Computational Commutative Algebra
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Lecture – 28
Saturation – Part 3

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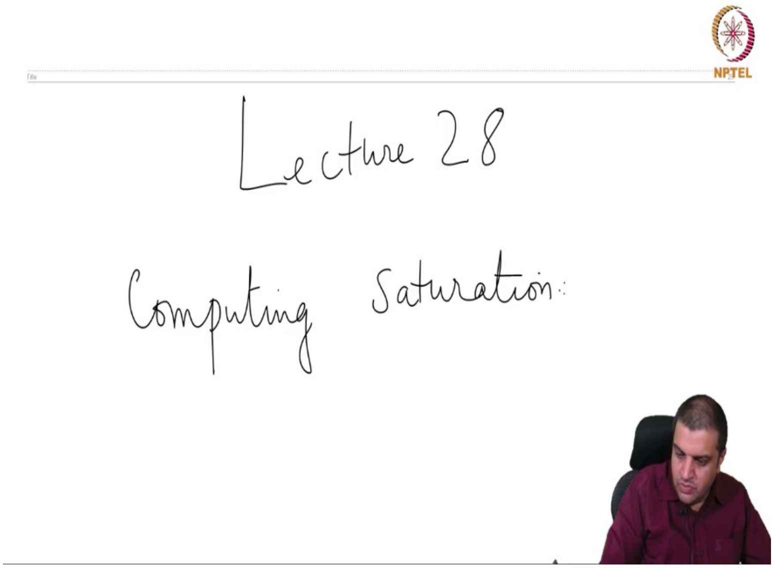

So, in this lecture we continue with studying Saturation and we look at a slightly more complicated example and again we will not be able to explain all the computations in it. Then, we will study a little bit about how saturations are done and then we can revisit this problem.

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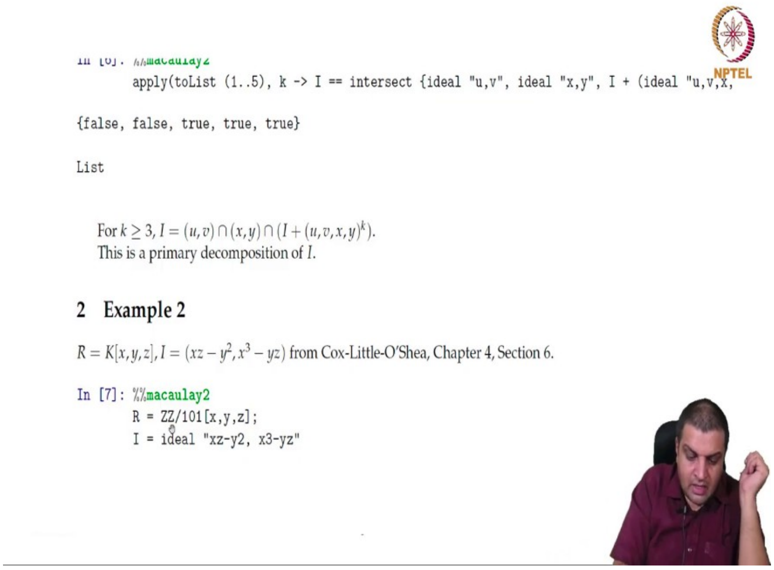

So, let us look at the example. So, this is taken from this book of Cox Little O'Shea, Chapter 4, Section 6. So, the ideal is generated by (xz - y^2, x^3−yz) in a polynomial ring in three variables. And so, we just do some field finite field of 100, this is a prime number. So, I is an ideal by this.

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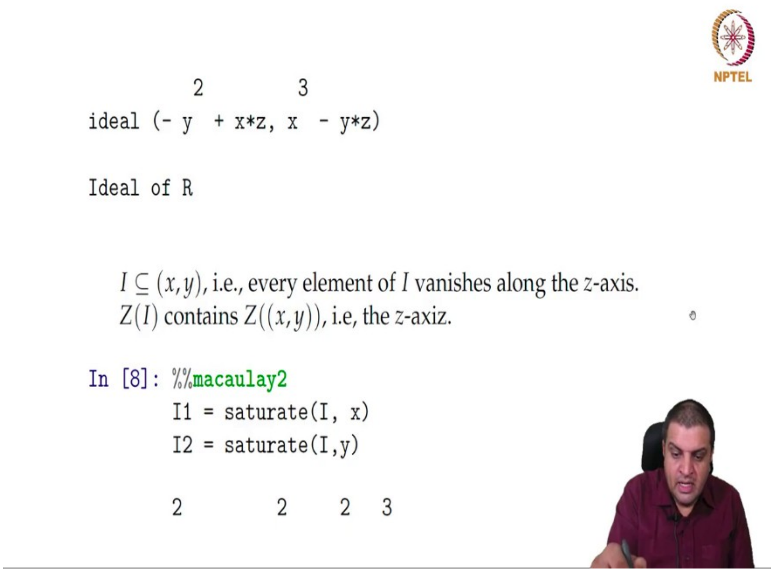

So, just by looking at this we notice that every term of every generator of I involves x or y. So, I ⊆(x, y). In other words, Z(I), so, when we say Z(I), it is not over this field, but it is over the algebraic closure of the field, where it was defined. It was defined; it was defined over some field, but when we talk about Z(I) we mean the algebraic closure effect, but anyway that is not in the calculations I it is just a remark.

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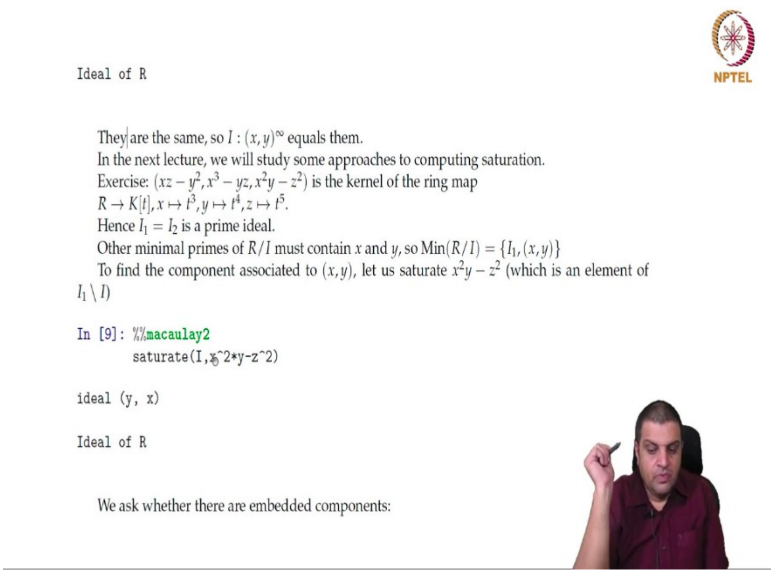

So, Z(I) will contain I is contained inside here. So, Z(I) will contain the I, so Z(I) contains this. But what is this, what is the point where x and y would vanish? It is a z -axis. So, we will try to saturate this ideal one variable at a time that is what we learnt. We need to remove the component corresponding to x, y then we need to saturate this ideal here. If we; I mean we will see this. So, we will do it for one generate at a time. So, let us saturate x and then I_2 saturate y.

So, then we just notice that I_1 and I_2 are the same. And so, these this and this is the given generator and we have a new generator x^2y-z^2, which is not in the ideal given by x and y. But, since these two saturations are the same, so, there it is equal to I:$(x,y)^\infty$.

So, in the next lecture or later in this lecture we will study some approach just to computing saturation. Right now, we will just continue with this thing. So, we need to understand this prime ideal this ideal now, what does it look like.

So, the observation that we make is this is the kernel of a ring map from R= K [x, y, z] $\to$ K[t]; x $\to t^3$, y $\to t^4$, z $\to t^4$and the exercises I will give a hint on how to prove this. So, in other words this given ideal I_1 or $I_2$with the which are same is actually prime ideal. So, what have we concluded so far?

We have concluded that here is a prime ideal and the any other prime; so, this I mean as soon as you saturate x we get this prime ideal or as soon as you saturate y, we get this prime ideal which means any other prime ideal any other minimal prime which is not this; which is not this I_1 or I_2 must contain x and y but notice that (x, y) itself is a prime ideal containing this ideal. So, there are only two minimal primes; $I_1$and (x, y).

So, now, we need we will; so, for the component corresponding to I_1,it itself is the primary component. I mean, associated prime and the primary component ideal are the same because it ok, but, for the other one let us take an element inside here. So, notice that, ¿ $x^2 y - z^2$ ¿ ¿) is not there in this ideal. So, if you saturate I with respect to this element, we would pick out the component corresponding to x, y and it just says (x, y).

So, therefore, we now have concluded that I = I_1 intersect this, I_1 intersect (x, y) and we asked I mean sorry I take it back. We have concluded that for the minimal components are I_1 intersect (x, y), but there could be an embedded prime, which we do not know.

So, we ask. So, we intersect I 1 and the ideal generated by x and y and we ask intersect that and then we ask is I equal to that and it says it is true. So, which means, we have now found the primary decomposition which is I = I_1 intersect (x, y).

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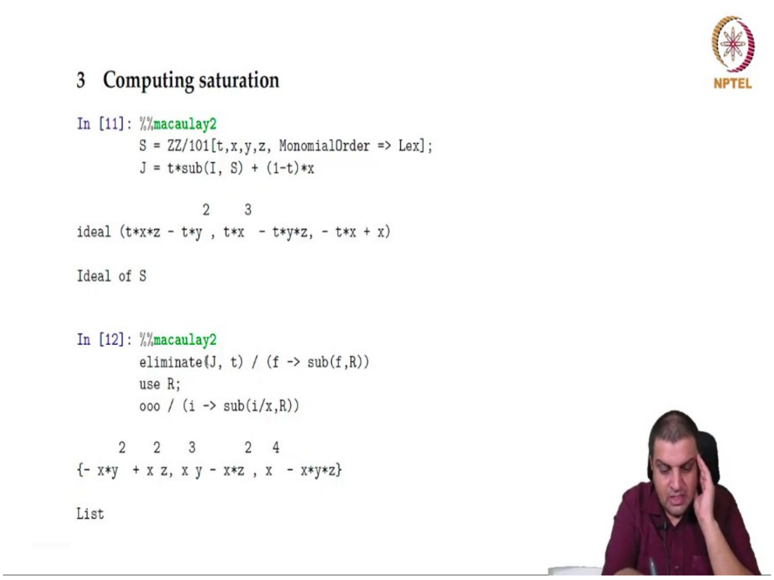

So, this example we will discuss after we study some techniques behind computing saturation. So, this is about computing saturation.

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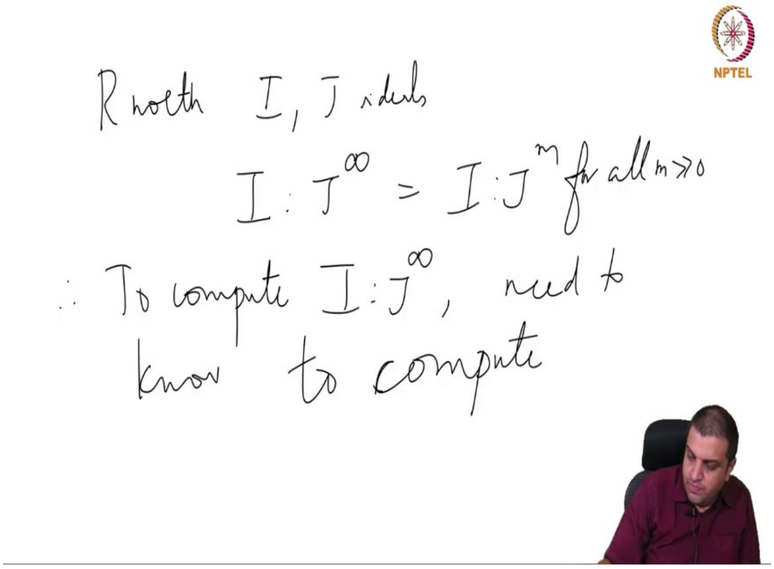

So, recall that if R is Noetherian and when we say computing it mostly refers to some polynomial ring or a quotient of polynomials ring, but we will develop these ideas in slightly generality. R - Noetherian, I and J ideals, then $I:J^{\infty}=I:J^{m}$, for all sufficiently large n, this is what we know.

So, if you want to find the saturation its sort of enough to understand how to compute colons. Hence, so, in in principle we could just compute some $I:J, I:J^2, I:J^3$and at some point try to guess whether it has stabilized and then use that as and decide whether that is a saturation.

So, that is the, this one we could do that. So, if you can just learn. So, to compute $I:J^{\infty}$, we need to know just I: J, I mean different J as J runs over various things. So, you know we need to know to compute I: J , some colon of two ideas.

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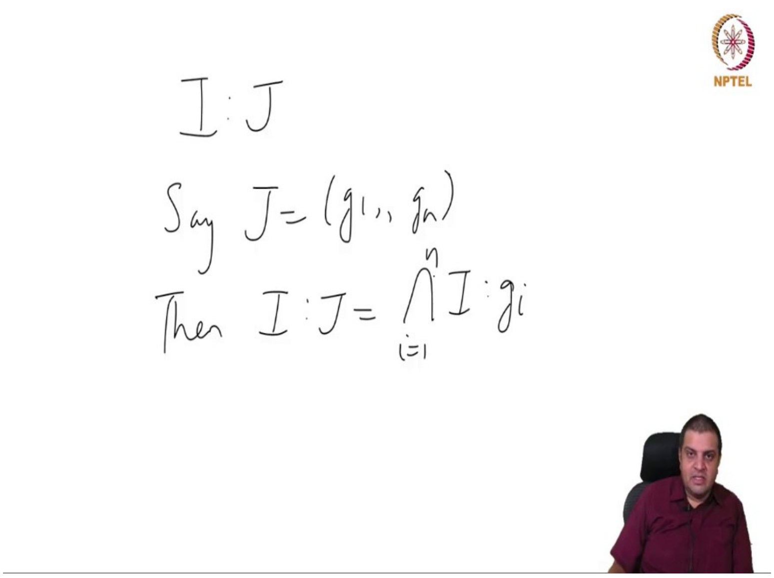

But how do we, I mean what about these? So, if say $J=(g_{1,\ldots,}g_n)$let us say, then $I:J=.\bigcap_{i=1}^{n} I:g_i$. So, algorithmically if we know how to intersect and how to take colons with one element then we can do this. So, therefore, we need to learn two things; how to intersect ideals, and how to take colons.

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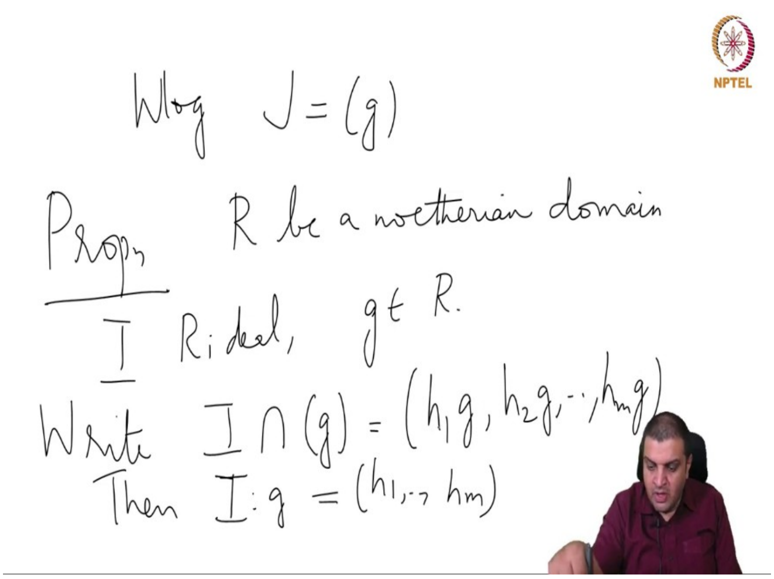

So, without loss of generality for now as far as colon is concerned we can assume that J is principal. So, proposition; let R be a Noetherian domain, I an R -ideal and g ∈R. Write

$I \cap (g) = (h_1 g, \ldots, h_m g)$ as, so every generator of this is going to be divisible by g as h_1 will be a multiple of g.

Then, $I : g = (h_1, \ldots, h_m)$. So, just let me read this once more. So, we have an ideal and an element g and we want to know what I: g is. So, to compute I: g we first express I intersect g.

So, $I \cap (g)$ would be would be finitely generated, and every generating set will be multiple of g. Every element in the generating set would be a multiple of g, because it is a subset of g and then the colon is just this h_1 throughh_m. So, we are not saying that h_1 through $h_m$is unique or anything. Just take any generating set, it will have this sort of a form and then just remove the g from them.

So, that is a proposition. And, just one more observation; this is not I mean we will not prove it, but notice that if you take I: g and then multiply them by elements of g it actually lands inside this intersection. It is inside I by definition, and it is also inside g because there are multiples of g. So, it is does land inside that intersection.

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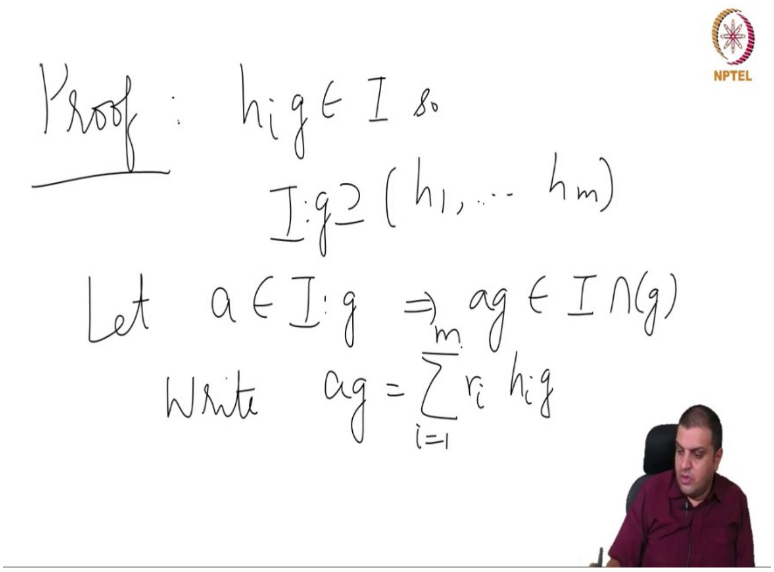

Proof: $h_1 g \in I$. Therefore, $I : g \supseteq (h_1, \ldots, h_m)$). So, this proves one inclusion. And this inclusion does not we did not use the fact that it is a domain, it is just. So, now let us do the converse.

So, let $a \in I : g$. This implies that $ag \in I \cap (g)$. Therefore, we can write it $ag = \sum_{i=1}^{m} r_i h_i g$.

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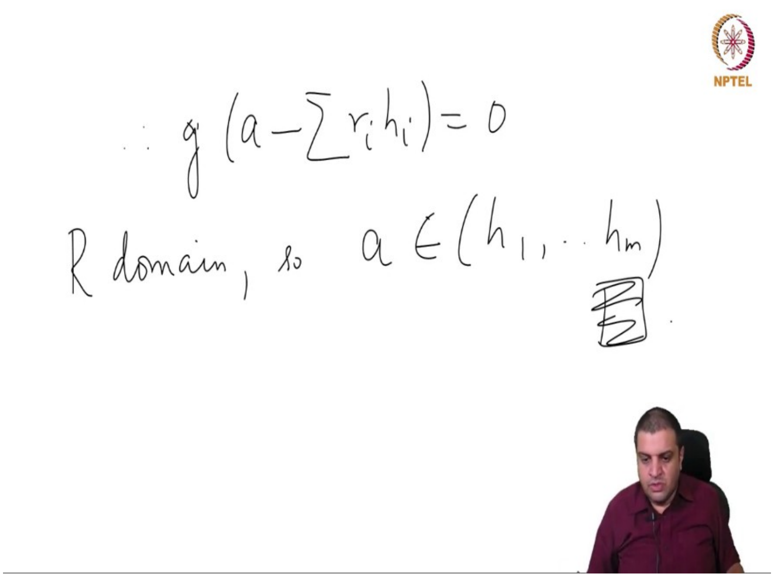

So, therefore, we can write $g\left(a - \sum r_i h_i\right) = 0$. Now R is a domain. So, $a \in (h_1, \ldots, h_m)$which is what we wanted to prove. So, this is where domain is used. So, this gives us a way to compute I:g. So, let us go back where we were we wanted, we wanted to be able to take colons well we wanted to do saturation.

We wanted to do saturation that we concluded that if you can intersect if you take if it if one can compute colon ideals arbitrarily then one can do saturation also. For colon ideals we have to do for taking colon ideals we need algorithms to do two things; one intersects ideals and then take colons of ideals of principle ideals.

So, the principle ideal part is what we just described doing one once, but that itself involves taking an intersection. So, it might be useful to learn to come to do intersection first. So, let us do that now.

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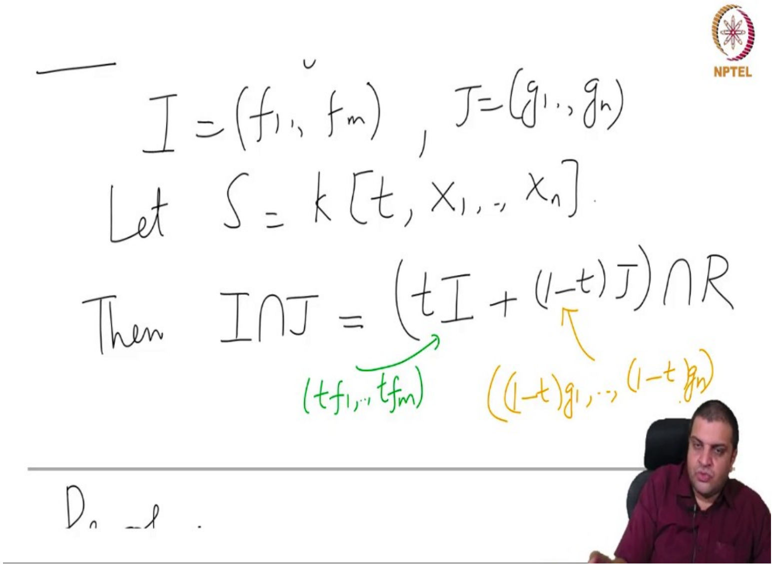

So, this is now, we will restrict ourselves to a computational setup where we can actually do these things. Let k be a field, $R=k[X_{1,\ldots},X_n]$. $I=(f_1,\ldots,f_m)$ and $J=(g_1,\ldots,g_m)$both are ideals.

Let $S=k[t,X_{1,\ldots},X_n]$. Then $I\cap J=(tI+(1-t)J)\cap R$, so this is an S ideal contracted to R. So, let us just read this.

So, this settles the question this gives us a way to compute an intersection of two ideals in a polynomial ring , which can be used now to just to address this part of intersecting I with the principle ideal and then somehow we have to divide this that is.. And then, it also takes care of this intersection of these colon ideals.

So, essentially if you understand this I mean if you can use this theorem then we will be able to do saturation by hand and with the help of a computer. Of course, I can ask you to saturate, but that is not what we meant. So, we will prove this theorem and then go back to that example.

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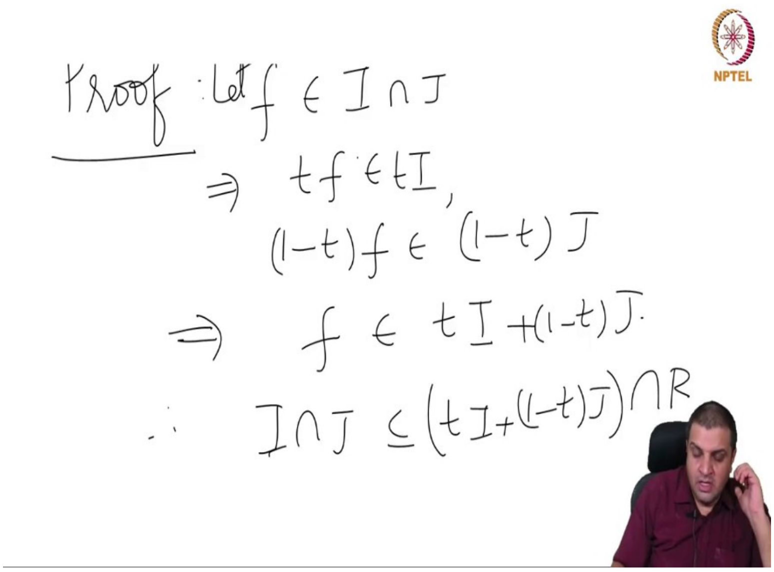

So, first of all I mean I hope this is clear what is meant here. Take, so this is the ideal generated, they should just write. $tI = (tf_1, \ldots, tf_m)$ and (1-tJ) = $((1-t)g_1, \ldots, (1-t)g_n)$. So, it is an S- ideal and that contractor R is would be I intersects J.

Proof: Consider $f \in I \cap J$. Now, so, what does it mean to say that $f \in I$? So, this now implies that $tf \in tI$ and $(1-t)f \in (1-t)J$, because f $\in$I$\cap$ J. So, now this implies that f which is the sum of these two things, $f \in tI + (1-t)J$. So, $I \cap J \subseteq (tI + (1-t)J) \cap R$. So, this is one f is an element of R; so, it is inside R, we get this; so, now the other direction.

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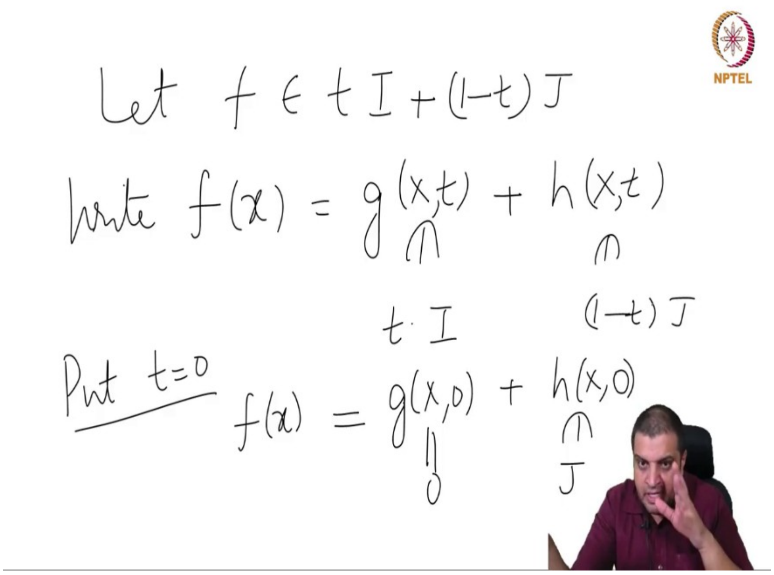

So, let $f \in tI + (1-t)J$. So, now, we can write because it is inside here, we can write f(X) = g (X, t) + h (X, t) where $g(X,t) \in tI$ and $h(X,t) \in (1-t)J$. We just saying that f is in the sum of these two ideals, so take one part from here and one part from there.

So, now, let us look at what happens when we put 0 for this; so, put t= 0. Nothing happens to X because t does not appear in it at all. We get f(X) = g (X, 0) + h (X, 0), but what are these? Well, this is so, g(X,0) = 0 and $h(X,0) \in J$, so it is going to be t times some element of I, but S some element t times some S linear combination of generators of I, not an element of I, but an element of the extended ideal.

So, S linear combinations of elements of I times t. So, when you put t equal to 0, we just get this to be 0, and here we would just get some h(x), which is one can check without much difficulty that this is actually inside J.

So, when you substitute 0 here. So, we can write h (X, t) as (1-t) times some element inside the extended ideal of J to S. So, it is linear combination of the g's with coefficients coming from S. When you substitute t = 0 this just becomes 1. So, we can take that out. Then, the generators are untouched and the coefficients we put t = 0, which means we just get their degree 0 part t which is just some elements of R. So, this is in J.

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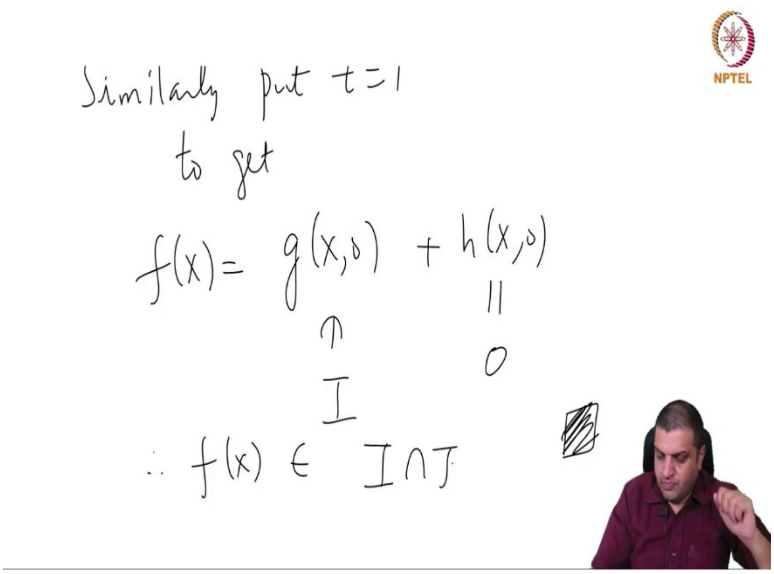

This says that $f(x) \in J$. Similarly, put t = 1, to get $f(x) = g(x, 0) + h(x, 0) \in I + h(x, 0) = I + 0$. So, f(x) $\in I \cap J$. So, this is the; this is the proof of this theorem. So, now, we have I mean an algorithmic plan which is what; so, if you if you want to intersect two ideals.

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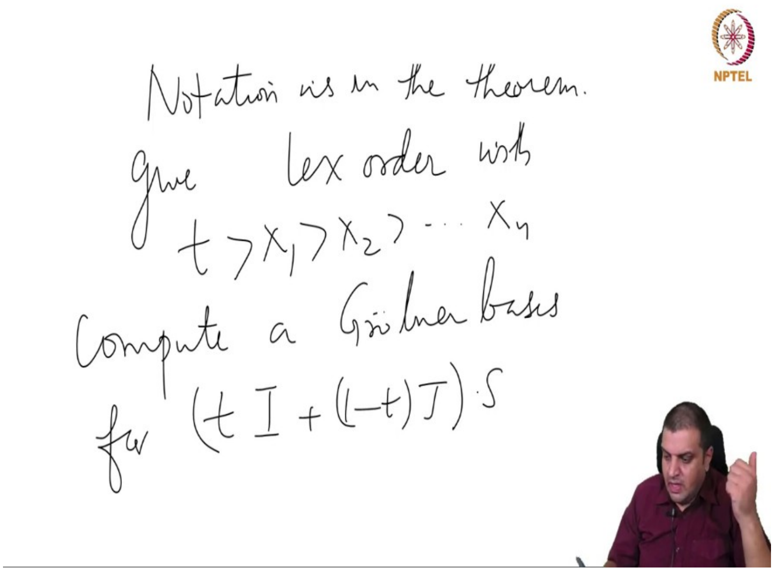

So, let so, with no notation as in the theorem. Give lex order with $t > x_1 > x_2 > \ldots > x_n$, then compute a Grobner basis for (tI+ (1-t)J) S.

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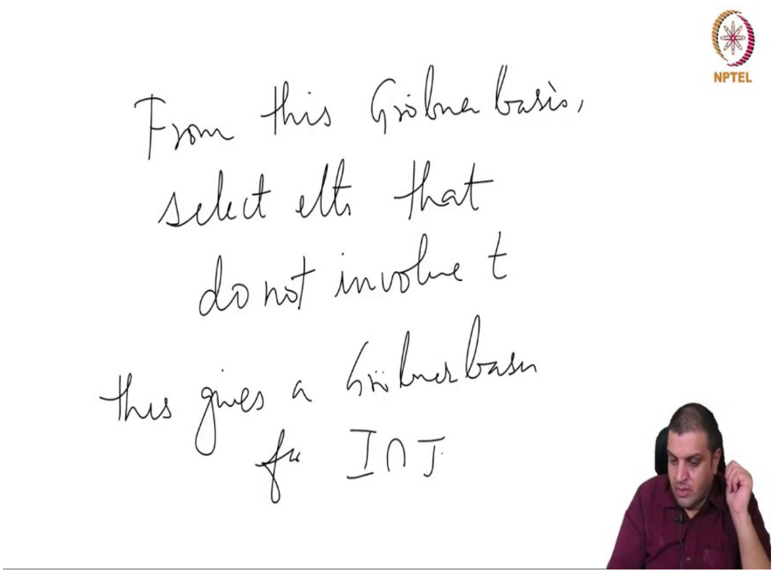

And so, this would now, imply, so from this now take from this Grobner basis, select elements that do not involve t and this gives Grobner basis, for I ∩J. And then, we can repeatedly we can do it for principal ideals their intersections etc. to now get semi colon and intersections.

So, let us just do this not the full thing, but in part in the previous example. So, this is again the same example continued example that we saw in the last lecture, which is R = ZZ / 101[x,y,z]. So, just any field not very big, so, the computations are faster; x, y, z; ideal is $\left(xz - y^2, x^3 - yz\right)$.

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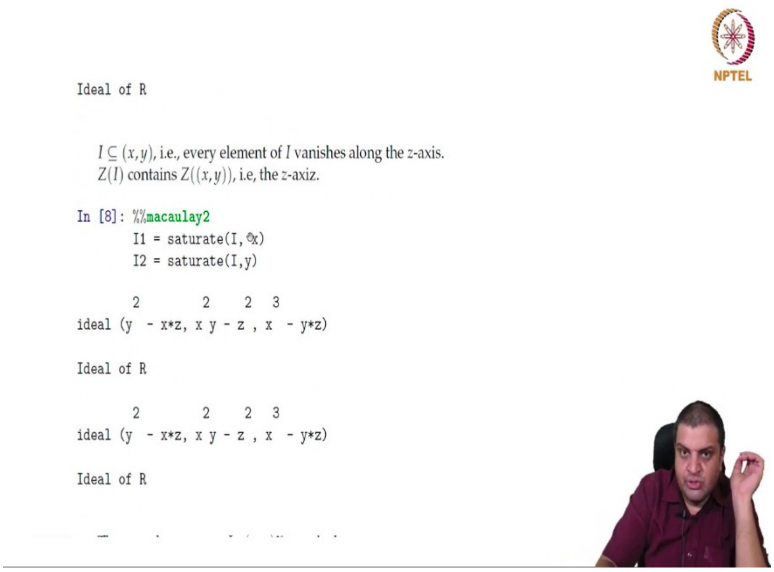

And, we would now like to saturate x and saturate y. So, let us see how what we just described now, we will let us try to do that, let us illustrate that. So, now, we construct this larger ring t, x, y, z, monomial order gets lex because we wanted to use it for elimination and then we just take t times. So, we need to use a sub command, this is the; if you have same set of symbols variables like x, y and z refer both in R and in S.

So, then we can if you just write sub (I, S), it would just extend the ideal and now treat x, y, z as elements of S. So, please look it up in this is sub is for substitute. So, please look this up in itself in my column and so, t times; so, this is just extended ideal t times the extended ideal of I times (1-t) x. So, the x here refers to x of S which is why we did not have to put a sub there, ok. So, we get some ideal here and then we would now we need to eliminate t from this ideal.

So, we asked for to eliminate and I did not print it, so what is this; eliminate (J, t)/f some function here. So, this is an abbreviation for the apply function. This is the same thing as up, so eliminate (J, t) will give some list and now to that list apply this function that is, so it gave some generators. Now, the ok, so this is just programming issue.

In all these in this context the variables x, y and z refer to those in S\R, but here we would like to construct an ideal R moving in any case t is not there. So, what are we doing here? Take, so take this list. So, yeah, so eliminate (J, t) applied to this function would just say would just rewrite the function, so eliminate (J, t) is this list, but then thought of as elements of S. So, we are just using a sub again to convert it to elements of R.

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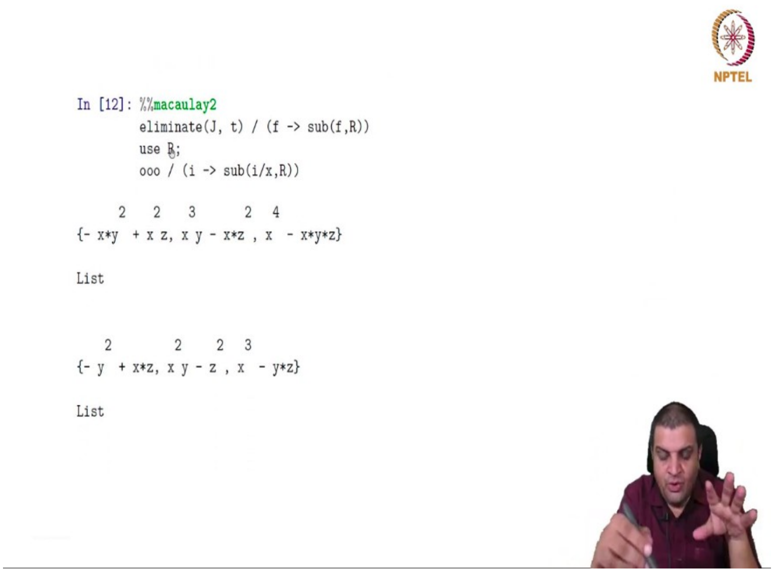

Use R just to say now the variables refer to element those of R, and then we ask it to divide these elements by; so, this is a, so I refers to not the previous output, the output before it which means to the output of this list here. So, to each element in that list each element I in that list divide I by x, but when we do this, it goes into what is called the fraction field of R.

So, given a domain integral domain one can localize at the 0, at the compliment of the 0 prime ideal and get a field which is this will be exercises this, so and then so, you will get an element in the fraction field of R, but of course, these were divisible by R by x, so that we know. So, therefore, we just convert it back to R.

So, for example, when we apply that run that command to this we would get you would drew x from here and x from there; so, $-y^2+xz$. Here x^2y-z^2 and this is what the saturation was for x. Notice that when you saturated this is what we have got; y^2-xz, z^2-x^2y and x^3-yz and notice that is exactly what we got here.

So, this is the end of this lecture and in the next lecture we are we will discuss start discussing a new topic which is about, so first would be sort of an introductory lecture on what these morphisms between rings being for Spec. And, then you will use that as a motivation to study what is called finite morphisms or integral morphisms.

And, in that course of discussing that we will prove theorem called Hilbert's Noetherian normalization lemma and from which we will derive a version of Nullstellensatz, from which we will prove the Nullstellensatz that we had used earlier.