Computational Commutative Algebra
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Lecture – 26
Saturation – Part 1

Welcome to lecture 26, in this lecture and the next will sort of wind up our understanding of primary decomposition. And, also learn about computing them or some strategies, well not exactly to program to compute them, but essentially how to use Macaulay 2, or some other computation algebra system to start trying to solve some problems. So, we will learn these techniques in the next, in this lecture and the next, and also see some examples being done on Macaulay.

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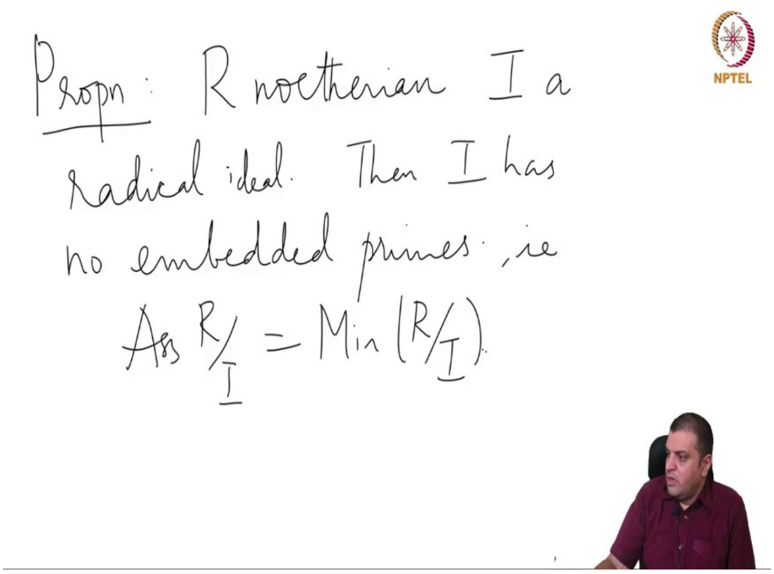

So, here is a proposition. So, most of this these results are not they are sort of building on in the previous lectures. And geared towards being able to compute associated primes primary components on a incrementally, like the example that we did in the last lecture. Suppose it R is a Noetherian ring, and I radical ideal. Then, I has no embedded primes.

That is the associated primes of $\frac{R}{I}$ so, typically there is a often used abuse of terminology when we say, I has no embedded primes what we really mean is $\frac{R}{I}$ has no embedded primes, so please keep this in mind. So, here we really mean $\frac{R}{I}$ has no embedded primes this is all of them are just the minimal associated primes.

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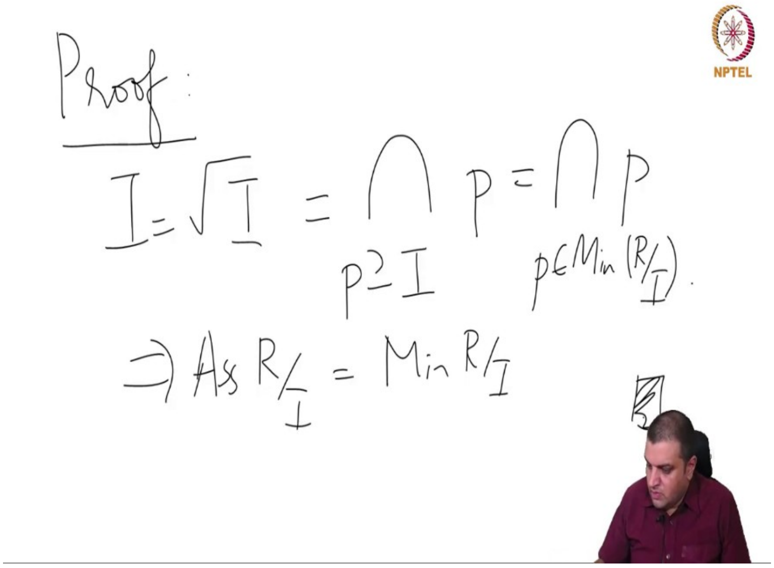

And, the proof is; the proof is based on the following observation. If we can write as an intersection of prime; if you can write an ideal as an intersection of prime ideals, which are themselves not comparable pair wise incomparable to each other to another, then that must be a primary decomposition.

So, note that $I=\sqrt{I}=.\bigcap_{I\subseteq p} p=.\bigcap_{p\in Min\left(\frac{R}{I}\right)} p$, but if you are intersecting then we only need to intersect the minimal primes, and that is exactly what we just said in the proposition. And, these are pair wise distinct and if we can write I as an intersection of these things, and it says that I does not have any; I does not have any other primary associated primes.

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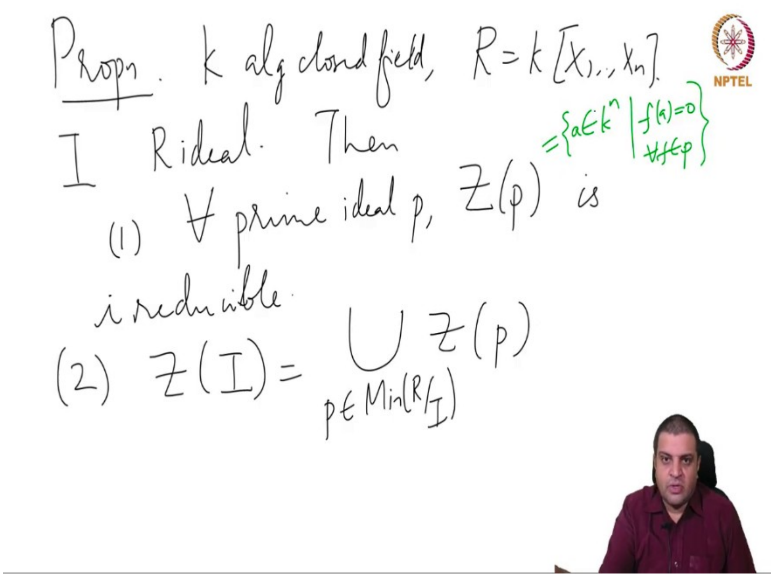

Next, so a geometric viewpoint so, this is also what we started with. So, let k be an algebraically closed field, $R = k[X_{1,}, \ldots, X_n]$, and I an R ideal. Then first statement, but first statement has nothing to do with I.

1) For every prime ideal p, Z(p) is irreducible, this is really not a new results I just wrote it down in this to complete this discussion.

So, you know recall Z(p) was this is what we had at the very beginning called V(p). So, this is $\{a \in k^n : f(a) = 0 \, for \, all \, f \in p\}$.

(2), $Z(I) =. \bigcup_{p \in Min\left(\frac{R}{I}\right)} Z(p)$, so, this equality is the irredundant irreducible decomposition of Z(I).

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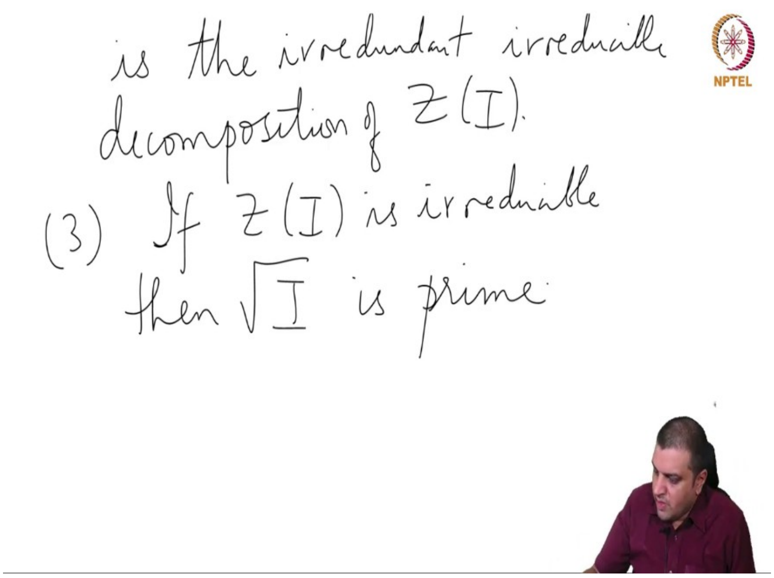

So, the primary decomposition gives information about, the irreducible decomposition of Z(I). And

(3), if this is sort of a converse to the earlier statement. If Z(I) is irreducible, then the $\sqrt{I}$ is prime. So, these are not exactly new results in the sense that, we are essentially going to rehash arguments that we used earlier to prove this. So, let us prove 1.

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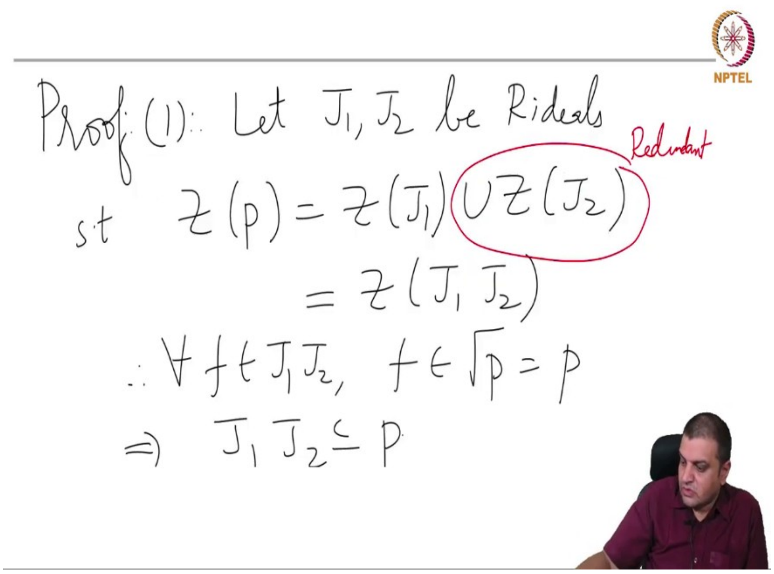

So, 1 is the statement that for a prime ideal Z(p) is irreducible. So, let J_1 and J_2 be R ideals such that so, whenever we say let me just remind. So, whenever we say topological statements about Z(I), it comes from the topology the Zariski topology on k^n, which is the same as because k is algebraically closed points of k^ncorrespond to the maximal ideals of this ring.

The maximal ideas of this ring is a subset of Spec of that ring. So, Spec has a Zariski topology, which the maximal ideals the set of maximal ideals the maximal spectrum of R, it gets as a subspace topology and that is identified so, maximal Spec as I identified with k, maximal Spec of R is identified with k^n. And, that gives a topology on k^n. So, this is always with respect to the Zariski topology.

The ideals such that $Z(p)=Z(J_1 \cup J_2)=Z(J_1 J_2)$. Therefore, every element inside here therefore, for all $f \in J_1 J_2$, $f \in \sqrt{p}=p$. So, in other words $J_1 J_2$, and last time we saw in an earlier lecture we saw that, if you have a product of ideals inside a prime ideal this means that one of them is inside $J_1 \subseteq p$.

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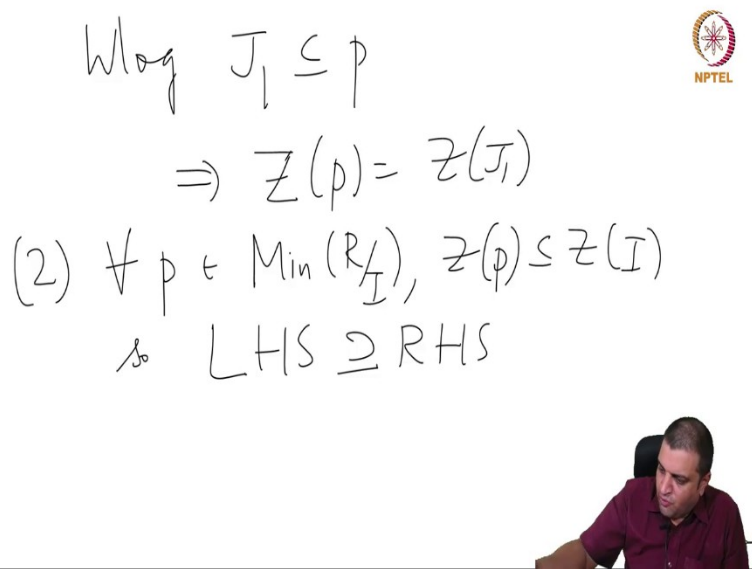

In other words, $Z(p)=Z(J_1)$,. And this is and therefore, it is not the other one is not this is redundant. So, then this becomes redundant. So, you can remove it and then it is irreducible. So, Z(p) is irreducible and the final statement is a sort of a converse to that, but let us prove 2 before that. So, we need to prove that two sets are equal Z(I) is the union of these things.

So, one is for all $p \in Min\left(\frac{R}{I}\right)$, $Z(p) \subseteq Z(I)$. So, the left hand side contains the right hand side, left hand side of the statement 2 contains the right hand side of the statement 2 and for the other direction. So, now we need to show that every point in Z(I) is inside Z(p) for some p.

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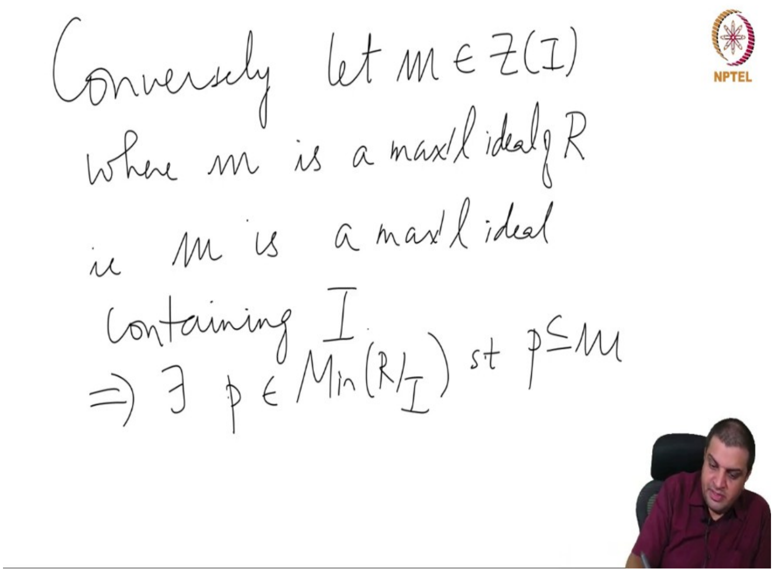

So, conversely let. So, here again it is an abuse of notation. So, we will write let $m \in Z(I)$, where m is a maximal ideal of R. So, at this point we do not distinguish between the collection of maximal ideals of R, and the set of points in k^n. So, it is an abuse of notation, but we will still do it. So, but so this means that it is a maximal ideal, that is m is a maximal ideal containing I.

That is because every polynomial in I vanishes at the point corresponding to m. And therefore, the evaluation map vanishes means when you evaluate it you get 0, but the maximal ideal is the kernel of that evaluation map.

So, therefore I must be in the kernel which is m, maximal ideal containing I which, now implies that this is a prime ideal containing I so therefore, there exists $p \in Min\left(\frac{R}{I}\right)$ such that, $p \subseteq m$ that is just, the I mean this is the maximal ideal, this is a; this is min, so by definition there is 1.

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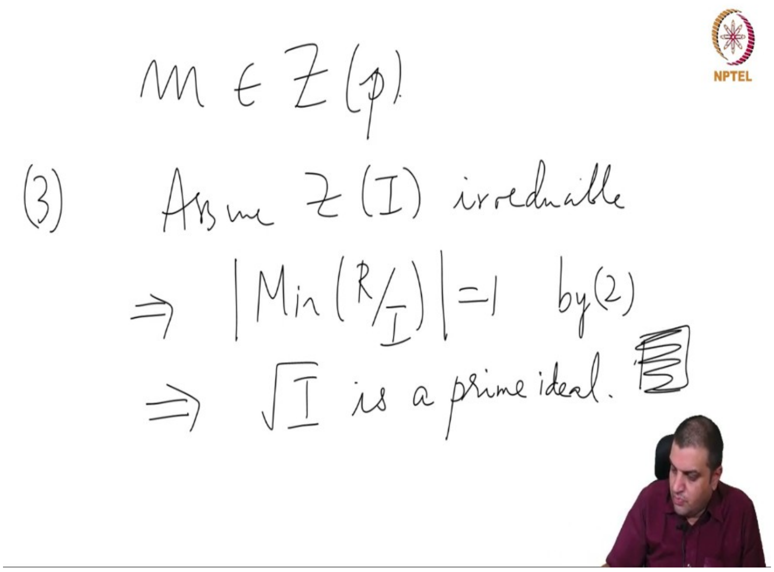

And, which now implies that, which is all that we wanted to prove, $p \subseteq m$ means that $m \in Z(p)$, which is what we wanted to prove. every point inside Z(I) is in Z(p) for some p, so we prove that. And 3,So, assume Z(I) irreducible, which means that. So, this is an we proved in 2 that this is an irredundant, we need every prime in every element of $Min\left(\frac{R}{I}\right)$ and we do not need anything further.

So, Z(I) is irreducible means that there will be exactly one such one element in this union, ok.

So, it is irreducible would mean by 2 that, $\#Min\left(\frac{R}{I}\right)=1$ by 2. And so, this minimal primes over I is the thing and which now implies that radical of I is a prime ideal. Because radical of I is the intersection of the primes in this collection, but there is only one so, it is prime, so that is the end of this proof. So, now yeah that is the end of this the proof.

So, now we would like sort of al maybe we would like to do to ourselves, but we would like that with the help of a computer we would like to given an ideal in a Noetherian ring, find its associated primes or the primary components or some partial information about in that direction. So, let us say we would like to understand how we can solve that problem.

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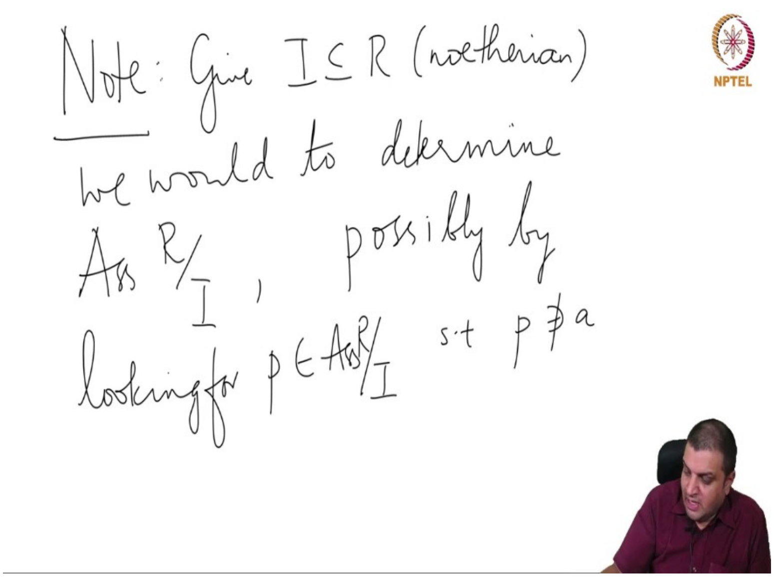

So, let us let me just briefly write these things so, that,. So, one approach which we did last time was, given an ideal we will try to say let us try to find associated primes that do not contain another way or that way we can will try to enumerate all of them. So, given I inside R Noetherian, otherwise lot of these do not apply.

We would like to determine $Ass\left(\frac{R}{I}\right)$, possibly by looking looking for $p \in Ass\left(\frac{R}{I}\right)$ a thing such that, p does not contain a for some suitable a. So, p does not contain for some.

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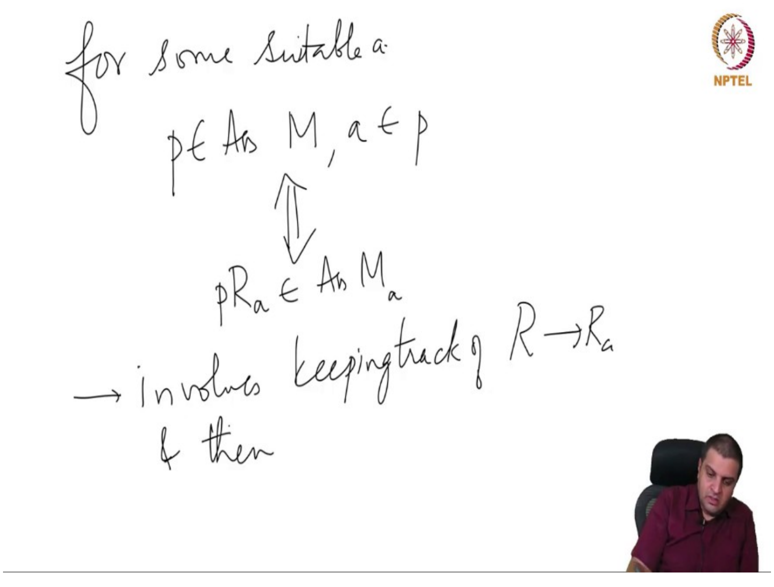

So, what is suitable, what is what should we use all this will depend on the problem. I am not suggesting some programming strategy I am just trying to give a strategy on how to solve for some problems essentially by hand, but with the help of a computer. That is suitable a and maybe not just a single a, but many of them. So, the so what is the I mean how did we know this thing. So, $p \in Ass(M)$, and $a \notin p$.

This we saw last time if and only if $pR_a \in Ass(M_a)$. this is one; this is one strategy, but the one of the problems with this is, it is not easy to describe to I mean localizing is not very easy. One element you can manage, but it still involves keeping track of this map. So, this is a problem involves keeping track of the map $R \to R_a$, and then if we want to find so this is.

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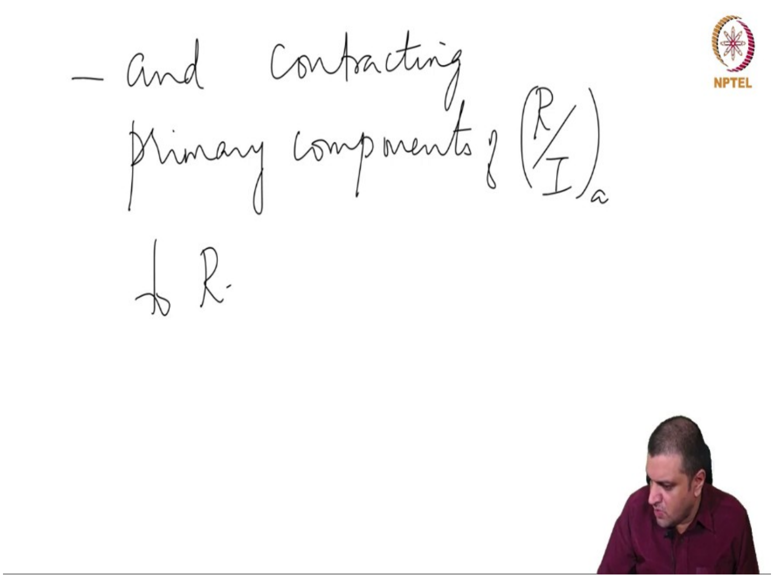

And, contracting primary components of $\left(\frac{R}{I}\right)_a$ to R. This is if you want to find primary components. So, this is even otherwise, we can let us say we can find, but what is p, which gives pR_a so, these things require some calculation; so, we will we would like to avoid doing this by trying to sit inside R itself, and working so that is what we would want, we want to do now. So, we introduce this idea of saturation.

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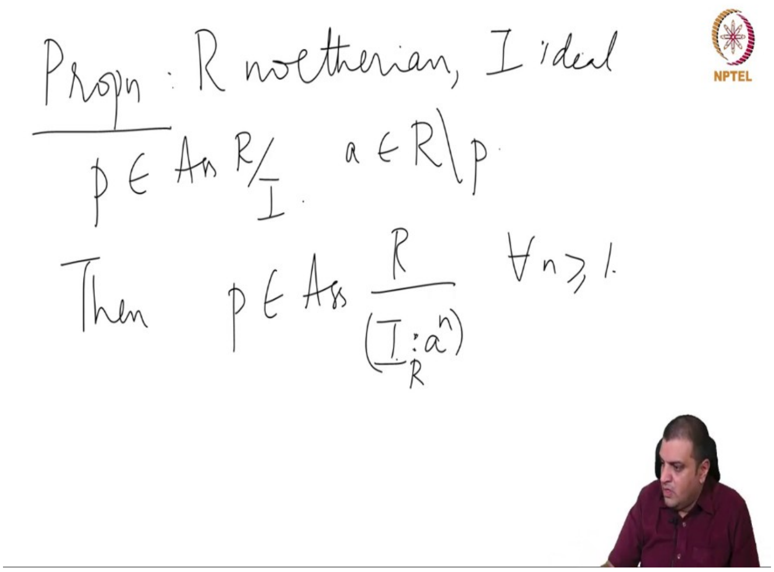

So, proposition R Noetherian, I ideal . Let us say $p \in Ass\left(\frac{R}{I}\right)$. And $a \notin p$ so we are taking; so we are trying to find or in other words we are trying to find some property of such associated primes. Then, $p \in Ass\left(\frac{R}{(I : a^n)}\right)$ for every integer n it is the way I go.

And, we will see that this is a way to identify primes that are not in that do not associated primes that do not contain a we are not there yet, but and this is in R, because this is an element of R this is a submodel of R the product has to be inside R .

So, I might forget to write the, we are only discussing ideals and rings so, the colon is always inside R unless I have to if it is different I will state it explicitly. So, proof is again one of these standard arguments using that one becomes familiar with after seeing this a few times.

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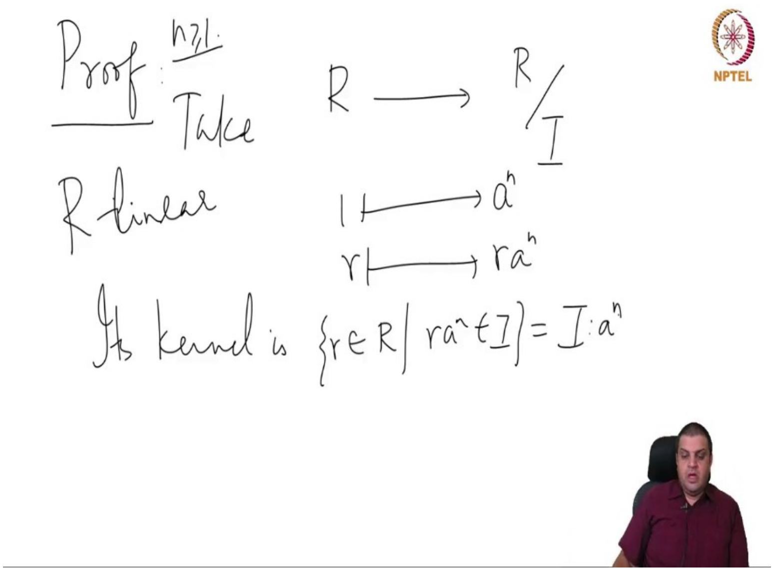

So, consider the map from $R \to \frac{R}{I}$, linear map not subjective map not the quotient ring map, but R linear map which sends $1 \to a^n$. So, some n ≥ 1 a as above,, consider this map. Now, it is kernel take this map, it is kernel is everything so, what is a map $r \to r\,a^n$.

So, it is kernel is the set of elements r in R such that, so this is remember this is R linear not algebra map not map of rings, but an of R modules $\{r \in R : ra^n \in I\} = I : a^n$. So, the kernel comes like this so, hence we get an injective map, R linear map $\frac{R}{(I : a^n)} \to \frac{R}{I}$, right.

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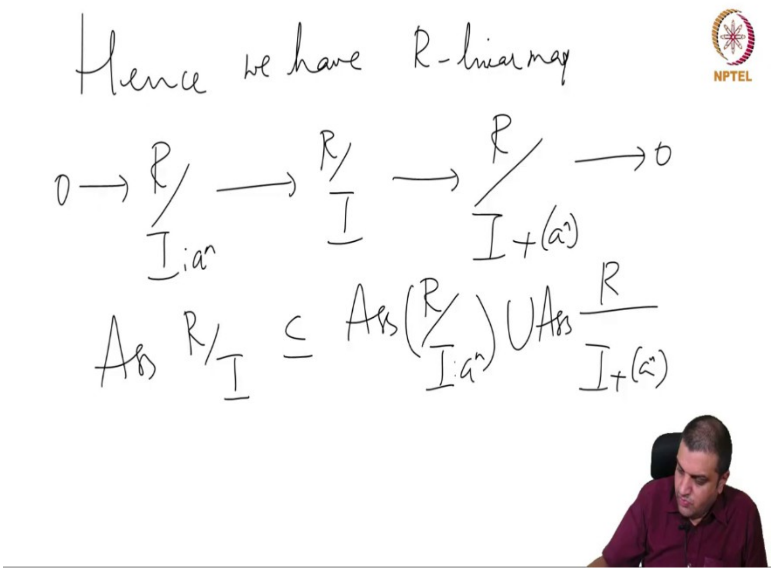

So, it now means that, and the quotient sorry the quotient is maybe one should write as a short exact sequence. So, let us go back here, so this is the kernel of this map the quotient is the co-kernel of this map, which is just $\frac{R}{(I:a^n)}$, sorry. Let me just write it in is a short exact sequence $0 \to \frac{R}{(I:a^n)} \to \frac{R}{I} \to \frac{R}{(I+a^n)} \to 0$.

So, this is what the map is. Now, let us look at so, this says that $Ass\left(\frac{R}{I}\right) = Ass\left(\frac{R}{(I:a^n)}\right) \cup Ass\left(\frac{R}{(I+a^n)}\right)$

therefore, p is an associated prime of R, and p does not contain a. So, where can p belong ? p can only belong here, because if p belongs here then p would contain I and a^n so, it will contain a.

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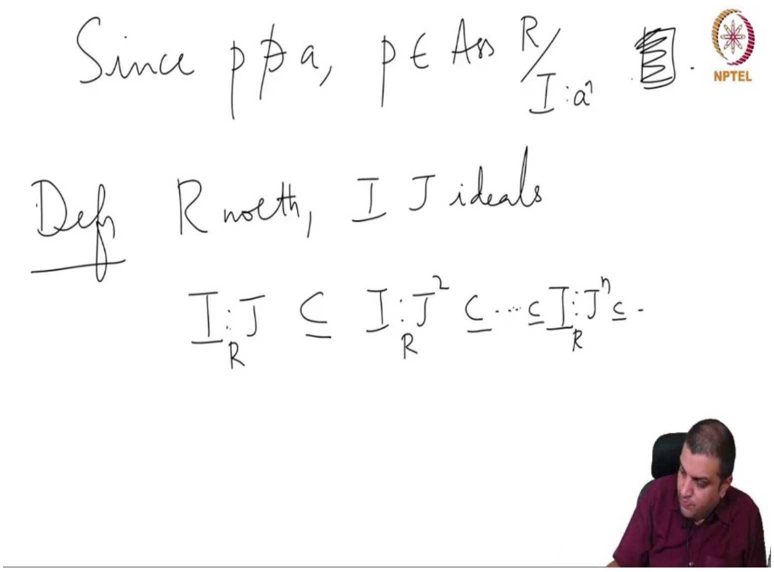

So, let us just since $a \notin p$, $p \in Ass\left(\frac{R}{(I:a^n)}\right)$ So, this is true for every n so, we would. So, now definition again R Noetherian and I and J ideals. So, because R is Noetherian we can look at the following $I:J \subseteq I:J^2 \subseteq ... \subseteq I:J^n \subseteq ...$. This is all colons are inside .

So, what is this? This is all the elements of the ring which multiply J into I. Anything that multiplies J into I will also multiply J^2, because J^2 is inside J, so we get this. And anything that multiplies J^2 will also multiply any higher power so, in particular, so we have this ascending chain of ideal, and these are ideals of R. So, we have an ascending chain of ideals in R, so it must stabilize.

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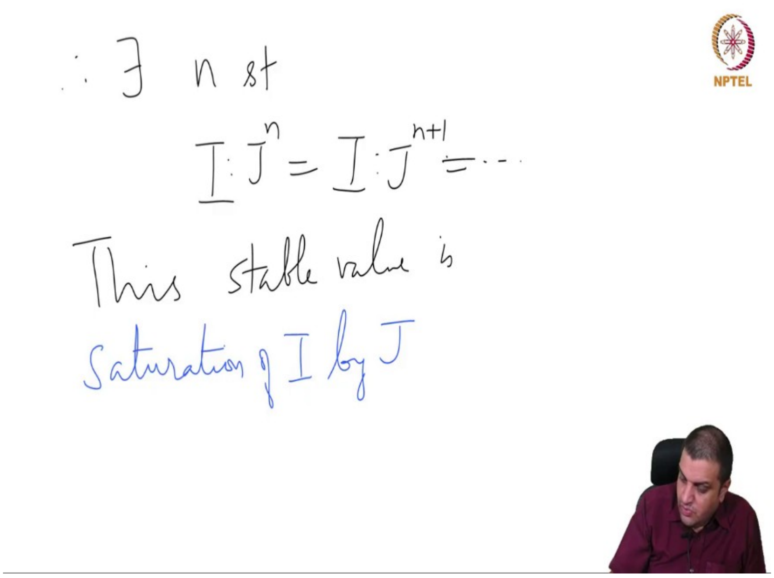

So, therefore there exists n such that sorry $I : J^n = I : J^{n+1} = \ldots$. So, there is a stable value so this stable value is called the saturation of I by J.

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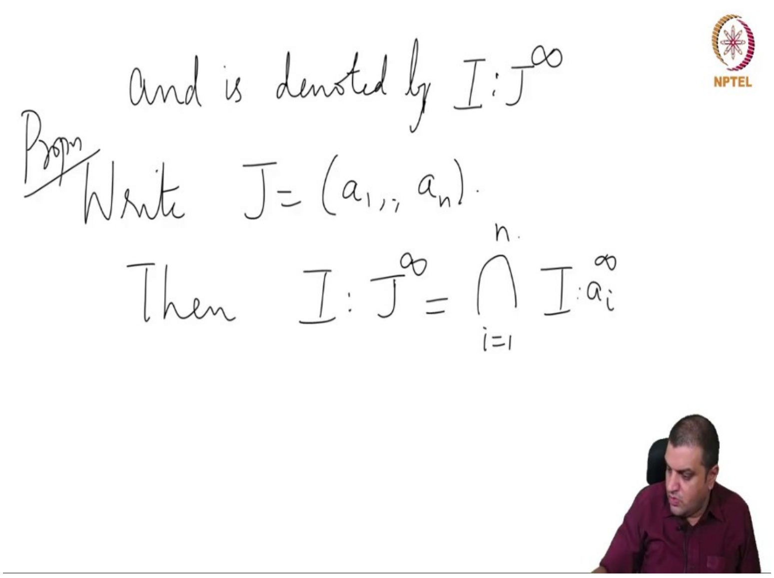

And is denoted; and is denoted by $I : J^\infty$, this is what happens when you take increasingly higher powers of J, and it infinity. And it stabilizes after a while. So, we would prove some properties about associated primes of saturations, but before that let us just make some observation that is required that will be useful improves, or when trying to solve problems.

So, write $J=(a_1,\ldots,a_n)$.Then, $I:J^{\infty}=.\bigcap_{i=1}^{n} I:a_i^{n}$. So, this is a notation just when you take principal element, we will omit the parentheses.

So, let us prove this sorry we should write proposition. So, this is convenient, because if you have to saturate an ideal altogether we can do it with one generate at a time. So, let us prove this.

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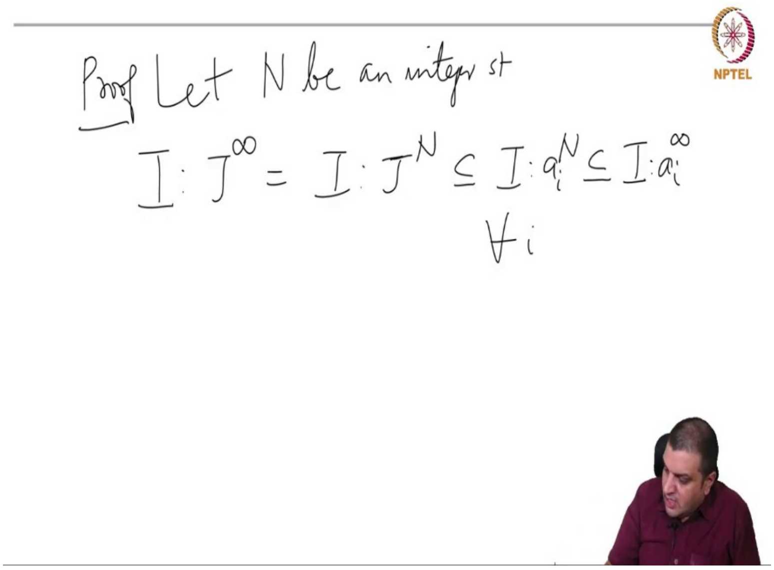

So, proof so recall from our earlier discussion that the discussion above this. This was actually equal to I: J to some fixed power, some J large enough power. So, let N be an integer, such that I : $I:J^{\infty}=I:J^{N}\subseteq I:a_i^{N}\subseteq I:a_i^{\infty}$, is anything that multiplies the Jth power of N will also multiply the Jth power of a_i because a_i^N is inside J .

And this is; and this is inside I : a_i^{∞} .So, the stable value of this family is this saturation. So, this we have this, so this is true for every I. So, this proves one inclusion.

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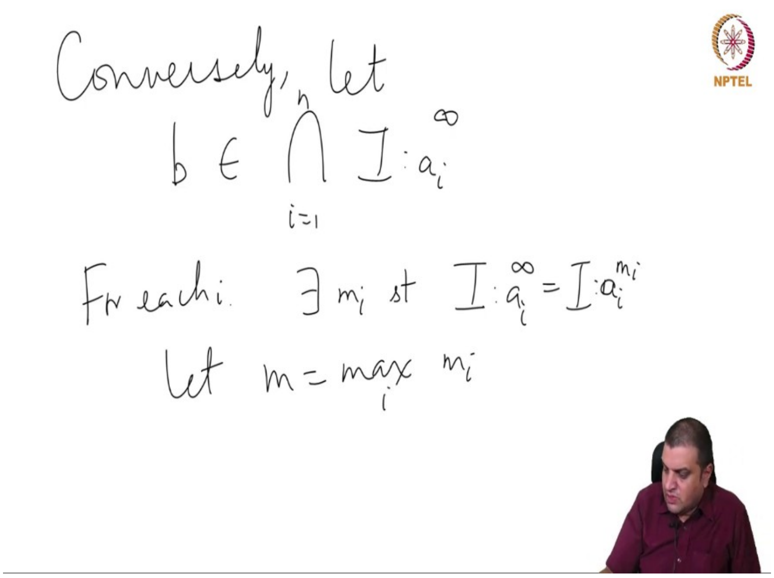

So, the other; for the other inclusion conversely we will take $b \in . \bigcap_{i=1}^{n} I : a_i^{\infty}$. So, each one of this is , so for each i there exists some $m_i$such that $I : a_i^{\infty} = I : a_i^{m_i}$

and so, this is the stable value so we get this equality.

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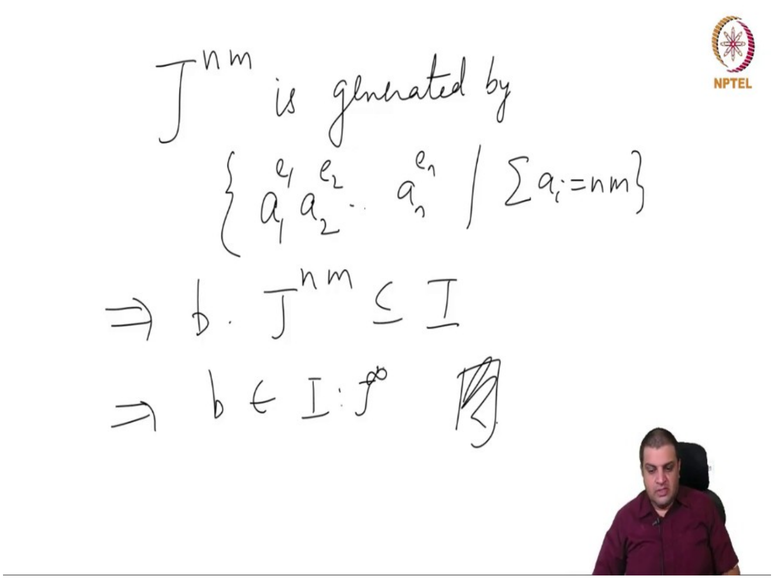

So, now let us consider and we can just take one single m so, let m = max$\{m_i : 1 \le i \le n\}$. So, let us just take one single m so, now consider J^{nm} , n is the number of generators $a_{1,\dots,}a_n$ and m is this exponent that you needed to go to saturate each one of those.

So, what is this? This is generated by; this is generated by $\{a_1^{e_1}, a_2^{e_2}, \dots, a_n^{e_n} : \sum e_i = nm\}$ where the exponents add up to nm therefore, at least one exponent must be at least m. So, this now implies that $b\,J^{nm} \subseteq I$, which now implies that $b \in J : I^{\infty}$

. So, this is the other direction.

So, we will stop here and we will continue our studying about saturation, and how it is related to primary, what information it would give for primary decomposition in the next lecture.