Computational Commutative Algebra
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Lecture – 25
Prime Avoidence

So, we discuss a few more properties about associated Primes in this lecture.

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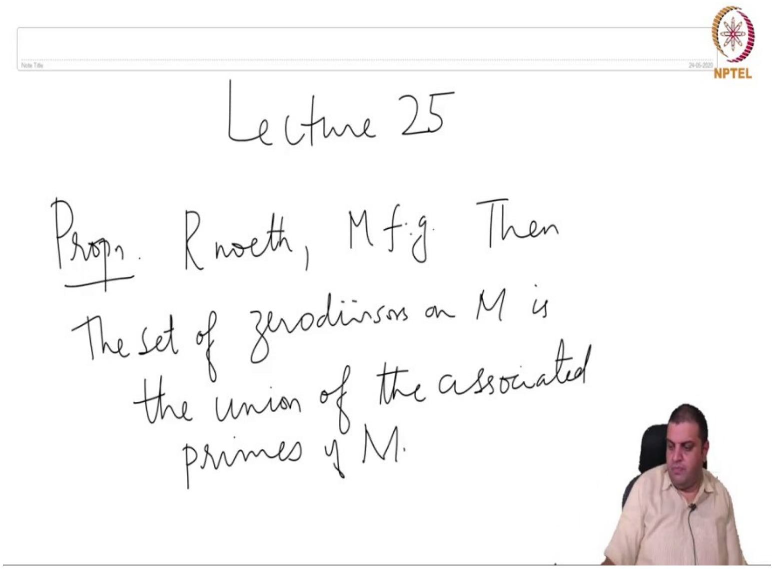

So, here is a proposition. Let R be Noetherian and M finitely generated, then the set of zero divisors on M is the union of the associated primes of M. What are we saying? So, what is the zero divisor?

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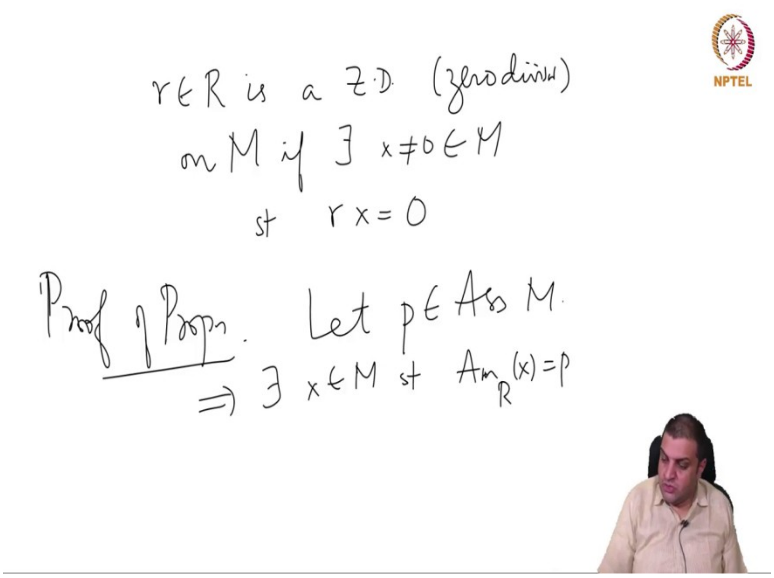

So, $r \in R$ is a zero divisor. So, this is zero divisor on M, if there exists an $x \neq 0 \in M$ such that $rx = 0$. So, what we want to show is that? Every such r belongs to to some associated prime and conversely every element of an associated prime is a zero divisor.

So, proof of proposition. So, let $p \in Ass\, M$, this implies that there exists $x \in M$ such that $Ann_R(x) = p$. And, if annihilator of element is a proper ideal, then it must be a non zero x.

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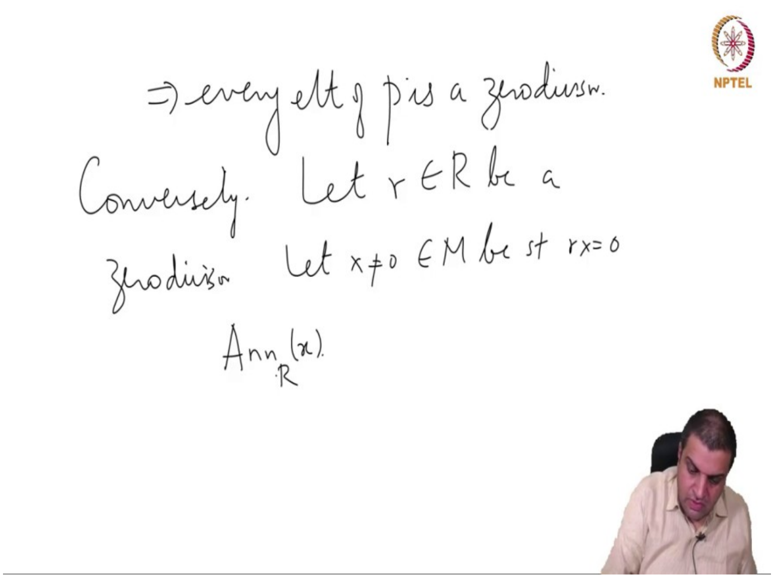

So, this implies that every element of p is a zero divisor. Therefore, ¿ $p \in AssM$ p is consist of zero divisors and nothing else.

So, now conversely let $r \in R$ be a zero divisor. Let $x \neq 0 \in M$ be such that $rx = 0$. So, now, consider the set $Ann_R(x)$.

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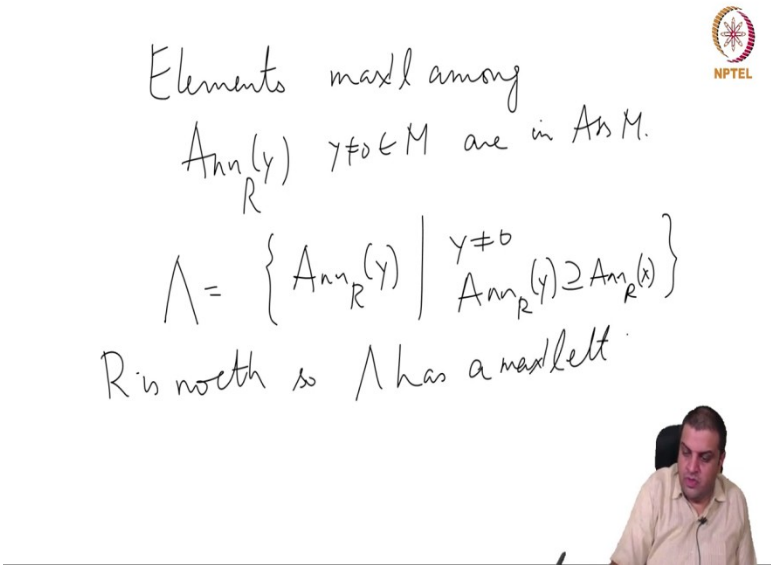

So elements maximal among annihilators of arbitrary y; $y \in M$ are associated to M.

Let $\Lambda = \{ Ann_R(y) | y \neq 0, Ann_R(y) \supseteq Ann_R(x) \}$. This is non empty because annihilator of x itself is inside here. Also R is an Noetherian. So, Λ has a maximal element.

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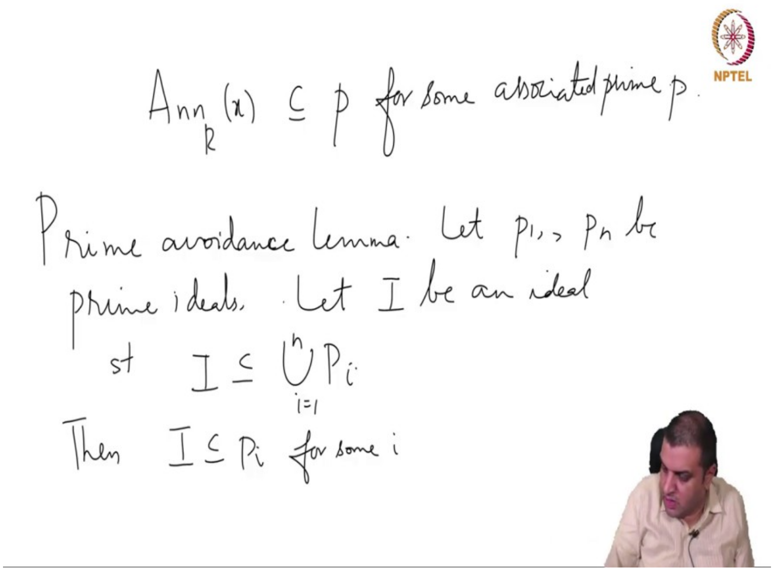

Which means that $Ann_R(x) \subseteq p$ for some associated prime p.

So, now, actually we can make this a little bit stronger. So, what did this argument say using the Noetherian condition? That annihilator of an element is inside some associated prime. We can actually prove a slightly stronger statement, which is that if I is an ideal that consists of zero divisors. Then I is in some associated prime. It will have to use this proposition and one more proposition called prime avoidance lemma.

So, here is a statement about arbitrary rings. Let $p_1, p_2, \ldots, p_n$ be prime ideals. Let I be an ideal such that $I = \cup_i p_i$. So, the union itself is not an ideal or may not be an ideal so I is inside. Then, $I \subseteq p_i$ for some i and this is not the most general version of this lemma one can relax these conditions. So, I put them as I will put them as exercises, but this is the version that we would need to use.

So, why is it called prime avoidance? It says that prime avoidance is usually the contrapositive statement which is that, if I avoids each of the p_i. I is not a subset of each of the p_i then I is not a subset of the union. So, that that is some ways sometimes that is how its used. So, this is why its called prime avoidance lemma.

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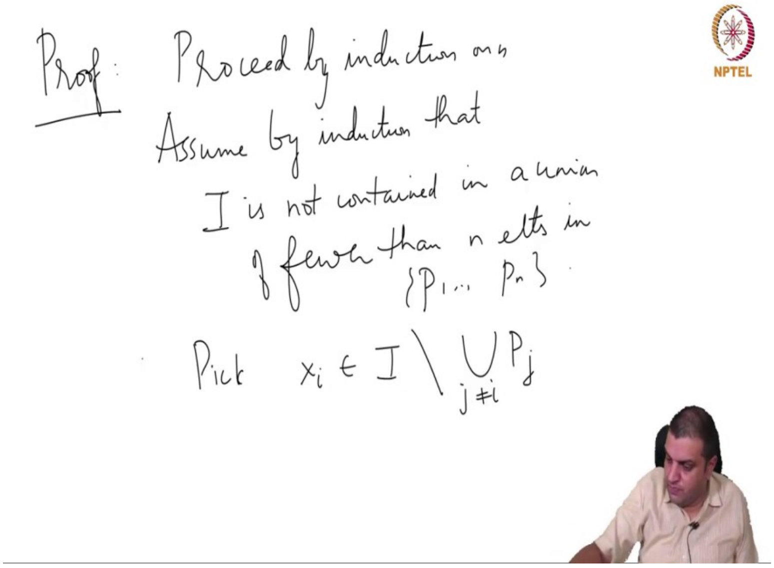

So, proof. So, I is contain the union of n primes and we want to show that I is none of them. So, we can assume that so, we will proceed by induction on n. If $n=1$ there is nothing to prove. So, what do we I think? So, we assume by induction that I is not contained in a union of fewer than n elements in $p_1, \ldots, p_n$.

Because, if that were the case then by induction we would know this. So we can assume that its not a container. So what happens when $n=2$. So, what is hypothesis? The hypothesis is that $I \subseteq p_1 \cup p_2$ and we want to show that $I \subseteq p_1$ or $I \subseteq p_2$.

So, since I is not contained in the union of n-1 things here, before we start we can even set up that notation. Pick $x_i \in I - \cup_{j \neq i} p_j$. Since I is not contained in the union we can pick such x_i.

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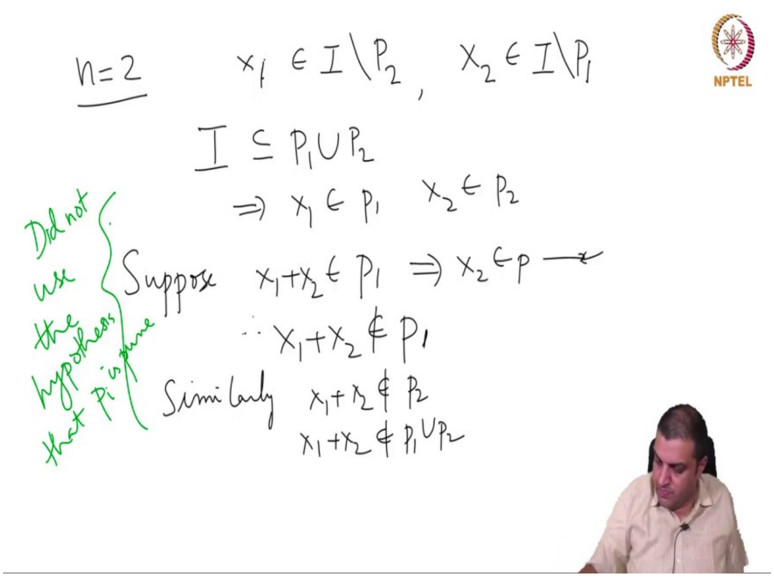

What does it mean when $n=2$? So $x_1 \in I\{p¿_1$ and $x_2 \in \{p¿_2$. $I \subseteq p_1 \cup p_2$. So, what about $x_1 + x_2$?

What does this say now? Now $x_1 + x_2 \in p_1 \cup p_2$ so $x_1 \in p_1$ and $x_2 \in p_2$.

Suppose $x_1 + x_2 \in p_1$. So, putting these two conditions we would get that $x_2 \in p_1$ which is a contradiction. Therefore $x_1 + x_2 \notin p_1$.

Similarly $x_1 + x_2 \notin p_2$. So what is the conclusion? The conclusion is that if $I \nsubseteq p_1, p_2$ then $I \nsubseteq p_1 \cup p_2$. So this is a conclusion. And one observation that we would like to make at this point is that which is not relevant for this proof but it is relevant for an exercise is that we did not use the fact that p_i is prime.

So, the first two ideals in that list p_1, p_2 need not be prime ideals. So, that is the observation that we take from this which is relevant for the one of the exercises.

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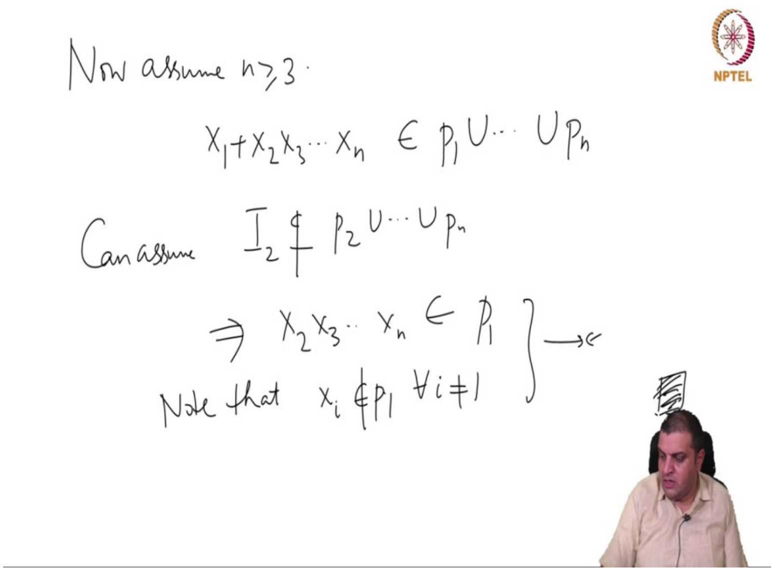

So, now, assume $n \geq 3$. So we have picked elements $x_1, \ldots, x_n$ satisfying this. And consider the element $x_1 + x_2 x_3 \ldots x_n$. So this element is inside $p_1 \cup p_2 \cup \ldots \cup p_n$ that is hypothesis this is an element of I.

Now $x_i \in p_i$ for all i. So we can assume by induction that $x_1 + x_2 x_3 \ldots x_n \notin p_2 \cup \ldots \cup p_n$. Else $x_1 \in p_2 \cup \ldots \cup x_n$ which is not true.

Then $x_1 + x_2 x_3 \ldots x_n \in p_1$but $x_1 \in p_1$ so $x_2 x_3 \ldots x_n \in p_1$ and $p_1$is prime. Note that $x_i \notin p_1$ for all $i \neq 1$. So, this is a contradiction. So, that is the proof of this prime avoidance lemma.

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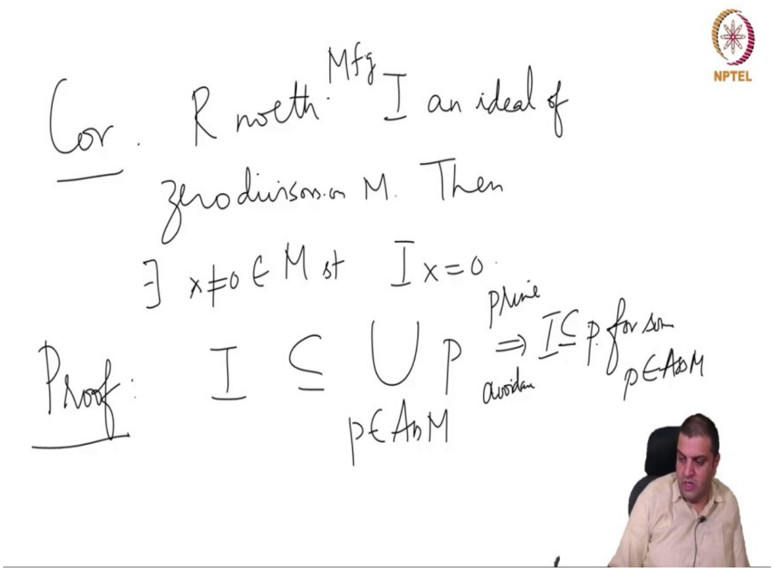

And using these things we can get the following corollary, R Noetherian, M ois finitely generated, I an ideal of zero divisors. Not necessarily killing the same x inside M. It does not mean I is a subset of the annihilator of some element it could be annihilator of various element. Then $\exists\, x \neq 0 \in M \; st \; Ix = 0$.

It just says that for every element $r \in I$, there is an x such that $rx = 0$ that is all that we are assuming here. But, we are saying we are switching the order of the quantifiers here. So, the proof is just using prime algorithms.

Notice that $I \subseteq \cup_{p \in Ass\, M} p$ this is a finite set. So, this now implies that $I \subseteq p$ for some p associated to M. But, p itself is annihilator of some element so, I annihilates the same element. So, that is the end of this proper statement.

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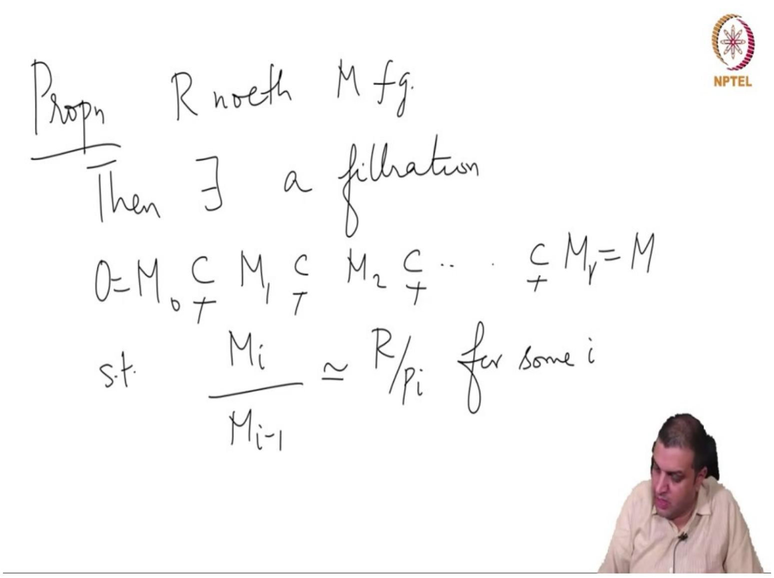

So, another useful proposition in this topic is the following. Let us say R is Noetherian M finitely generated. Then, there exists a filtration $0 = M_0 \subsetneq M_1 \subsetneq M_2 \subsetneq \ldots \subsetneq M_r = M$ such that $\frac{M_i}{M_{i-1}} \cong \frac{R}{p_i}$ for some i.

So, this is not going to go very far it going to stabilize somewhere, but we can actually do this we can get such a filtrations. So, a successive stage is the quotients are $\frac{R}{p_i}$.

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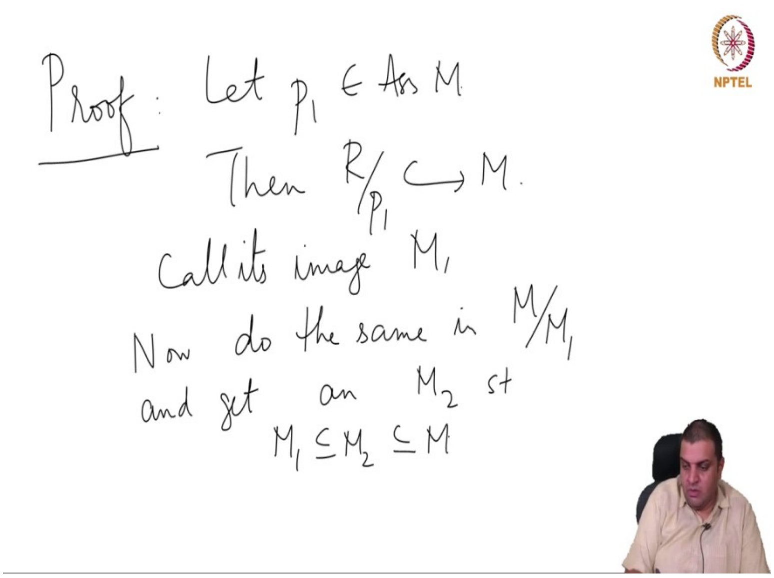

Proof, So, what is the first module? The first module is just $\frac{M_1}{M_0}$, which is just M_1 and we are saying that that should look like $\frac{R}{p_1}$. So, in other words we are saying that there is a there is a cyclic module, quotient by a prime ideal that sits inside M and we know one place to look for such a thing which is an associated prime.

So, let p_1 be an associated prime. Then $\frac{R}{p_1}$ injects into M call its image M_1. Now, we go modulo M_1 and construct an M_2. Now, do the same in $\frac{M}{M_1}$ and get an M_2 such that; I mean we will get some module here with the same argument. Look at its pre image such that $M_1 \subsetneq M_2 \subsetneq M$.

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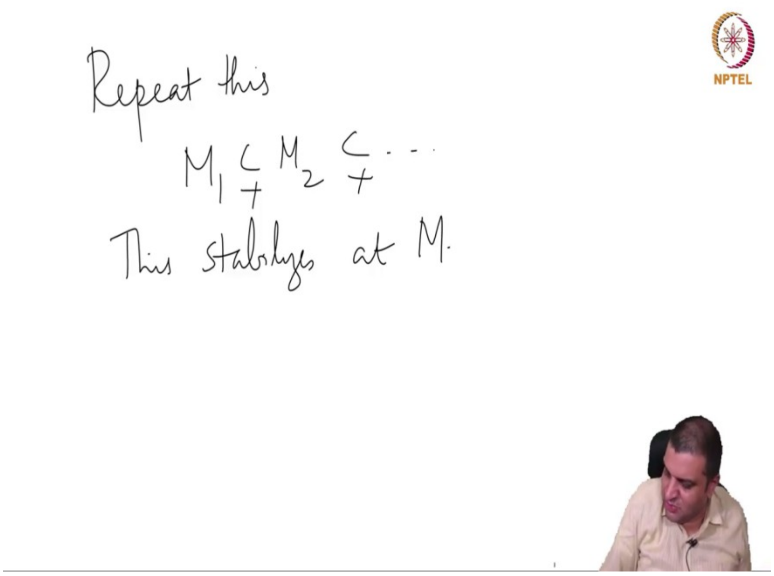

And we can repeat this to get a sequence. This must stabilize, but the only place it can stabilize is at M; because, if it stabilizes elsewhere just go modulo that and do the same argument. So, do this and this stabilizes exactly at M, it cannot stabilize strictly inside M; because, then you go modulo that stable value and then repeat the argument and then you would have constructed one so, this is.

And often such as because, this gives us sort of a inductive handle to answer many questions about modules by just looking at quotients by prime ideals and we know that I mean in some sense this might be easier to handle than arbitrary modules. So, this is an some proposition that will get used sometime later. So, I conclude this lecture with one example.

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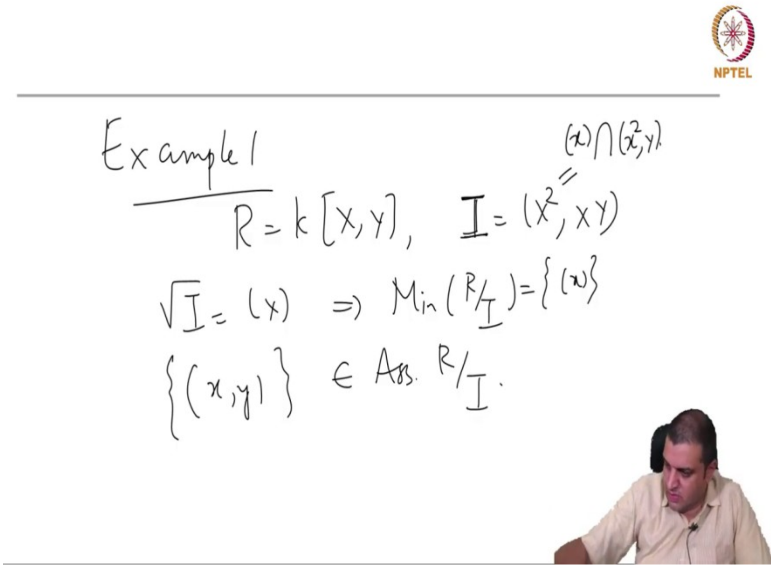

This the other one we already worked out. $R=k[x,y]$, $I=(x^2,xy)$. We would like to determine its minimal and embedded components.

So, then we know that $\sqrt{I}=(x)$. So, $Min\left(\frac{R}{I}\right)=\{(x)\}$. And then we also saw that this ideal is associated to $\frac{R}{I}$, we know the irreducible and primary decomposition. but how do we get the uniqueness? So, we just found one how do we know that is unique? So, how do we do that argument?

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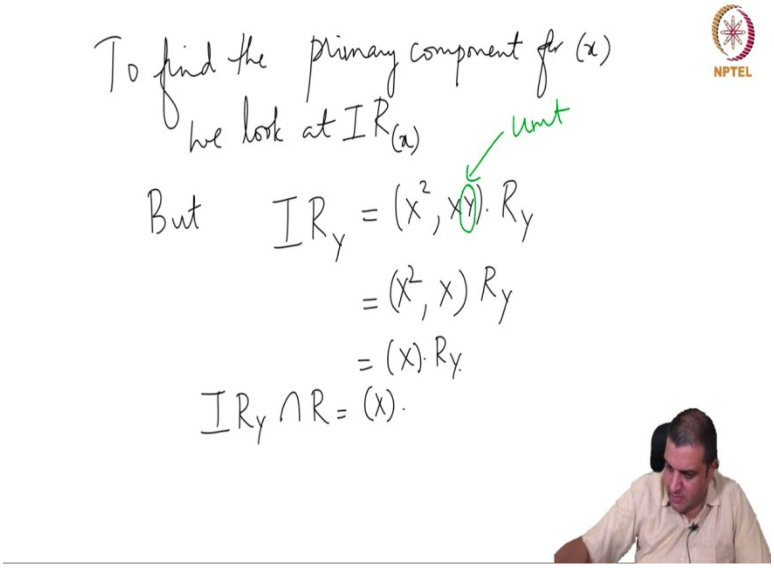

So, to find the primary component for (x) we look at $IR_{(x)}$. But localizing many elements outside x is not a very computationally convenient thing. So, one we could do is we could just first invert y and then worry about inverting other things. But, if you just invert y, so y is outside (x). So, in the process of inverting every element outside (x), one step would be to invert y.

So, now what is this? So, the ideal contains x^2 and xy. In the ring R_y, y is a unit. So, you do not need to put this unit. So, xy and x will generate the same ideal after y is inverted. And once x is there so, maybe you can write this (x^2, xy). So, I said this is unnecessary now, in this ring its not going to change the ideal. So, this is the same as. What I meant is? This is a unit now in that ring.

So, this is the same as $(x^2, x) R_y$ and of course if x is there then you do not need x^2; so this is $(x) R_y$. So, therefore, $IR_y \cap R = (x)$. So, that is why the minimal component corresponding to this. So, one can determine minimal components like this. This is a way to find that.

We will do one more example which is a little bit more complicated.

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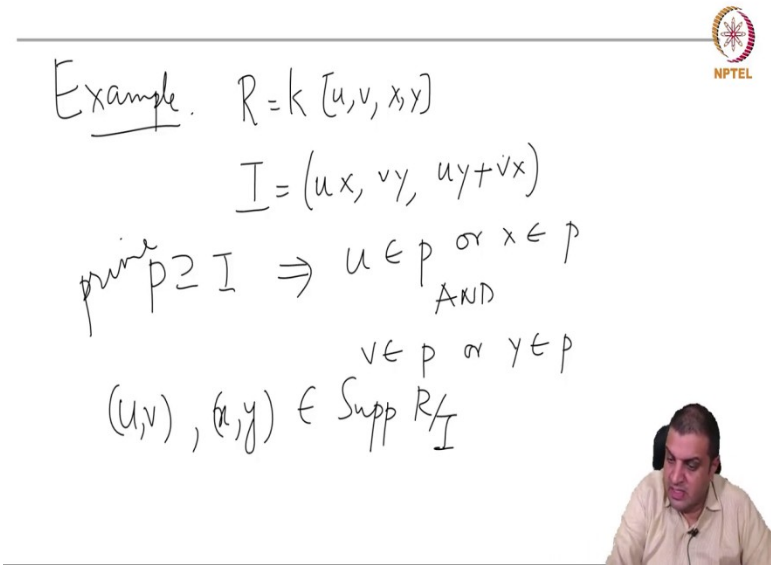

So, R is the ring in four variables $R = k[u, v, x, y]$ and $I = (ux, vy, uy + vx)$. So, there are two sets of variable and then we are taking all four products, but except in the third term we are taking the sum I mean we are taking two products separately and then taking the sum. So, this is then we would like to know what this is. So, first we would like to understand the support of $\frac{R}{I}$.

So $p \supseteq I$ implies $u \in p$ or $x \in p$ and $v \in p$ or $y \in p$. So, first of all there are now (u, v) is a possibility. So, you can just explicitly check enumerate them (u, v) this term is there this term is there and this term is there. Similarly, (x, y) are both in support $\frac{R}{I}$. So, this is there.

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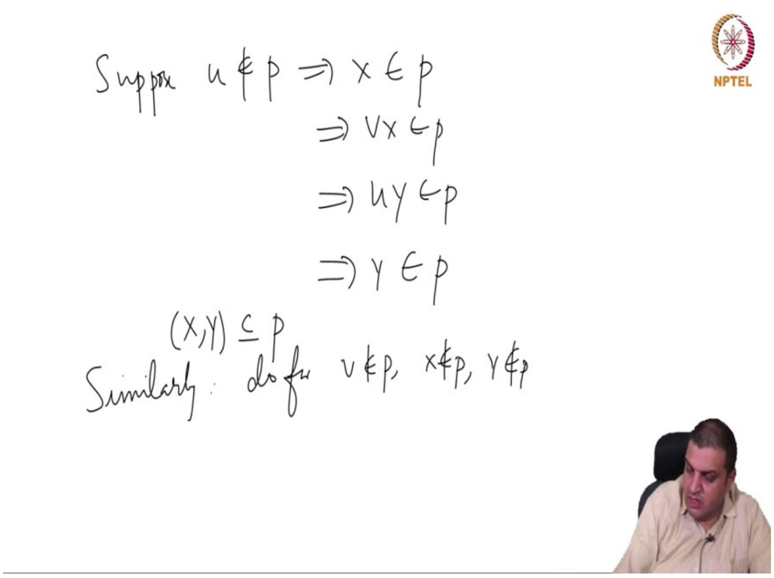

Now, we can suppose $u \notin p$, p is some primeideal. If u is not there then x must be there and if x is there then this term is there. So, this implies that $x \in p$ if $x \in p$ then $vx \in p$ this implies that $uy \in p$.

So, what is what I am saying here? $x \in p$ means that the second term here $vx \in p$ which means that the first term is also inside p, but u is not there. So, $y \in p$. So, in other words $(x, y) \subseteq p$. Similarly do for $v \notin p$, $x \notin p$, $y \notin p$.

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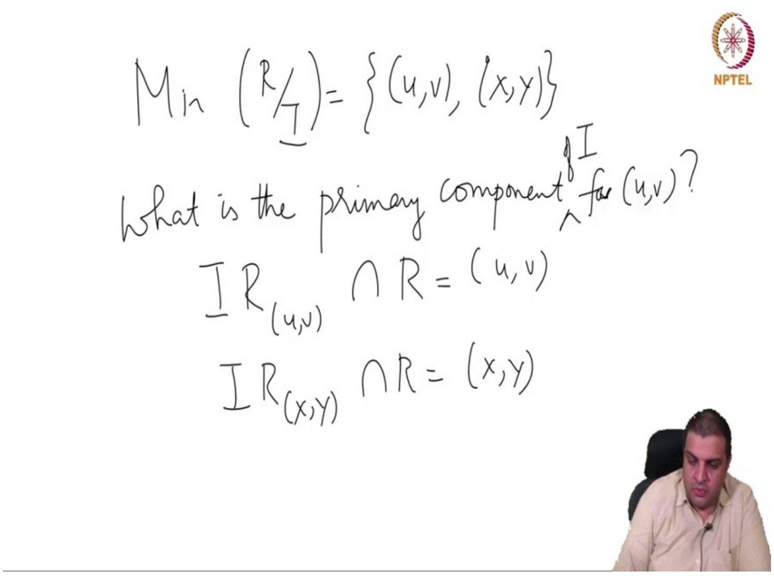

So, the conclusion of all of this argument is that the minimal primes $Min\left(\frac{R}{I}\right)=\{(u,v),(x,y)\}$.

Now, we can ask what are the other associated primes and if there are what are they. So, let us first evaluate the primary components corresponding to these primes. So, what is the primary component? So, we do not need to know the entire associated prime to determine the primary components for minimal primes; that is one advantage of that theorem. Primary component of I for (u,v).

Well we just localize it. So, this is $IR_{(u,v)}\cap R$, but again localizing it this is not that easy to I mean its not easy to describe that ring, but we could just start by inverting x and y first. So, then so this is the same as. So, so this is inverting everything outside (u,v).

So, when we invert just x and y we already get u and v, if we invert x then as we argued in the previous example this becomes a unit and then we just need to put an u, similarly y becomes a unit. So, we just need to put in v and once u and v are there all the terms are there. So, then we can just conclude that this is (u,v). And similarly $IR_{(x,y)}\cap R=(x,y)$. So, these two are two primary components.

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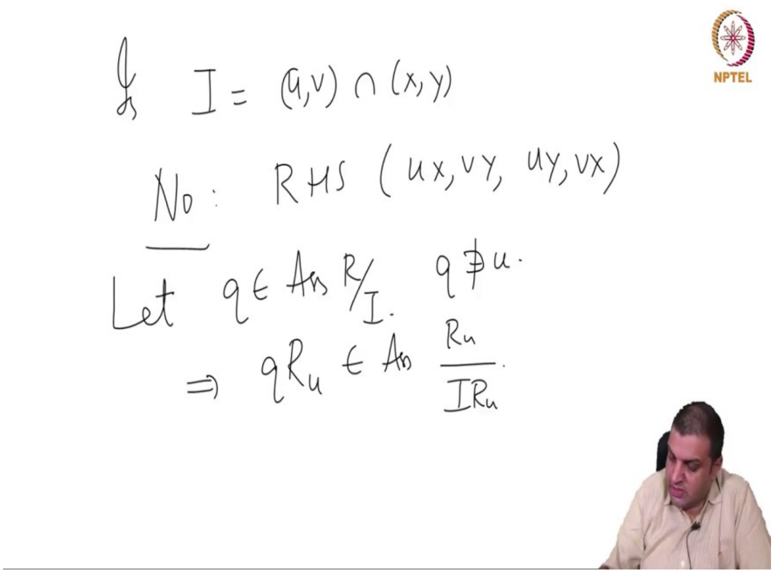

Now, we check if $I=(u,v)\cap(x,y)$. So, the answer is no because the right hand side is (ux,vy,uy,vx) , there are four terms here while in I it was just the sum. So, that is a smaller ideal. So, this is the right hand side is no which means that there are other associated primes and we need to find them ok.

So, what are the other associated primes? So, let q be an embedded prime meaning a prime that is not minimum. We could just say let q be any prime let q be an associated prime such that q does not contain u. So, we are going to do this. So, which means that $q R_u \in Ass \frac{R_u}{I R_u}$.

But what is $\frac{R_u}{I R_u}$? So, we will just invert u here and if you invert u we would get x and if you invert x, then it just becomes v and y and so, if you invert u then u gets killed. And then if you invert y v will get killed and sorry if you sorry in R mod I if you if you invert u then x gets killed.

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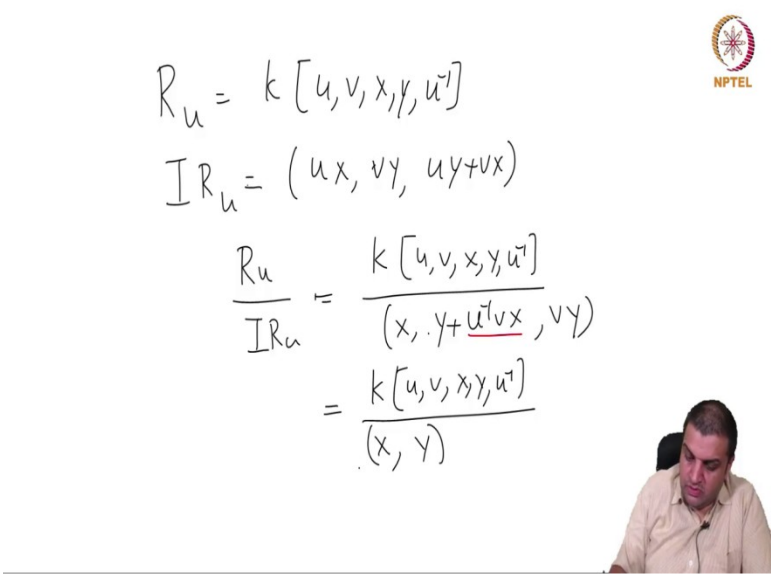

And then so, let us write this out. So, $R_u = k[u, v, x, y, u^{-1}]$. $I R_u = (ux, vy, uy+vx)$, $\frac{R_u}{I R_u} = \frac{k[u, v, x, y, u^{-1}]}{(x, y+u^{-1}vx, vy)}$. So, we just needs x and use a unit. So, we can remove this $uy+vx$ we can rewrite this as $y+u^{-1}vx$.

So, what have I done? The first element you do not need the u because, its the units the remaining things is needed only needed in the ideal in this form we use a unit. So, I can multiply both sides by u inverse and get what rewrite this as $y+u^{-1}vx$.

Now, in this ideal we have x here and if x is there, then one does not need this second term here because, this is just a multiple of x. So, one just needs y and if one has x and y one does not need this I mean one has y one does not need this term. So, in other words, but this is a domain this has only one associated prime.

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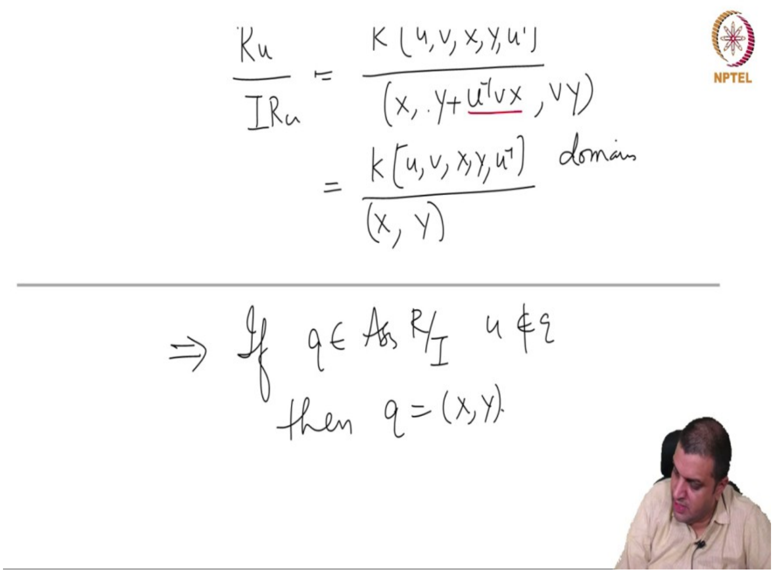

So, this is a domain this is just killing x and y here. So, its just $\frac{k[u,v,x,y,u^{-1}]}{(x,y)}$ that is a domain has only one associated prime which is this. So, in other words so, this implies that if q is an associated prime of $\frac{R}{I}$ such that $u \notin q$, then $q=(x,y)$ that is what this conclusions this argument says. So, we can do this for the remaining other variables also.

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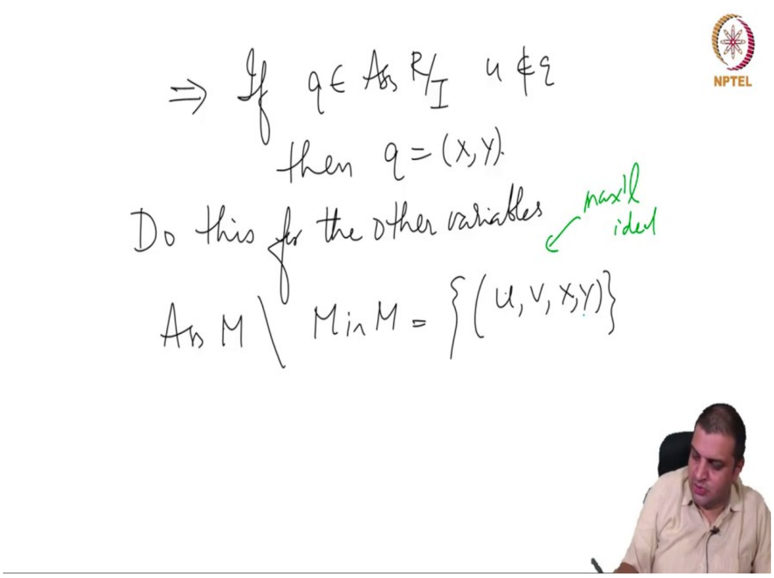

Do this for the other variables do this for the other variables. Now what? So, the conclusion would be that first of all there is an associated prime which is not minimal. And it has to contain if it does not contain u its a minimal prime similarly for v and x and y. So, a conclusion is that $Ass\,M$ ¿ $M=\{(u,v,x,y)\}$.

So, let me just repeat what I how we got this? If you have an associated prime that does not contain u, then it must be this if you as the same argument to say if here associated with does not contain v it must be this. Similarly, if you omit x and y we will get u and v. If we if you localize x and localize y you will get u and v.

However, we know that there is an associative prime which is not minimal because the minimal components do not give I, its something bigger than I. Therefore, there is only one solution to this problem now that associated prime there is an associated prime which as all the variables and this is a maximal ideal.

So, therefore, associated primes is minimal primes over I plus this maximal ideal. And now how do we write a primary decomposition?

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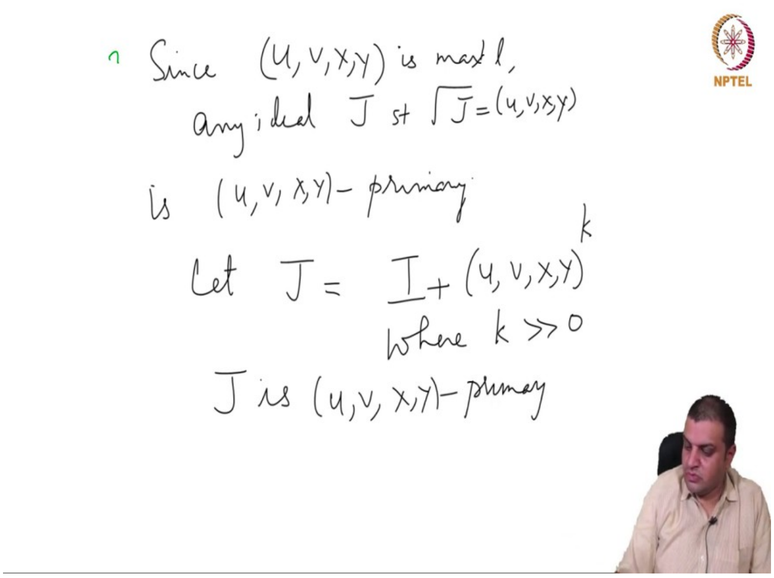

Well because this is a maximal ideal we have we are lucky, any ideal J such that $\sqrt{J}=(u,v,x,y)$ is (u,v,x,y) primary. So, now, what do we do? Well we play a little trick which is that, let $J=I+(u,v,x,y)^k$ where k is very large. So J is (u,v,x,y) primary.

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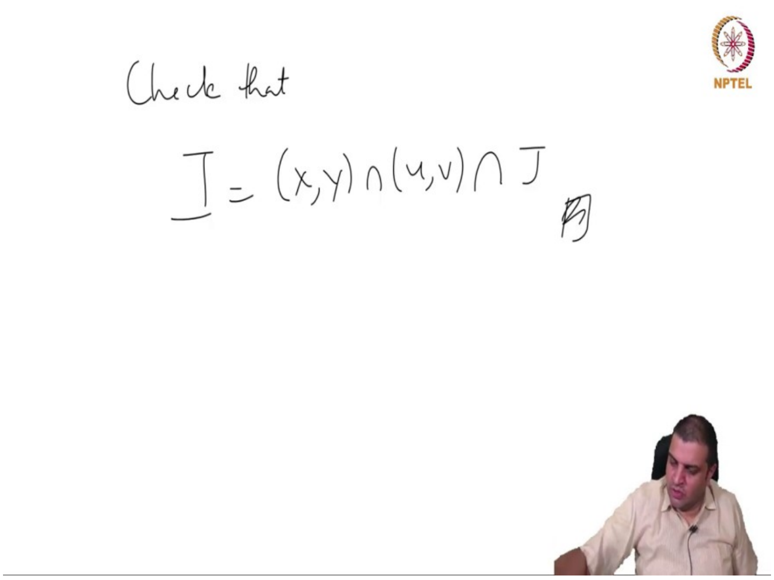

And then check that; I is indeed the components for the minimal primes which we already worked out and J and its independent of that exponent k, for whenever its very sufficiently large.

So, this is one example which sort of goes back and forth between various ideas that we learned and then let us compute. There the finally, there is a trick, but apart from this trick here everything else some in one way or the other we have seen in earlier lectures.

So, this is the end of this lecture.