Computational Commutative Algebra
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Lecture - 27
Saturation – Part 2

Welcome to lecture 27. So, we continue our study about Saturation and try to come up with some way to identify primes that have certain containment properties and then we will see some Macaulay2 examples and in the next lecture we will look at some ideas behind the behind computing them algorithmic Computation of Saturation.

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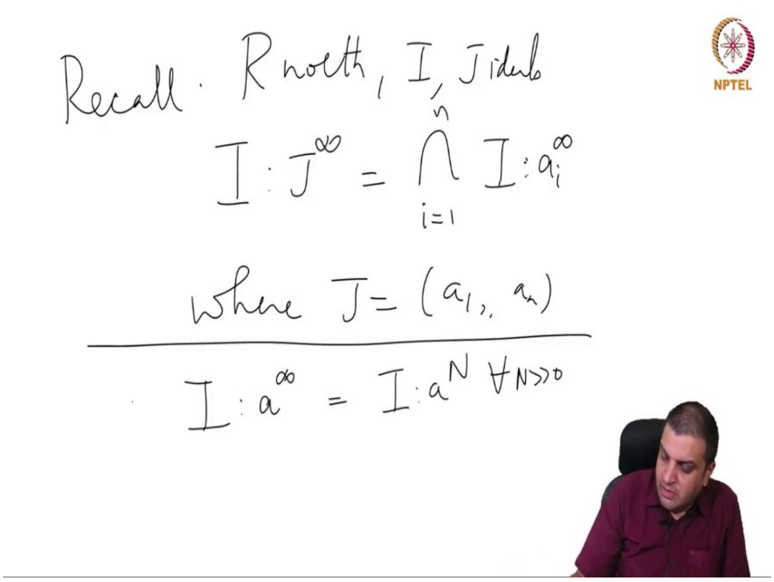

So, in the last lecture recall that $I:J^{\infty}=.\bigcap_{i=1}^{n} I:a_i^{\infty}$. So, this is all in R- Noetherian ring and I and J are ideals and this is J is generated by $\{a_{1,}\dots,a_n\}$, this we saw last time.

So, if we want to find a way to saturate elements it might be useful to understand saturating with respect to 1 element. So, how large and we also know that we also saw that, also $I:a^{\infty}=I:a^N$, for all N sufficiently large. So, we saw these two observations, I mean these two points in the last lecture and how large should this N be?

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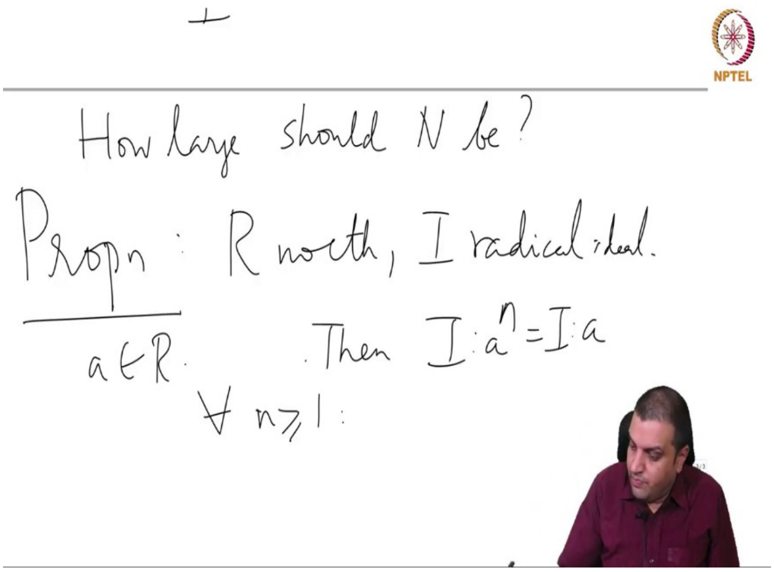

So, I will give one example which is relevant to these geometric discussions, but there are other ways one can understand this and more will be in some other variations of this will be there in the exercises.

So, let us prove one proposition in this. R - Noetherian and I - radical ideal, a $\in$ R. Then $I:a^n = I:a$ for every n $\geq$1. So, I will get saturated by a at the first step itself for a radical ideal.

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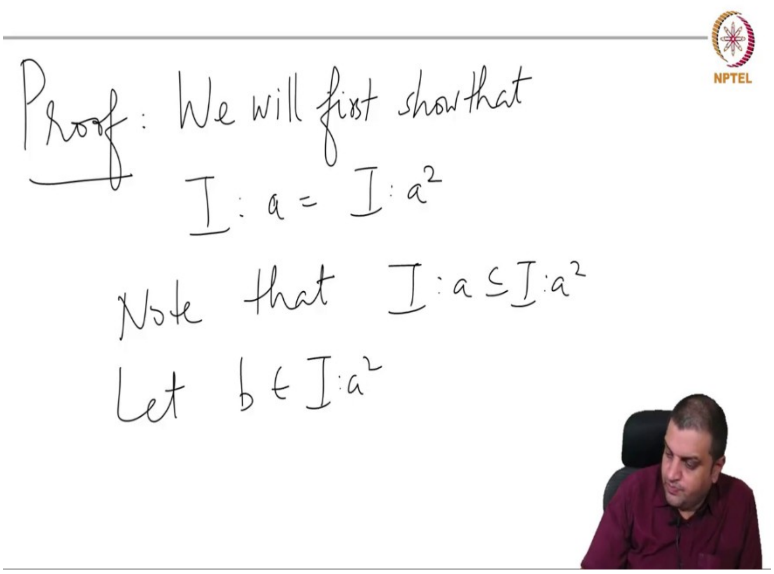

Proof: So, this sort of an argument that we see in this proof is somewhat typical in arguing about colon ideal. So, this will give you some familiarity to solve some of the related problems in the exercises, . So, we will first show that $I:a^2 = I:a$. So, what does I mean how does that follow?

So, first of all there is always one containment note that anything that multiplies a into I will also multiply a square into I because I is an ideal. So, $I:a \subseteq I:a^2$. So, let $b \in I:a^2$. Then $b a^2 \in I$, but I is an ideal.

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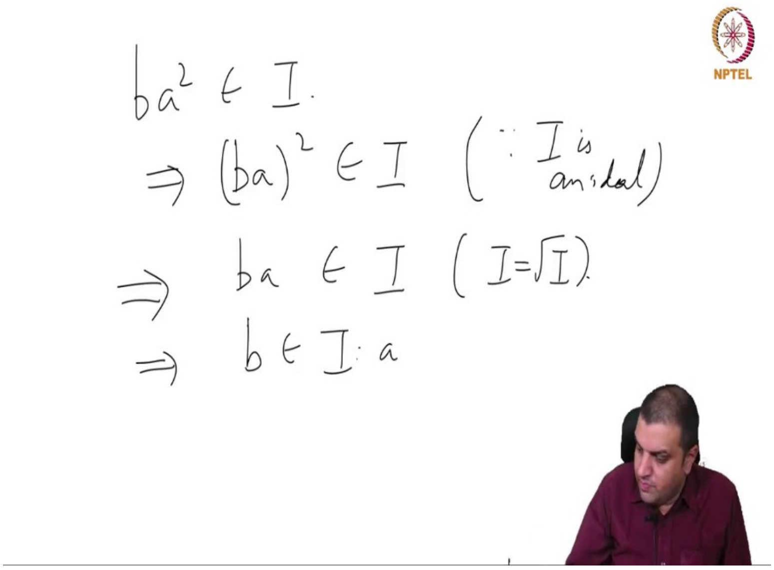

So, this is because I is an ideal and this implies that ba ∈ I. So, this is why I is its own radical and which was what we wanted to prove which is that $b \in I:a$.

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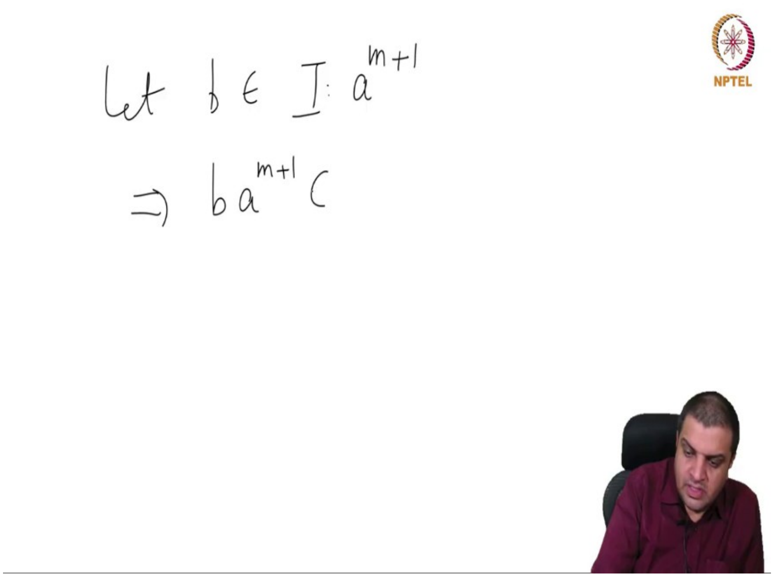

So, now let us look at now let us look at let $b \in I : a^{m+1}$ .This implies that ba^{m+1} is inside sorry, let me take a different, sorry once you have done the above argument it is easier to argue it (Refer Time: 05:56) this way.

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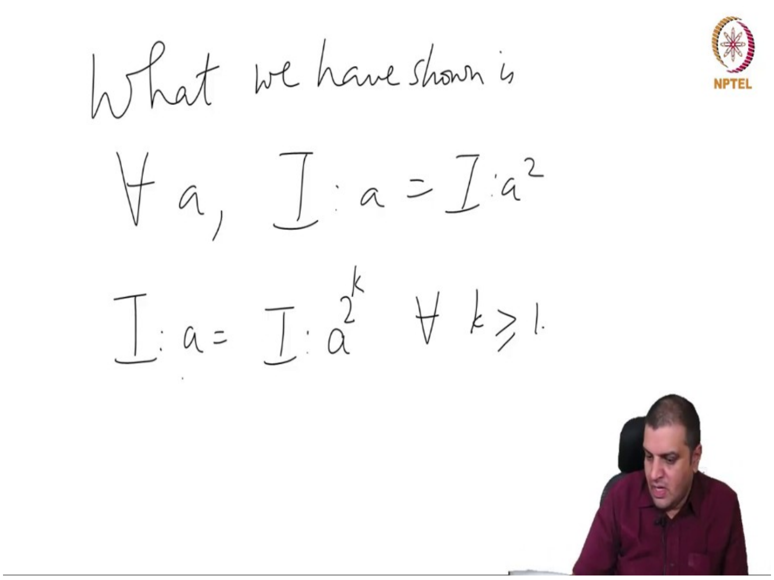

So, what does that give us? So, for every as what we have shown is for every a, $I : a = I : a^2$. Now apply this to a squared because that can also be done. So, then we get $I : a = I : a^{2^k} \forall k \geq 1$. because if you do it with this the square of a square would be a to the fourth and square of that and so on and so we it will go on like that.

So, we will get that there is this for an infinitely many exponents of a I: a to that exponent is I : a .

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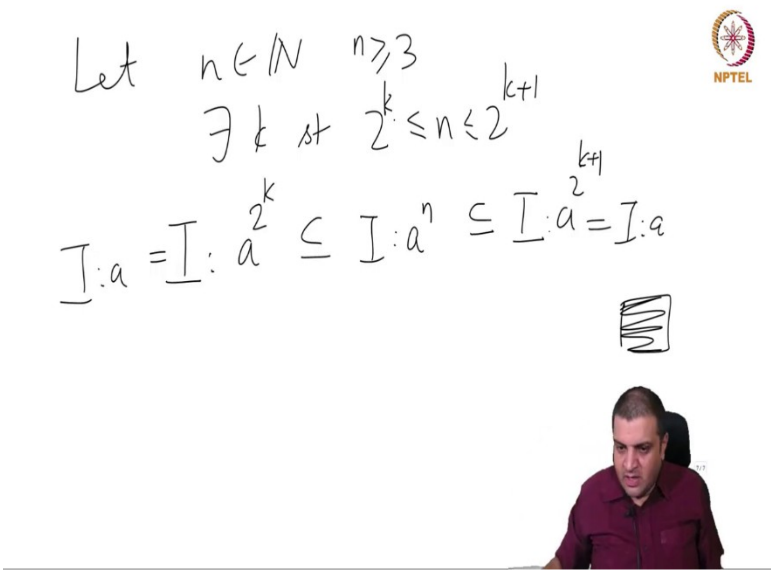

So, now given any n. So, let $n \in N$, $n \geq 3$ then there exists k such that $2^k \leq n \leq 2^{k+1}$.

So, then $I:a = I:a^{2^k} \subseteq I:a^n \subseteq I:a^{2^{k+1}} = I:a$ that is the proof.

So, this is I am not saying that going to this exponents of tools typical of such of these arguments, but these sort of manipulations that we look at some element then try to manipulate and extract some more information from it and so on. This is sort of one has to do these sort of arguments in trying to understand these problems. So, this proves one example one case where one does not have to do lot of saturation the first colon itself uses the saturation.

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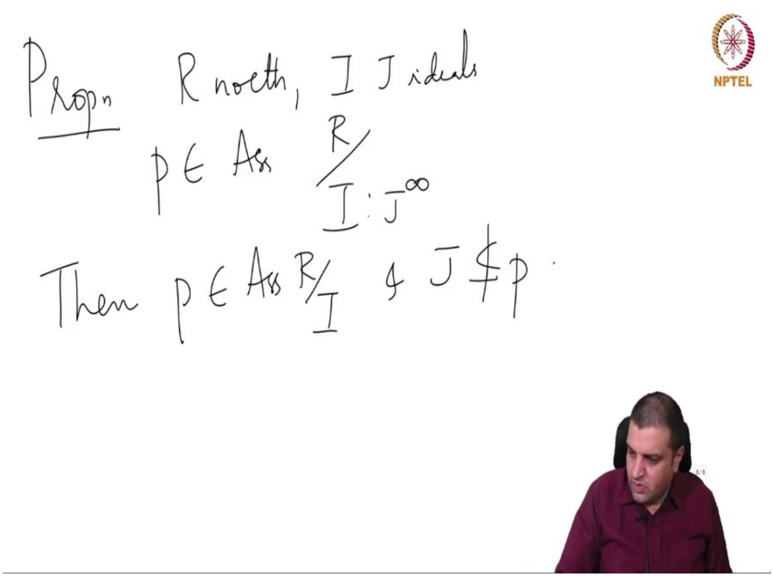

So, now let us go back to looking at associated primes proposition. So, R - Noetherian, I and J ideals , $p \in Ass\left(\frac{R}{(I:J^{\infty})}\right)$, then $p \in Ass\left(\frac{R}{I}\right)$ and $J \not\subseteq p$. I mean p that are associated to the saturation of the J saturation of I are associated to I and will not contain J. So, this is a way of picking out certain associated primes .

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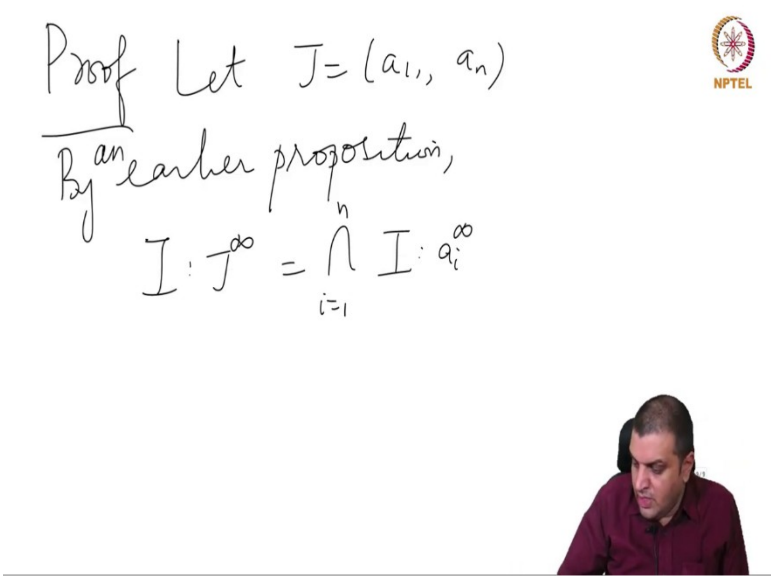

So, by earlier proposition of proof . So, by an earlier proposition we know what the saturation is, we just have to this is just the intersection of saturation of individual generating set . So, let J be generated by n elements by earlier proposition by an earlier proposition there have been many. $I:J^{\infty}=.\bigcap_{i=1}^{n} I:a_i^{\infty}$. And we have seen that if we have intersection of finitely many ideals.

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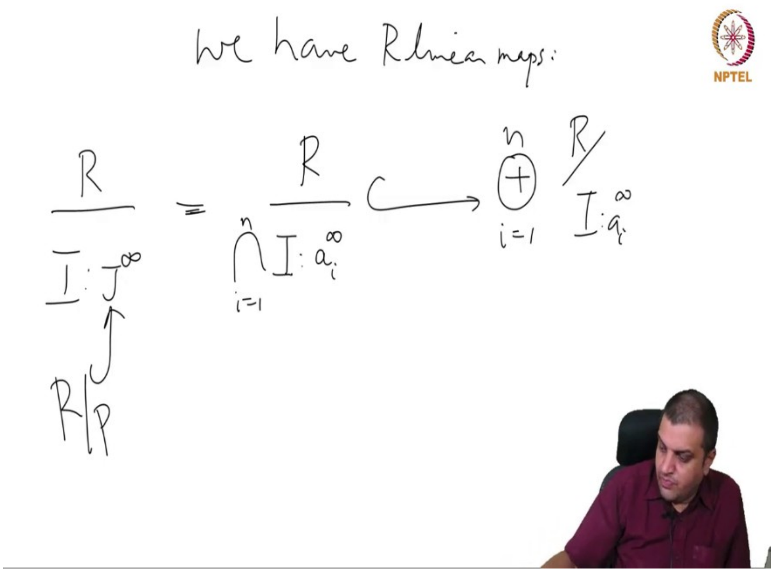

So, let us write it here $\frac{R}{(I:J^{\infty})}=\frac{R}{.\bigcap_{i=1}^{n} I:a_i^{\infty}} \to \bigoplus_{i=1}^{n} \frac{R}{I:a_i^{\infty}}$

. So, we have this is may be. This is we have R linear maps .

This is an isomorphism or equality because these ideals are the same and then this is an inclusion map . p is associated to this. So, $\frac{R}{P}$ injects into this. So, it injects into this and if it has to inject into one of those things it must be associated to one of those elements , because the associated primes of this is the right.

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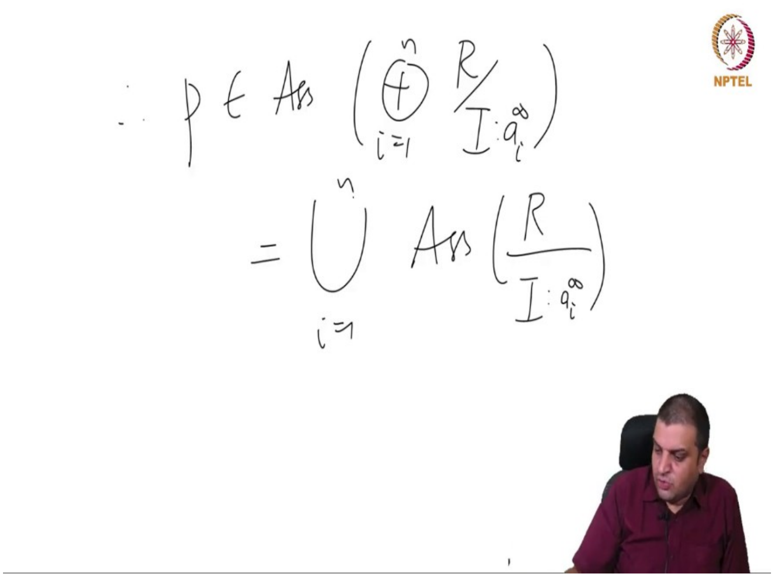

So, this we saw associated. So, therefore, $p \in Ass\left(\overset{n}{\underset{i=1}{\oplus}} \frac{R}{I:a_i^{\infty}}\right) = p \in . \bigcup_{i=1}^{n} Ass\left(\frac{R}{I:a_i^{\infty}}\right)$. And if we have a direct sum of modules then the associated primes are the individual, the union of the individual associated primes 1 and therefore, p is associated to 1 one of them.

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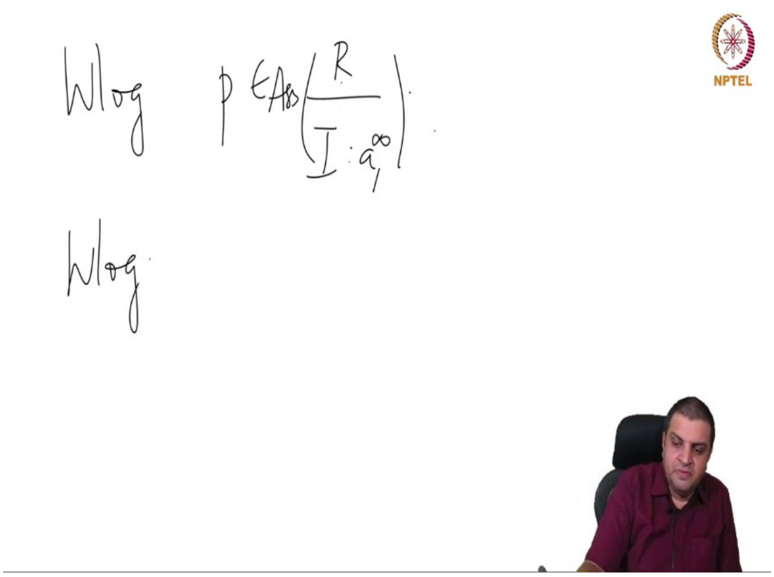

So, without loss of generality $p \in Ass\left(\frac{R}{I:a_1^{\infty}}\right)$.

So, we can replace J by a principal ideal. So, without loss of generality we just have to prove that a_1 is not inside. So, let us. So, we if we show the theorem for a principal ideal then a_1 will not be inside p and hence J will not be inside p .

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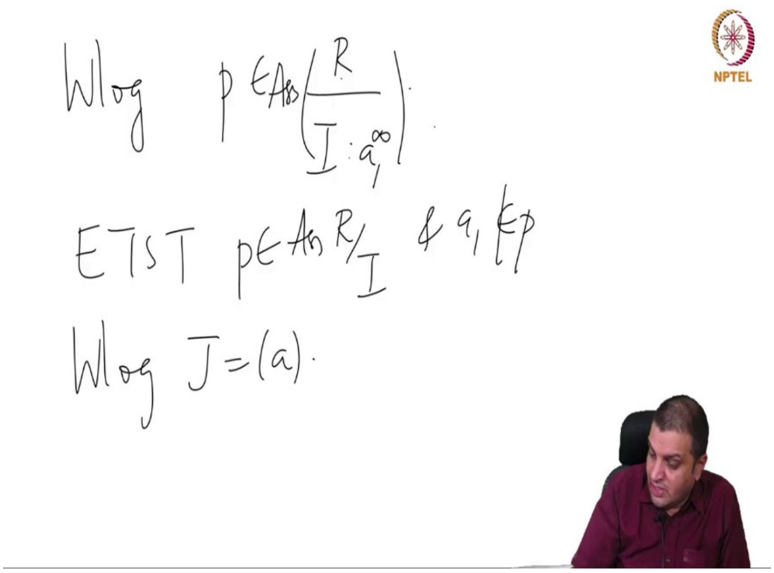

So, enough to show that $p \in Ass\left(\frac{R}{I}\right)$ and $a_1 \notin p$ that would be proof for J also. So, without loss of generality J =(a) . So, instead of calling it a_1 we will just call it a.

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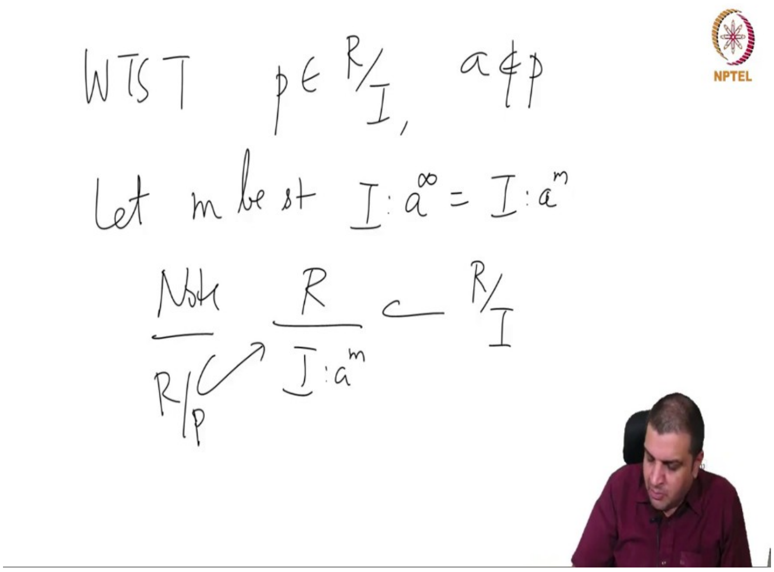

So, then so, we want to show that $p \in Ass\left(\frac{R}{I}\right)$ and $a \notin p$.Let us again this is sort of the usual again a rehash for various arguments that we have used earlier.

So, let m be such that $I : a^{\infty} = I : a^{m}$. Then arguing as earlier again note $\frac{R}{(I : a^{m})} \rightarrow \frac{R}{I}$ and the quotient is I + a^{m} that is not allowed for us now and p is associated to this $\frac{R}{P}$ is associated to this. So, I mean $\frac{R}{p} \rightarrow \frac{R}{(I : a^{m})}$.

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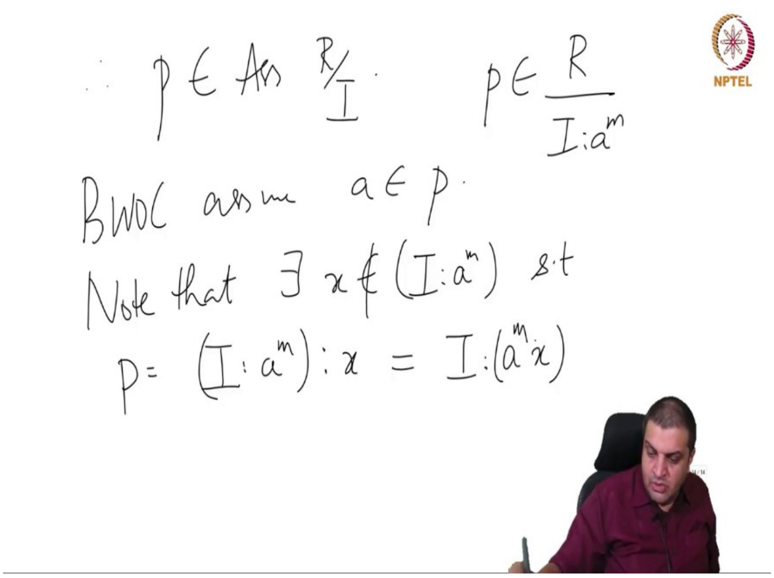

So, therefore, $p \in Ass\left(\frac{R}{I}\right)$. So, we prove the first part. We want to prove the second part now which is not in p. So, by way of contradiction assume $a \in p$. Note that there exists some $x \notin I$ such that the annihilator of x in R/I which is the same thing as annihilator of x in R mod thought of it is an R modulo. So, which is the same thing as I : x. sorry I apologize I made a mistake here sorry there exist.

The assumption is that $p \in Ass\left(\frac{R}{(I:a^m)}\right)$. We do not need to use this conclusion just use this I mean start from there itself there exists an $x \notin I:a^m$ such that $p = (I:a^m):x$. but what does this say so, but $(I:a^m):x = I:(a^m x)$ this is a property about colon ideals which ah, not just principal ideals here any ideal which you will work out in the exercises.

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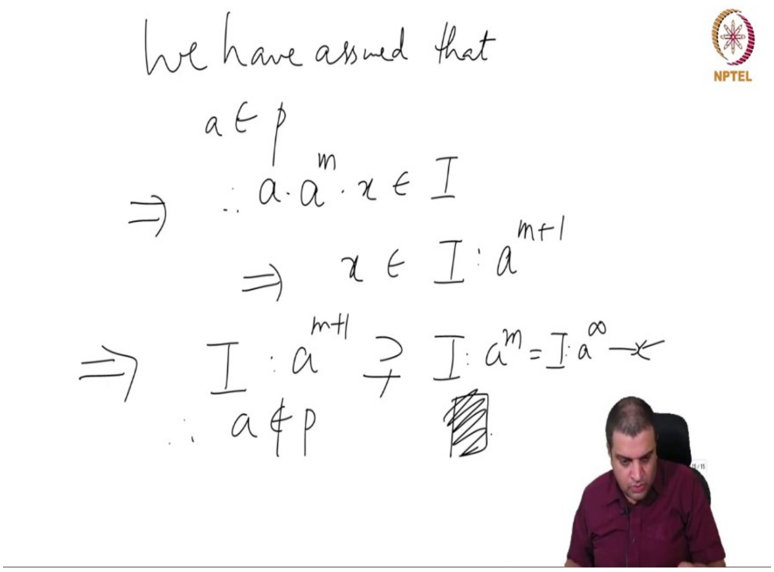

So, we have this $p = I : (a^m x)$ but we assumed that. So, by we have assumed that $a \in p$. So, which now implies from here that $a\,a^m x \in I$ hence $x \in I : a^{m+1}$.

but where did we pick x? $x \notin I : a^m$, but this now implies that $I : a^{m+1} \supset I : a^m$ which is a contradiction because this was a stable value and the contradiction came because of this assumption, contradiction came because of this assumption, this assumption here. So, therefore, a is not inside p. So, that is the end of this proof.

So, this gives us a way to construct to sort of incrementally determine associated primes of an ideal by sort of looking at saturations. So, one more proposition this is mostly of some geometric trying to reinterpret all these things in terms of the geometry coming from nullstellensatz.

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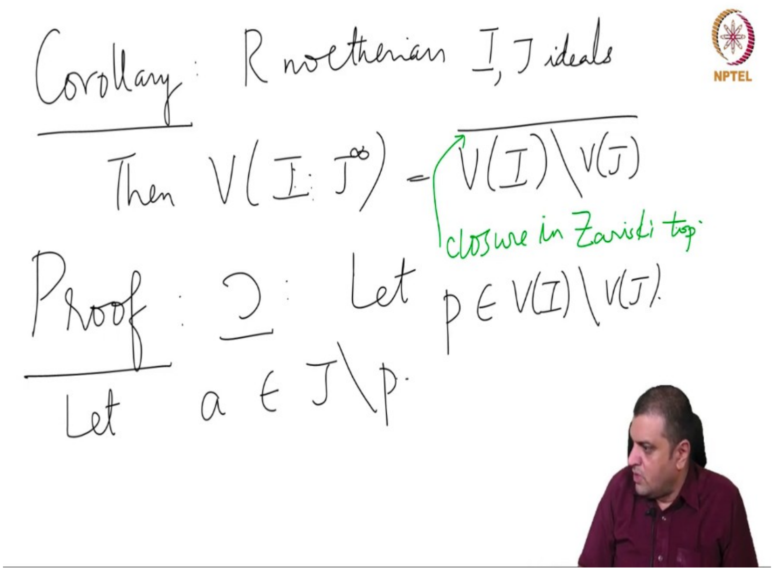

Corollary again R- Noetherian, I and J ideals . Then, $V(I:J^{\infty})=\overline{V(I)\setminus V(J)}$closure in Zariski topology. So, this is not entirely surprising given this proposition that, sorry given this proposition that by saturating out J we have removed associated primes that contain J. So, this is not really surprising that we have removed some part of V(I).

So, I have written this in terms of an arbitrary Noetherian ring, but one can specialize the same situation where we take R to be a polynomial ring over an algebraically closed field and. So, instead of writing V here we can write Z and prove the same I mean same statement . So, that is not difficult .

Proof: So, we will prove that the two inclusions . So, this containment left is inside right, other way around sorry this containment. So, let us take a p that is inside here. So, let not inside here because remember this is a closure of some set and this is a closure in Zariski topology. So, this is closure in Zariski topology .

So, if we take an element inside here and then forget the closure just the V(I)\V(J) then if you show that inside is inside the set this is closed. So, it is closure is inside here. So, that is all we will need to prove. So, let $p\in V(I)\setminus V(J)$. So, let $a\in J\setminus p$. So, now, we will try to saturate this element.

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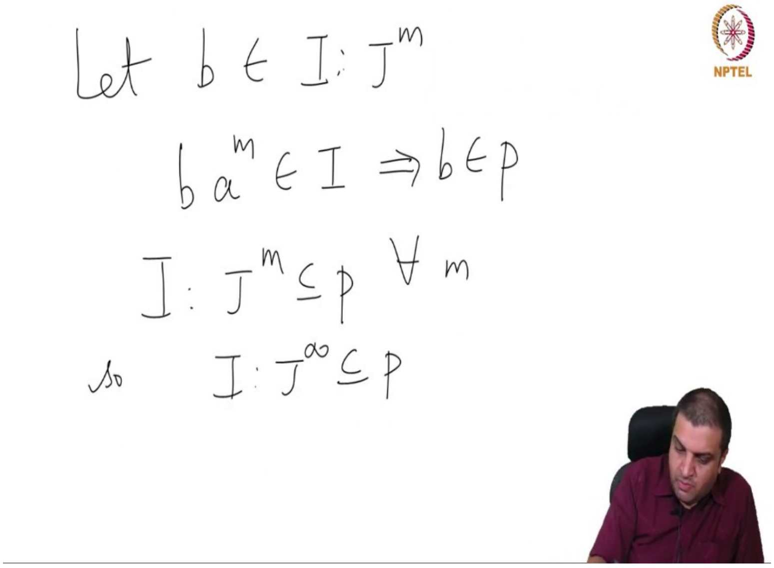

Let $b \in I : J^m$some m. So, $b\, a^m \in I$, but $a^m \notin p$. So, $b \in p$, a is not inside p. So, what does a conclusion we chose an element b that is inside I.

So, this one says that $I : J^m \subset p$, p was chosen here for every m. So, $I : J^\infty$, J saturation of I is inside p and that is exactly the statement .So, therefore, therefore, $p \in V\left(I : J^\infty\right)$.

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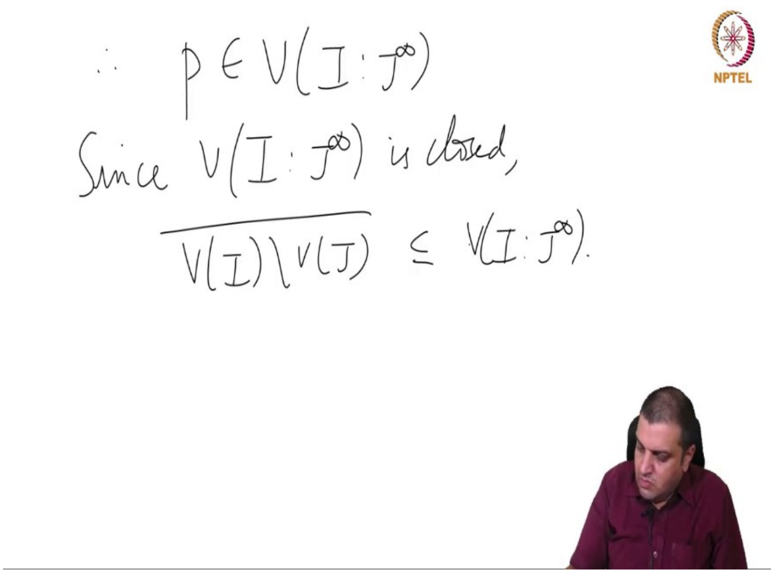

Since V of an ideal is always closed we see that V(I). So, what we just showed is V(I)\V(J) is inside this thing. So, it is closure is also inside. So, this is one direction, right side is inside the left side .

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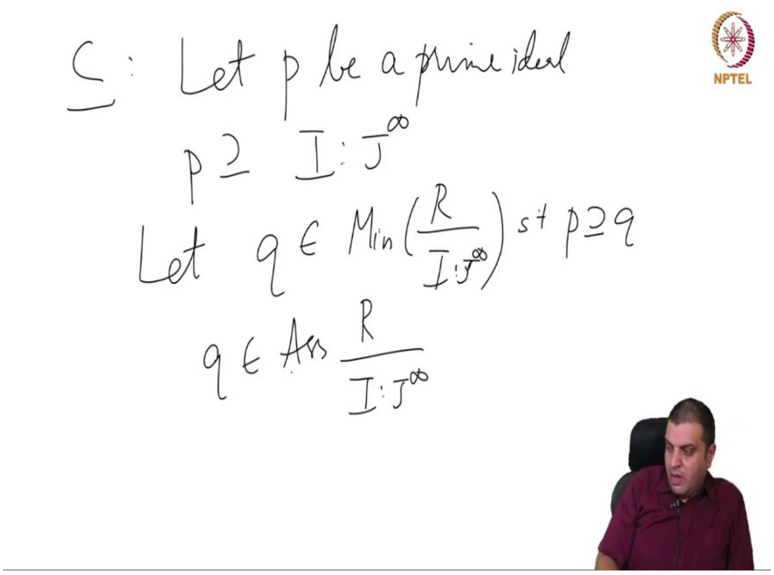

So, now the other direction . So, now, let us take a prime containing this. So, let p be a prime ideal, such that $I : J^{\infty} \subseteq p$, So, what does that mean? It means that there should be a minimal prime containing this which contains I : J . So, this implies.

So, let $q \in Min\left(\frac{R}{(I : J^{\infty})}\right)$, such that p contains q, p is some prime ideal containing this therefore, there must be a minimal prime containing this which is inside p. So, let us take q such a $q \in Ass\left(\frac{R}{(I : J)}\right)$ saturation every minimal prime is associated.

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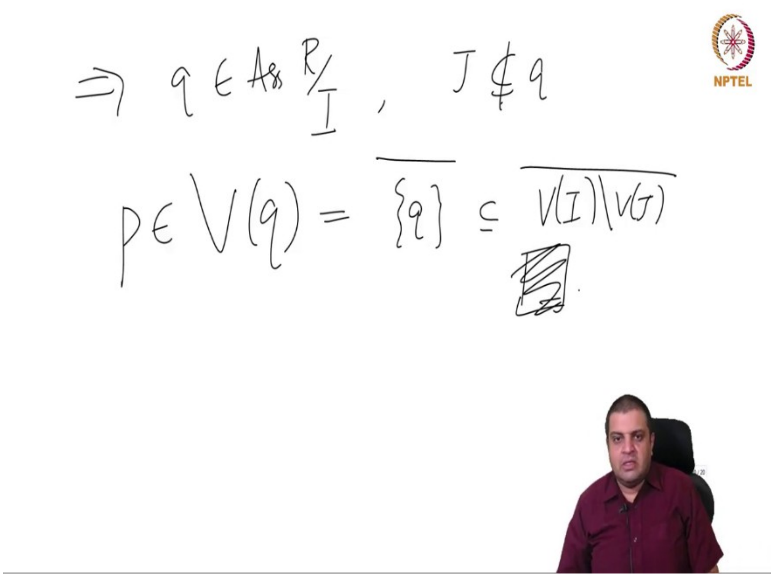

But this implies now that $q \in Ass\left(\frac{R}{I}\right)$ and of course, J is not inside p. So, this is J is not inside q. This is by the previous proposition , but now what is p. So, $p \in V(q)$ that is because p is a prime ideal containing q, but how do we I mean how do what do we know about.

So, let us see. So, $p \in V(q) = \overline{\{q\}} \subseteq \overline{V(I) \setminus V(J)}$, but what is V(q) in terms of a closure operation, it is the closure of the singleton set just containing q, that is because this is just definition means all prime ideals containing q .

So, this is V(q) is a closed set and it is the closure of this singleton set and, but this is inside. So, $q \in V(I)$ and $q \notin V(J)$. So, this q the set containing q is a subset of V(I)\V(J) and it is closure therefore, is inside here also and that is what we wanted to prove.

p that was in V(I) an arbitrary p that contained I : J^{∞} is inside this. So, one can also rephrase this to study irreducible components of varieties inside k^n, but that we just restating this the same .

So, we will do a quick example about saturation going from the previous from last in one of the earlier lectures, we will do it in Macaulay more thoroughly what we did it by hand and then we look much more slightly more elaborate example in the next lecture.

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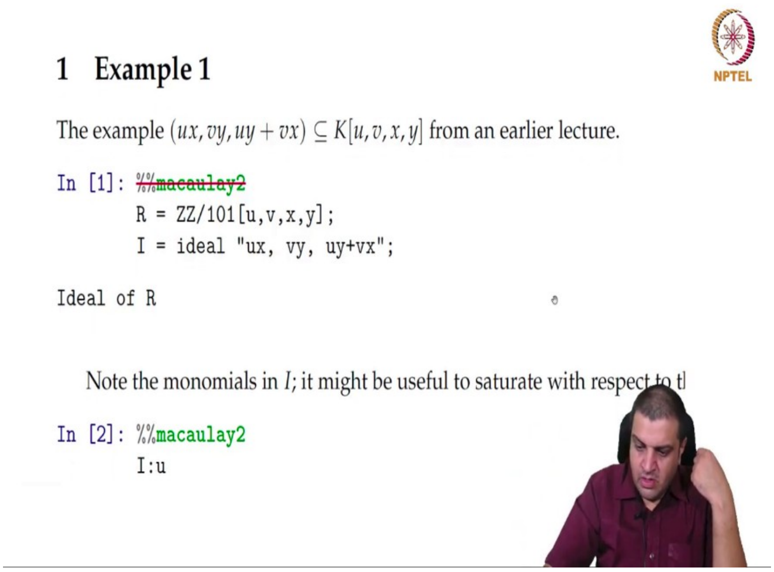

So, this is the example from an earlier lecture. So, as I. So, we have a polynomial ring in. So, ignore these lines polynomial ring in four variables u, v, x, y same thing as what we were discussing last time. Ideal in this the output was it says it is an ideal. So, these lines are output the offset lines are input .

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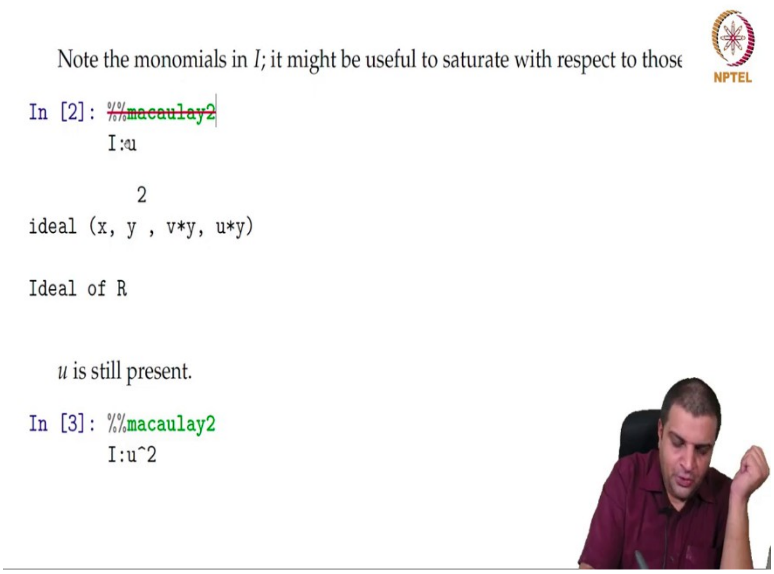

So, then we asked for I : u. So, then it says x, y^2. So, it would have in some way calculated we will discuss how these things would these things are calculated in the next lecture or maybe after that. And so, it just calculates this it says x is there x is there and if x is there, then y should be there, u, y should be there . So, it saturates this one, before we saturate we just I : u. So, if we do I : u we get this thing.

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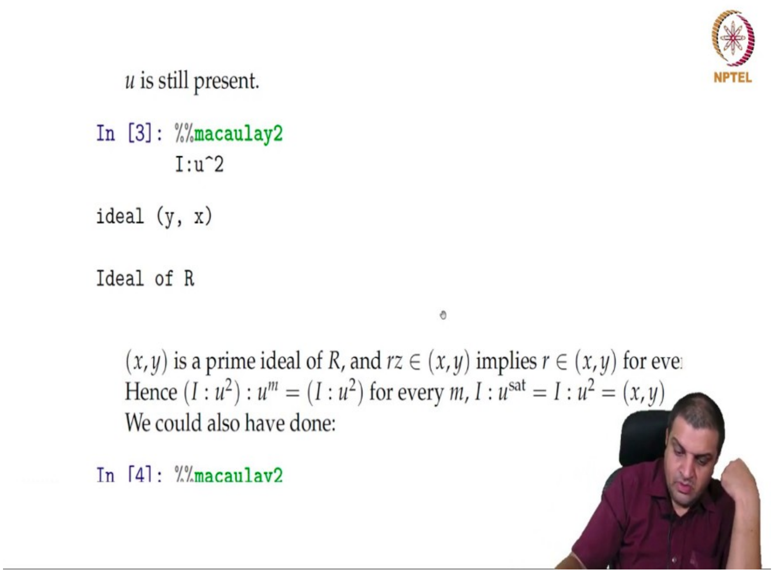

So, this is still a u. So, this is not the saturation. So, let us do it I : u^2. So, I : u^2here and then we get this output which is (y, x) ideal of R.

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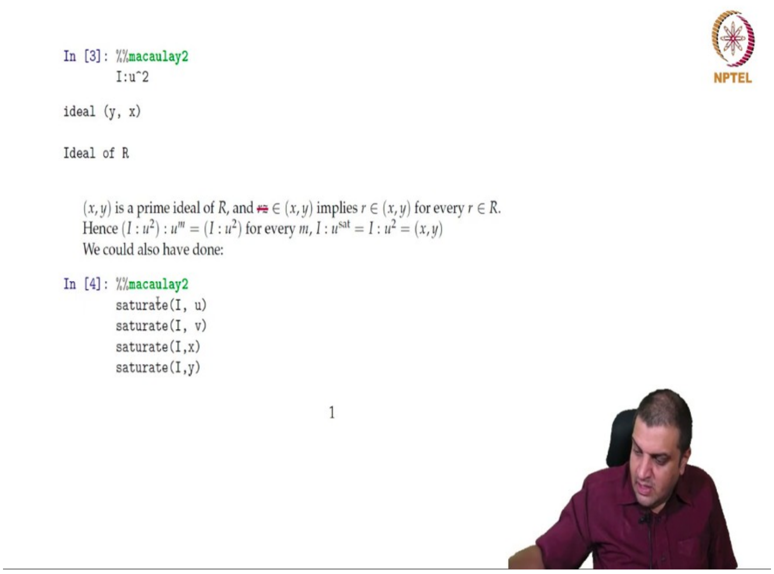

So, now let us see. So, (x, y) is a prime ideal of R and if ru ∈ (x, y), it would say that r ∈(x, y) for every r ∈ R. So, after this (I: u^2): u^m = $I:u^2$for every large m.

So, that is the saturation . So, this is one way of taking these colon successively and figuring out when it stabilizes in small examples we can do and this is what we did. We could also just give the Macaulay2 command called saturate and so, this is what we did here .

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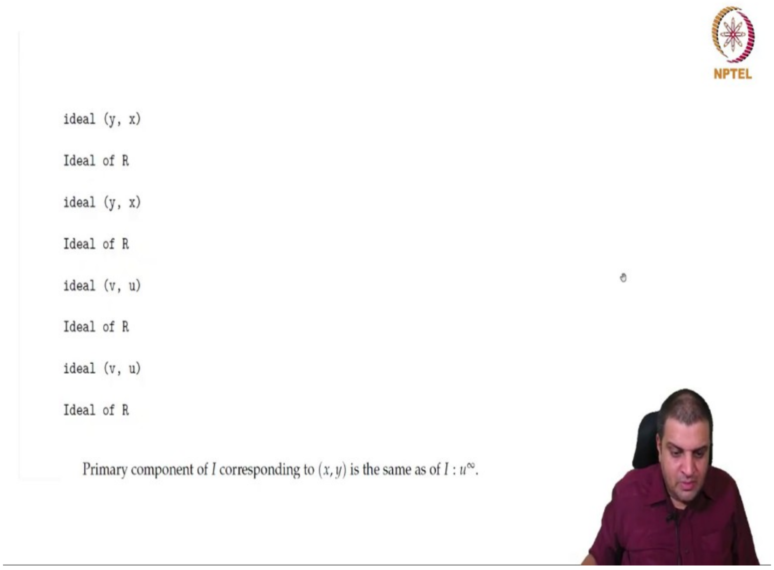

So, we just asked to do saturate with respect to all four variables and here is the output, saturation with respect to u, saturation with respect to v, saturation with respect to x and then y as expected. So, this is what .

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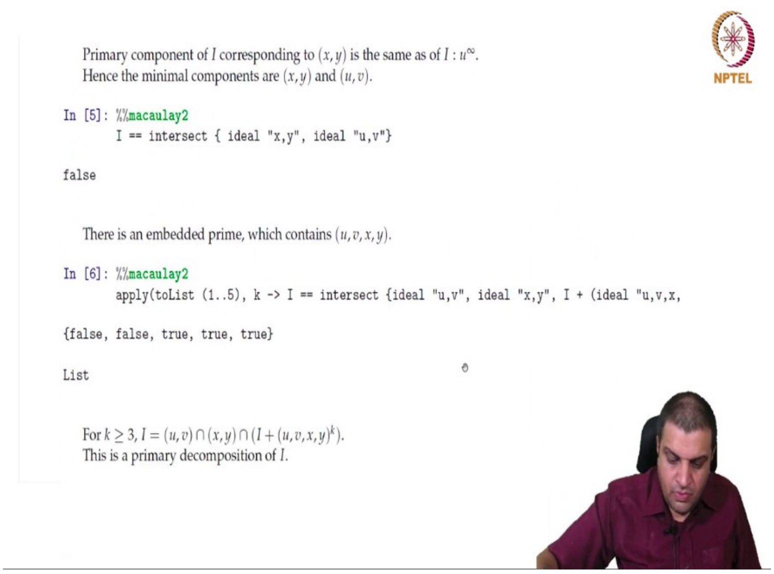

So, now with the stuff that we have studied , the primary and some exercises, the primary component of I corresponding to (x, y) is the same as that of I : u^{sat} . So, therefore, the components the primary components corresponding to the minimal primes are (x, y) and (u, v).

So, this part requires you to do some exercise which says that not just associated primes, but the primary components themselves are preserved the same relation would be there for when you do saturation. So, after that exercise this will be clear .

And then we asked to intersect the ideal “x , y” and “u, v” then we asked is it same as I and Macaulay says no. So, which means that there must be a non minimal embedded non minimal associated prime what is called an embedded prime , but let us see what the calculation says.

Calculation says every there is only one associated prime that does not contain there is only one associated prime that . So, there is only one associated prime that does not contain u which is (y,x) similarly for v just (y, x) . So, every other associated prime other than these two must contain all the four variables which is what we asked we see we conclude here.

There must be an embedded prime which must be it is just means must contain all the four variables, but once it contains all the four variables it is a maximal ideal and therefore, it is a it must be a prime associated prime, there is no there is something bigger .

So, then we just I we . So, this is code which I sorry I will type this line in the exercises I it has gone outside. So, I will type this in the exercises apply to. So, we are running a apply command which is from elements of this list.

So, to list 1..5 means construct I mean constructed list which elements 1 2 3 4 5. So, take 1 2 3 4 5 and to k we ask this question is I equal to something here . The something here is intersection of u, v, x, y and a third ideal which is I plus the ideal of all the variables to the kth power, sorry that is gone outside this fail. So, I will describe this example in the exercises again. So, there is. So, what we take is I plus this ideal to the kth power that is what this expression is.

And then we ask whether I is equal to this intersection and for the first two values one and two the answer is false and third, fourth and fifth onwards it is true and what we. So, this is what I what I have written for k ≥3, $I=(u)\cap(v)\cap(x,y)\cap I+(x,y,u,v)^k$, I mean the ideal generate by the variables to the kth power.

This is a prime ideal, this is a prime ideal with distinct different associated primes and this is a primary to the maximal idea. So, this is also primary and hence this is a primary decomposition of k. I mean we did this example I mean by hand earlier and it shows that primary decomposition is I mean except for the minimal primes this is very non unique .

So, we will stop here, we will discuss the next example in the next lecture.