements of $\frac{R}{p}$, $\frac{R}{p}$ is already a field, ok. So, inverting the non-zero elements does not yield anything new. So, this is an important point in this which will be useful when we go back to analysing that parabola.

So, then, it would just what we will just compute here is if p is a maximal ideal, we just have to compute $\frac{R}{p}$ and $\frac{S}{pS}$, that is all that we need.

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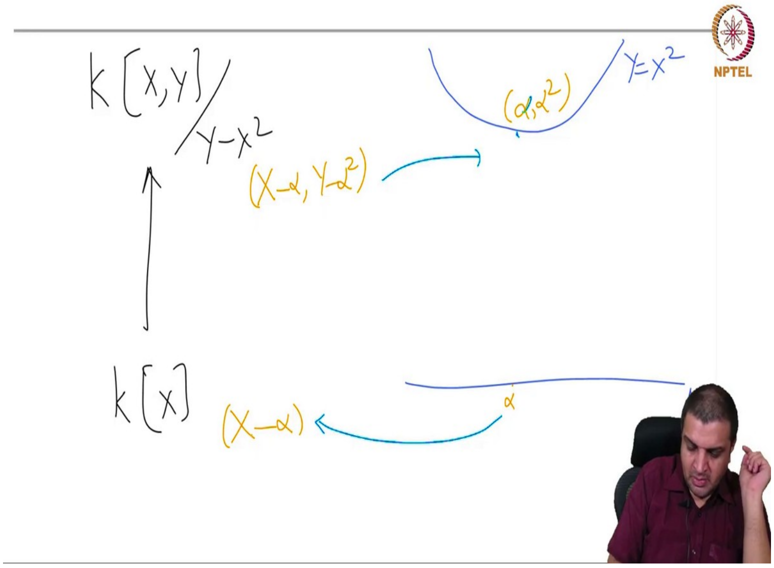

So, now let us go back to the example, ok. So, the map is $k[X]\to\frac{k[x,y]}{(y-x^2)}$, what does? So, here is the parabola, maybe this is not a very. Sorry, let me because I want to project to the X axis, let me reorient the diagram. So, here is this, where I this maxSpec of this is just one line. Meaning, it is just k^1. Here is this parabola $y=x^2$. let us take some point here, α that corresponds to the maximal ideal $(x-\alpha)$ here.

So, the order in which we are doing is we start with a point here then we got this maximal ideal. Then, we do this computation $\frac{R}{p}$ and $\frac{S}{pS}$ the reason, we do not need to do this bit here. So, now, let us take this ring and extend this ideal there. So, what is k?

So, not just extend go modulo of that $\frac{S}{pS}$. So, this ideal modulo, the extended ideal from $(x-\alpha)$.

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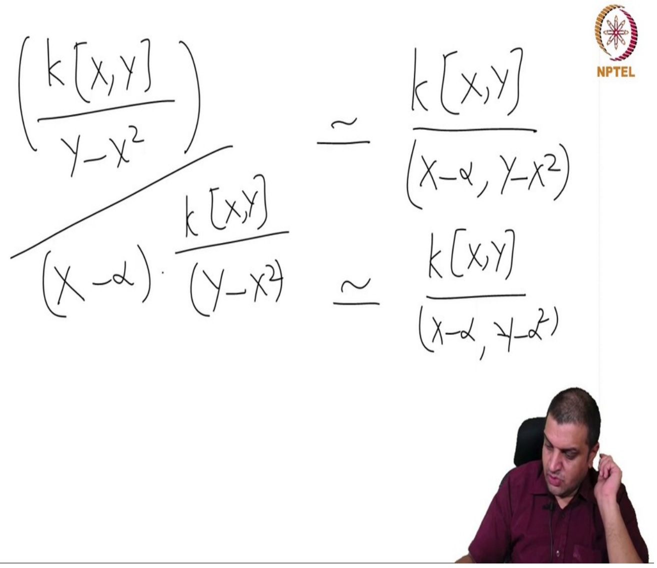

So, $\dfrac{\dfrac{k[x,y]}{(y-x^2)}}{(x-\alpha)\dfrac{k[x,y]}{(y-x^2)}}$, and this is just easy to check that this is isomorphic to $\dfrac{k[x,y]}{(y-x^2, x-\alpha)}$.

And again, this is not very difficult to check that this is the same thing as $\dfrac{k[x,y]}{(y-\alpha^2, x-\alpha)}$. So, now, so let us go back to this calculation here.

So, the maximal ideal that we have here the fibre over this thing is corresponds to the ideal corresponds to $(y-\alpha^2, x-\alpha)$ and that corresponds to the point here α, α^2 . So, that is how we this calculation we do. So, we start with an α , we consider this prime ideal here. Extend it and see; what is the quotient ring that we get. So, it is a quotient ring of $k[x,y]$ modulo this ideal.

So, inside k^2 , it is the point (α, α^2) and so, this is ok. So, now let us do the same thing for the Y-axis, where the picture is slightly different, ok. So, there I will try just to give the picture in our mind, I will try to do it. I will change the orientation of the diagram.

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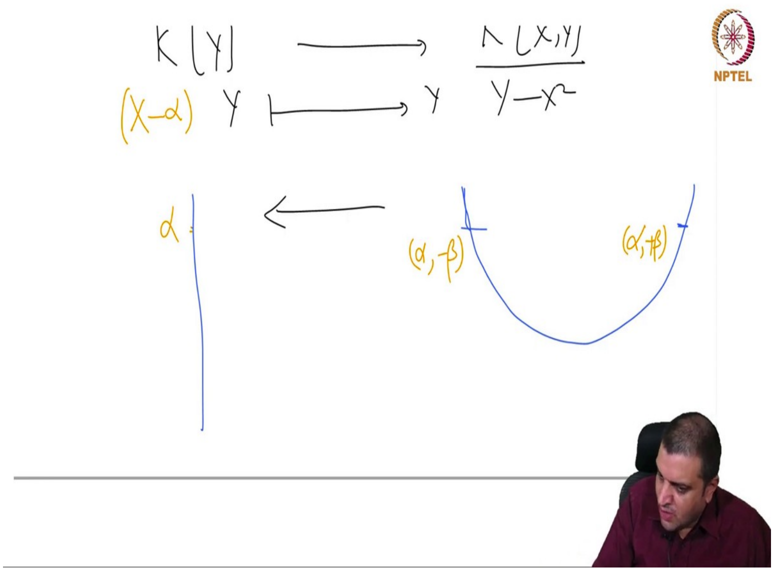

So, now, let me write $k[Y] \to \frac{k[x,y]}{(y-x^2)}$ and the map is Y goes to y just the usual map a natural map. So, it is k[y], then I join an x and set that x^2 is y, that is we do, ok. So, this is still the same, the parabola; this is the Y axis. So, I let me just for a picture let us try I mean, ok.

So, that it corresponds to this picture. Let us draw in this way, ok. So, then let us continue, I mean let us ask this thing again. So, we take a point α here, we take a point α there; alpha corresponds to the prime ideal $(X-\alpha)$ maximal ideal $(x \, minus \, \alpha)$ here, then we look at its image there.

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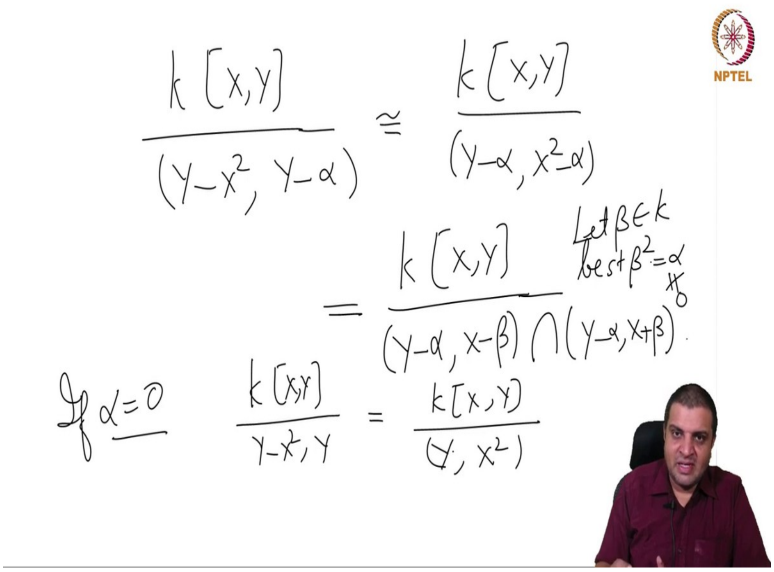

So, let us work that out; $\frac{k[x,y]}{(y-x^2,y-\alpha)}$ isomorphic to $\frac{k[x,y]}{(x^2-\alpha,y-\alpha)}$.

So, this is not a maximal ideal the reason is so we its. So, for now let us just assume that characteristic is not 2. We are working over complex numbers let us say, then this has a decomposition .

So, let $\beta\in k$ be such that $\beta^2=\alpha$. So, $x^2=\alpha$ which says that $x^2=\beta^2$ and it has two solutions let us say we have complex numbers. So, it has two solutions; $x-\beta$ and $x+\beta$. Even if you are in characteristic 2, it would have one solution of multiplicity 2.

So, the description of the ideal will be different that is assumed in case $(Y-\alpha,x+\beta)$, . So, this ideal decomposes like this, and each of them are maximal ideals. So, this is the maximal ideal that is a maximal ideal. So, Spec of this consists of two points; one corresponding to this maximal ideal and one corresponding to this maximal ideal.

So, let us go back to this picture. So, for this α, there will be two points which is which corresponds to the x value takes the two square roots of the y value. So, this will be let us say this is $(\alpha,-\beta)$. So, all of this picture is only approximate, because we are drawing the picture in R, while all of this is happening in some algebraically close field.

Sorry not this is, sorry, ok. So, here would be the point (α, β). So, we have these two points which are in the fibre of this map.

So, the map here is remember is in this direction. These two points map to α. And, here if you take an α, it will appear from this picture as if you take an α here its fibre is empty, but that is only that is false, because we have this is only some real picture that we are trying to write to understand, what the actual situation is however, there is just one point that we should look, ok. What if $\alpha = 0$?

So, this assumes that there are two distinctive thing α non-zero. So, this is only if it is nonzero, ok. If $\alpha = 0$, then we would just $\frac{k[x,y]}{(y-x^2,y)} = \frac{k[x,y]}{(x^2,y)}$.

So, if you think of it as a point it is only a single point, because this corresponds to one maximal ideal (x,y). But as a ring theoretic situation, we do not want to think of it as one point; we want to think of it as a point with some extra structure and which is that there are some extra functions. so, this is strict I mean the point there that you are thinking about is not exactly a variety, it has some extra functions and extra structures on it, which we will not pursue it any further.

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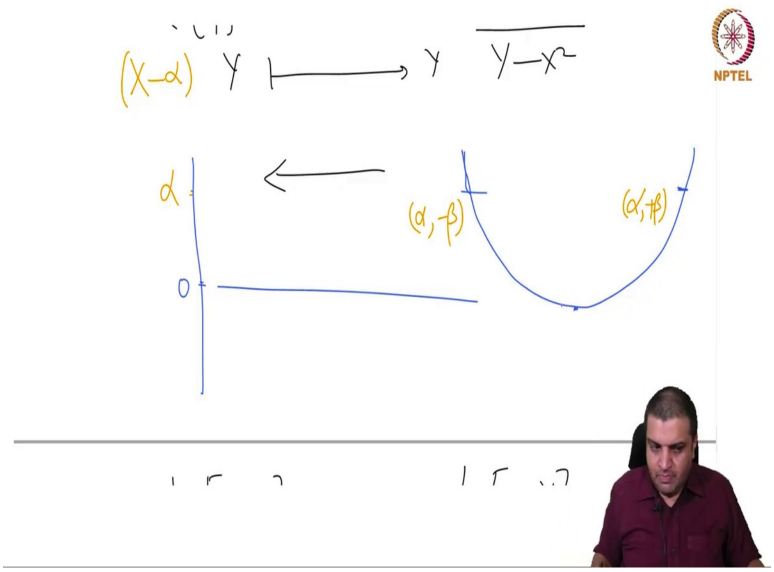

But at least, we should keep that in mind, the fibre over 0 here is a singleton point; is a single point, but it has some extra structure, it is all. So, I mean these are two distinct points. So, the structures on individual points is simple, but here that is not the case. So, this is the example of this what these maps mean.

So, in the next lecture, we will look at the same a similar picture. So, this is just where depending on where we are projecting the nature of the map changes and we will take that as a starting point to understand morphisms between two rings in somewhat greater detail.