Computational Commutative Algebra

Prof. Manoj Kummini
Department of Mathematics
Chennai Mathematical Institute
Indian Institute of Technology, Madras

Lecture – 24
Associated primes

This is lecture-24. In this, so we start with this simple proposition which we had sort of discussed earlier, I just want to clarify it, so we discuss that in a minute.

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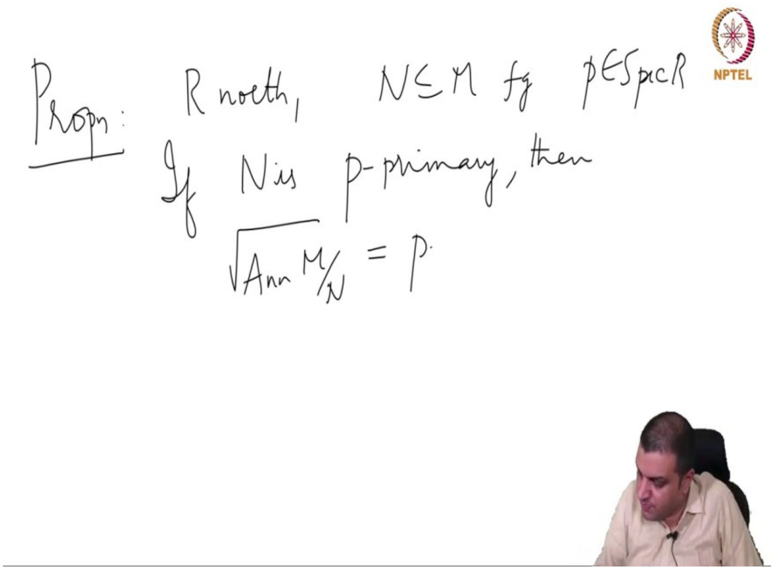

So, R noetherian; $N \subseteq M$ finitely generated R modules. If N is p-primary, then $\sqrt{Ann \frac{M}{N}} = p$, this we saw earlier. Now it is not true that any ideal with a radical p. So, what does it say for ideals?

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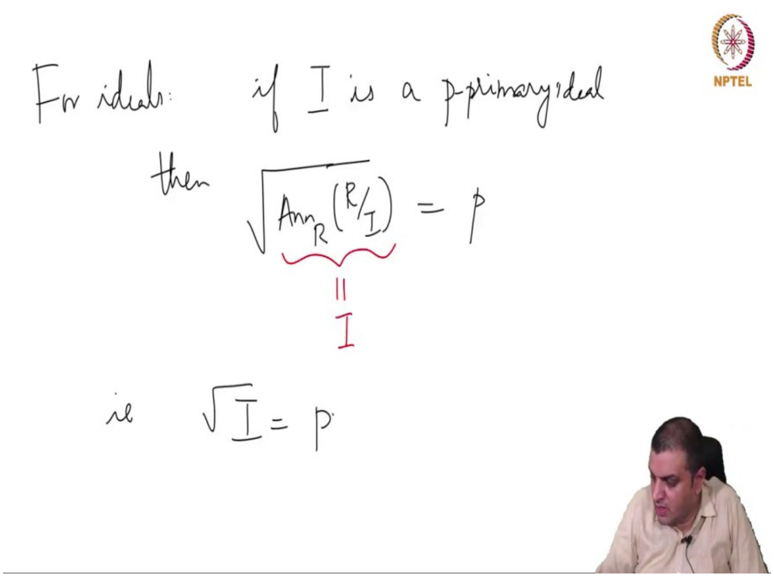

For ideals, it says that if I is a P-primary ideal, so it means that p-primary sub module of R then $\sqrt{Ann_R(\frac{R}{I})} = p$, this is what it says. But what is the annihilator of $\frac{R}{I}$, so this is just I. So, in other words that is $\sqrt{I} = p$ this is what we have.

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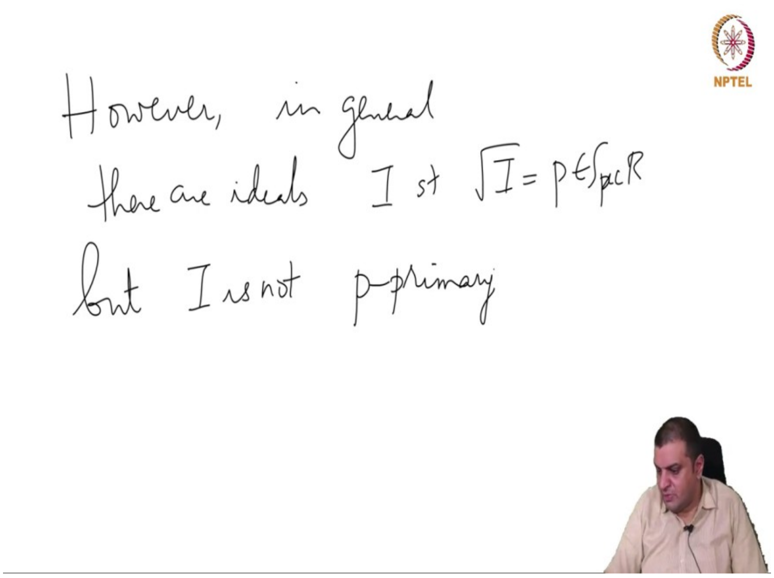

However, in general there are ideals, I mean in noetherian itself I such that $\sqrt{I} = p \in Spec\, R$ but I is not p-primary. Let us look at an example.

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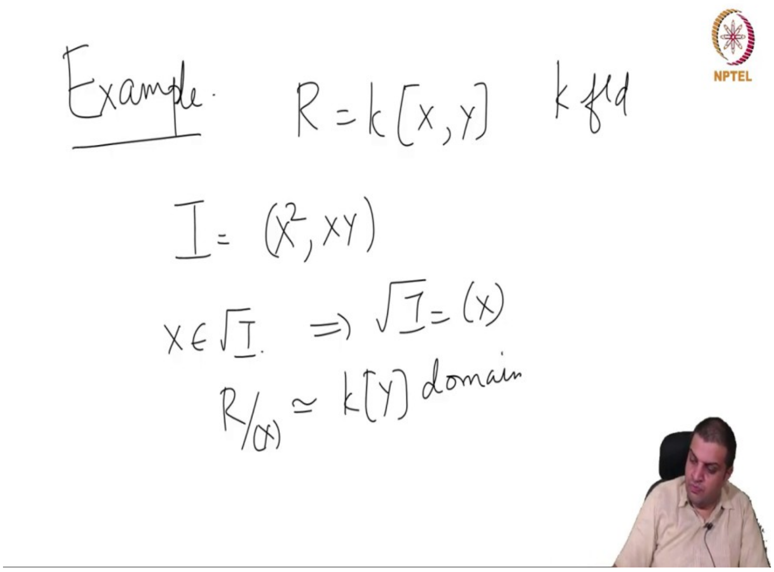

Let R be the polynomial ring in two variables over a field k. Now, let $I = (X^2, XY)$, what is the radical of I? Well, X must belong to the $\sqrt{I}$, because $X^2 \in I$ but if X is in $\sqrt{I}$ then XY is automatically there. So, this implies that $\sqrt{I} = (X)$. And $\frac{R}{(X)} \cong k[Y]$ which is a domain.

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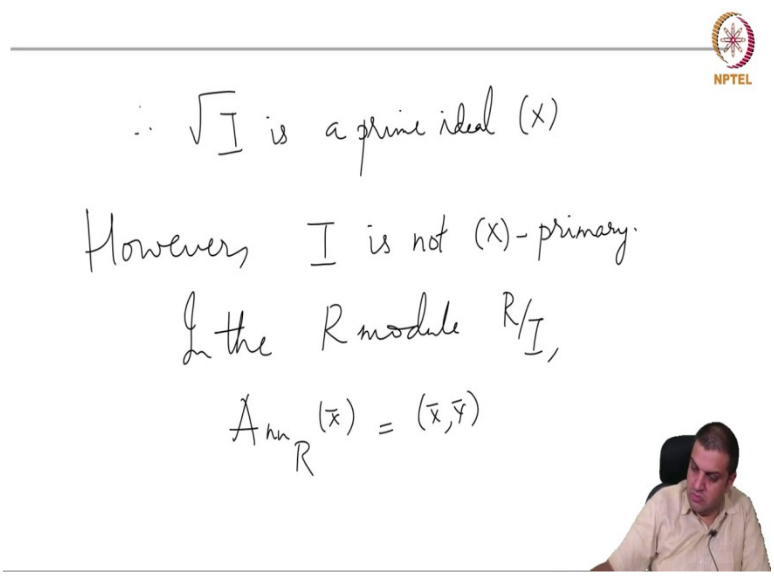

So, therefore, $\sqrt{I}$ is a prime ideal. However, I is not (X)-primary and why is that? So, what is the annihilator of $\overline{X} \in \frac{R}{I}$? This is the ideal (X, Y).

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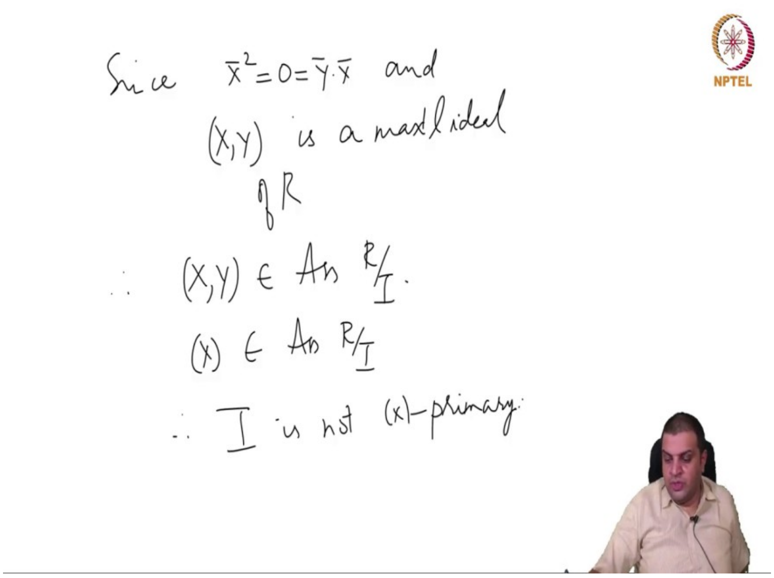

Since $\overline{x}^2 = \overline{y}\,\overline{x} = 0$ and (x, y) is a maximal ideal of R. So, annihilator contains this maximal ideal, but it does not contain 1; so it must be equal to the maximal ideal.

So, therefore $(x, y) \in Ass\,\frac{R}{I}$. And of course, the minimal prime $(x) \in Ass\,\frac{R}{I}$, so there of two distinct primes, so therefore I is not (x) primary. So, in general there are radical ideals which I mean, there are ideals whose radical is a prime ideal, but the ideal itself is not primary.

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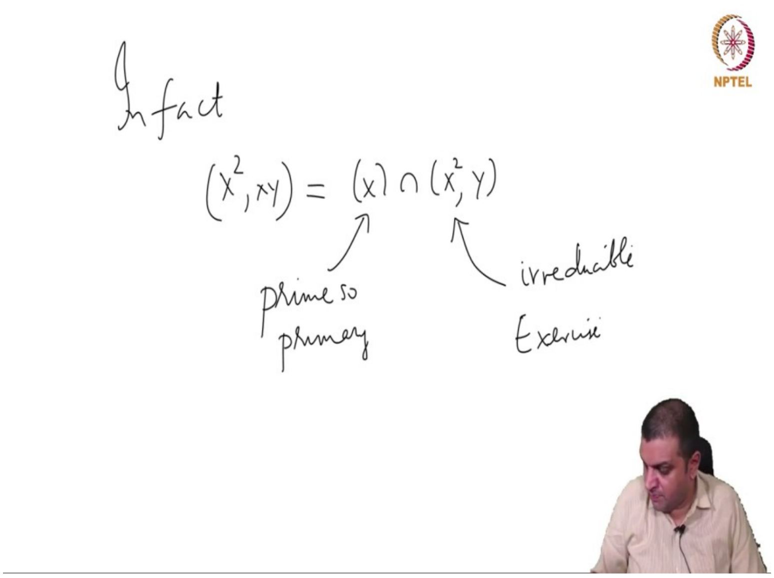

And in fact $(x^2, xy) = (x) \cap (x^2, y)$. So, here it says it must be so one can check that this is equal, this is clearly the right side contains the left side and right side contains the left side.

On the other hand, if you have something on the right side it must be divisible by x and it should be divisible it is written as a linear combination of $x^2$and y. And then you can remove the x terms outside and show that there must be an xy term, so this is equal. And both of this is prime, so primary and also irreducible I mean all of that thing that we want this is irreducible, this is exercise.

So, here is a decomposition into to an irreducible decomposition with two distinct associated primes. The associated prime of this is (x) associated prime of this is (x, y) distinct associated primes, so this is a irreducible decomposition of (x^2, xy).

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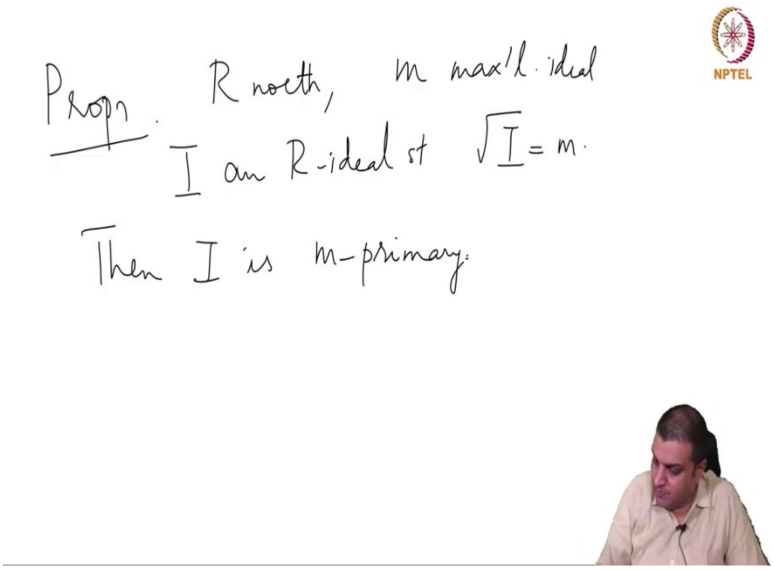

However, we have the following proposition R noetherian, m maximal ideal not necessarily unique maximal ideal; I an R ideal such that $\sqrt{I} = m$, then I is m primary. So, quite often at least in the context of noetherian rings and a maximal ideal, quite often people will interchange these two terminology; just say that I be an m primary ideal. So, these are equivalent notions for ideals whose radical is a maximal ideal.

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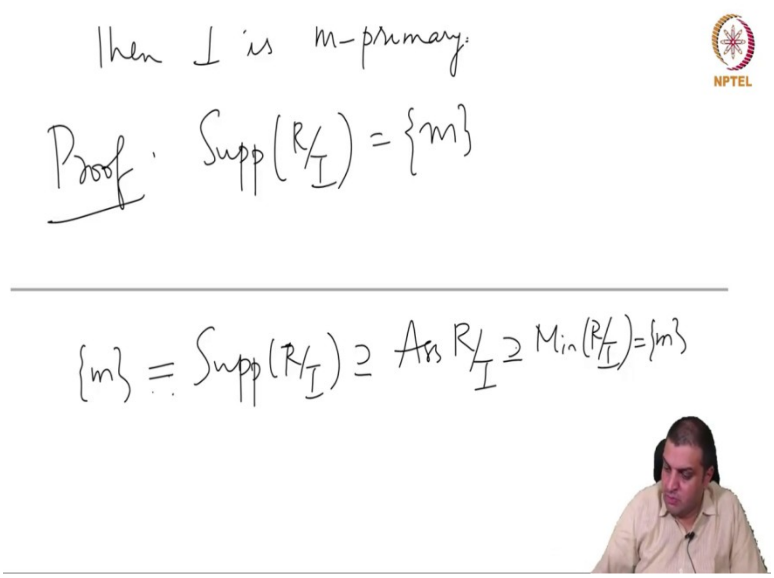

So, proof consider the support $Supp(\frac{R}{I})$. So, this must be inside the radical, so these are primes that would contain the radical, so this is just m. Therefore $Supp\left(\frac{R}{I}\right) \supseteq Ass\left(\frac{R}{I}\right) \supseteq Min\left(\frac{R}{I}\right) = \{m\}$ so remember we have this inclusion, this only a singleton element here which means its own minimal. So, this is also equal to m, because this is singleton so that is the unique minimal element and all of these are equal.

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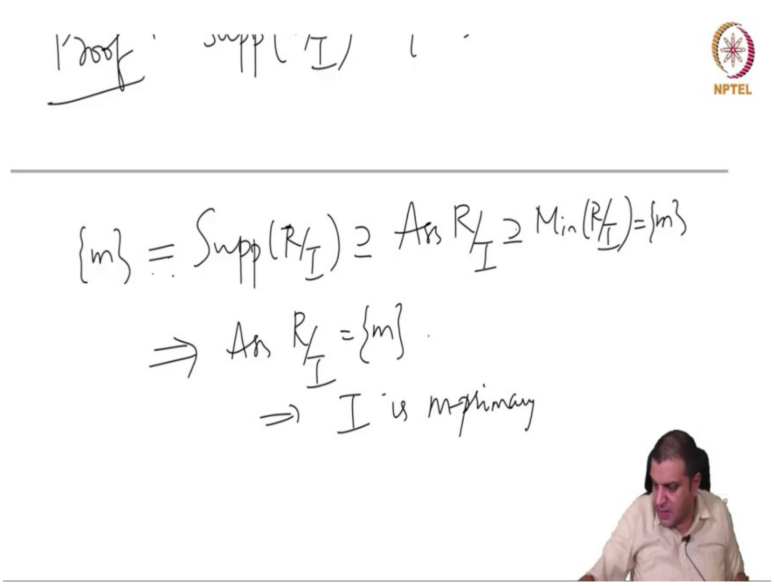

And that says that I is m primary. So, these are some general remarks about m primary ideals. So, now we would like to understand, so let us step back and look for a minute what have we proved.

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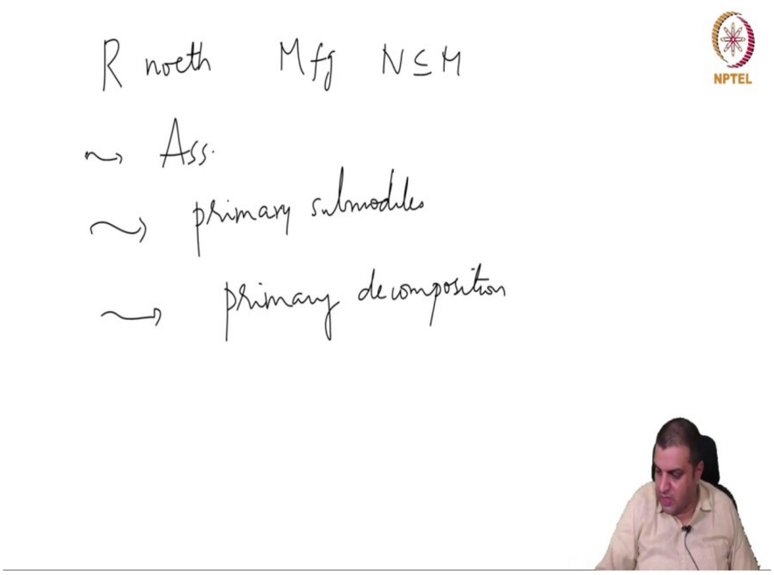

So, context is R is noetherian, M finitely generated and $N \subseteq M$. We introduced the notion of an associated prime, what does it mean to say it is an associated prime? Prime is associated to M, prime is associated to $\frac{M}{N}$ etc. We also introduce a notion of primary sub modules, what does it mean to say $\frac{M}{N}$ is primary, and what does it mean to say it is p-primary, and we also got a primary decomposition.

Let me just summarize we know what notion of associated prime says, we know what primary sub modules are I mean at least we are we I should not say we know, but we actually have seen them, and also we know that there is a primary decomposition. So, this just means that there is a decomposition in which each factor is primary. So, they have a unique associated prime, but what about the associated primes of $\frac{M}{N}$ for R if it is not primary.

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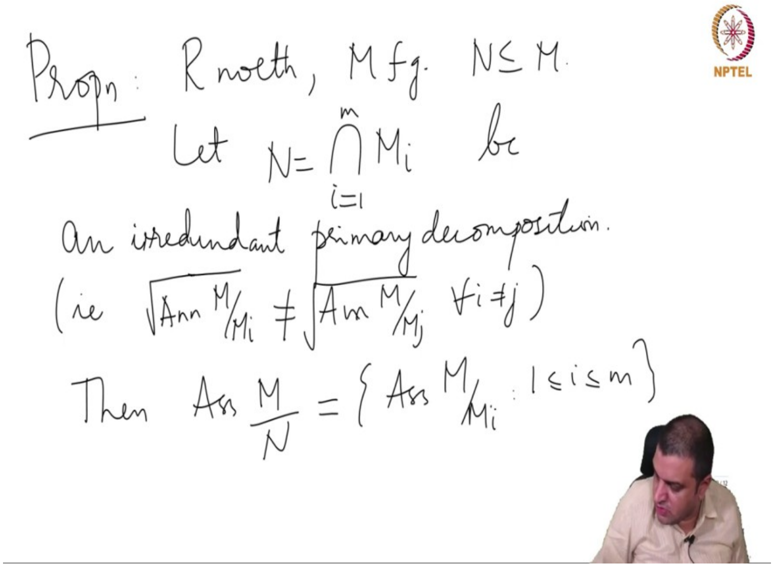

So, here is the proposition; R noetherian, M finitely generated, $N \subseteq M$. Let $N = \bigcap_{i=1}^{m} M_i$ be an irredundant primary decomposition, so each of these are primary. So, in other words $\sqrt{Ann\frac{M}{M_i}}$ is a prime ideal and M_i is primary for this prime and this is different from the corresponding prime for a different ideal $\forall i \neq j$.

This we saw given a irreducible decomposition we can combine the irreducible sub primary, so we can combine the reducible factors corresponding to the same associated prime and rewrite them and such a thing would be an irredundant primary decomposition that is irredundant means this, redundant primary decomposition.

So, then $Ass\frac{M}{N} = \left\{ Ass\frac{M}{M_i} \middle| 1 \leq i \leq m \right\}$. So, each of these factors here $\frac{M}{M_i}$ have contribute one associated prime and distinct ones and that is exactly the list of associated primes of $\frac{M}{N}$, so this is proposition. So, every associated prime of $\frac{M}{N}$ gets a contribution from this and so this is the proposition.

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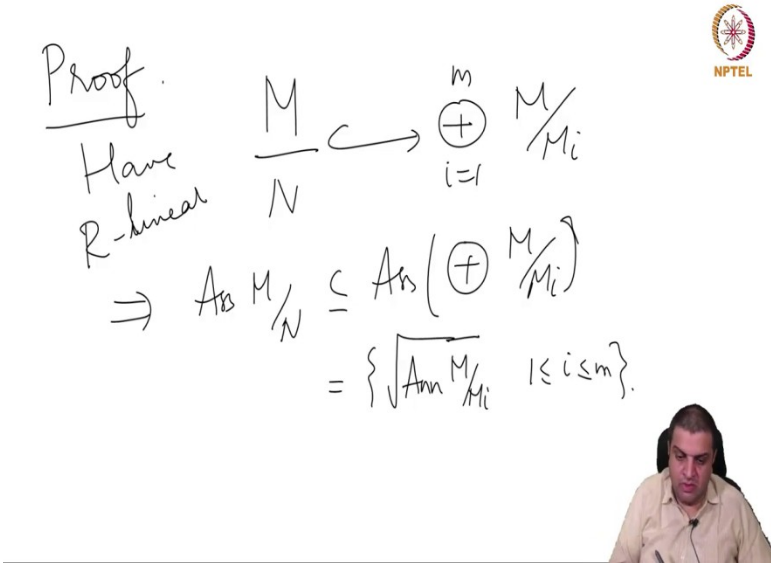

So, so we go back to a sort of argument that we used once used earlier. So, we have an injective map from $\frac{M}{N} \rightarrow ¿ i \frac{M}{M_i}$, so we have R linear injective map. The reason is there is a map from M to each one of these factors which gives a map from M to the direct sum and the kernel is $\cap_i M_i$ which is N, so this becomes injective.

So, this now means that associated prime of $\frac{M}{N}$ is inside the associated primes of the direct sum, but these are themselves primary. So, this is just exactly the annihilator of $\frac{M}{M_i}$. And this is a direct sum of modules therefore, the associated prime just one can take the union; so from each factor in the union one gets exactly one count.

So, the associated primes of $\frac{M}{N}$ is a subset of the associated primes on this side, so this is just slightly easier direction.

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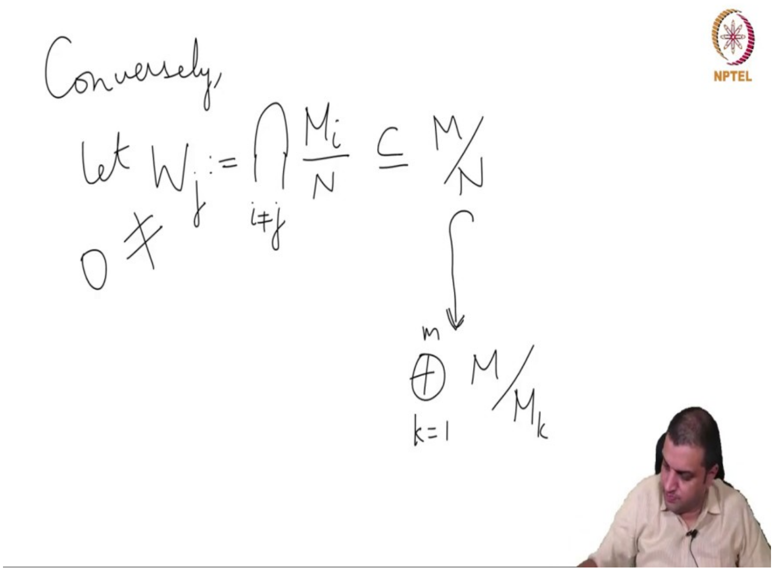

Now, conversely let $W_j = \cap_{i \neq j} \frac{M_i}{N} \subseteq \frac{M}{N}$. Now, each $M_i \subseteq M$ and they contain N, so $\frac{M_i}{N} \subseteq \frac{M}{N}$ and the intersection is also therefore inside $\frac{M}{N}$. And because this decomposition is irredundant, we cannot really remove any of them if you remove any one of them we will not get N.

So, this is really not 0; this is a non-zero module. And now from $\frac{M}{N} \to ¿ k \frac{M}{M_k}$. So, if you take this map and look at where this lands, so let us take a look at this thing at elements here belong to $\frac{M_i}{N}$ different from j. So, in this factor k different from j, it gets killed; so image of W_j inside $\frac{M}{M_k}$ is 0 unless k equals j, this is also injective.

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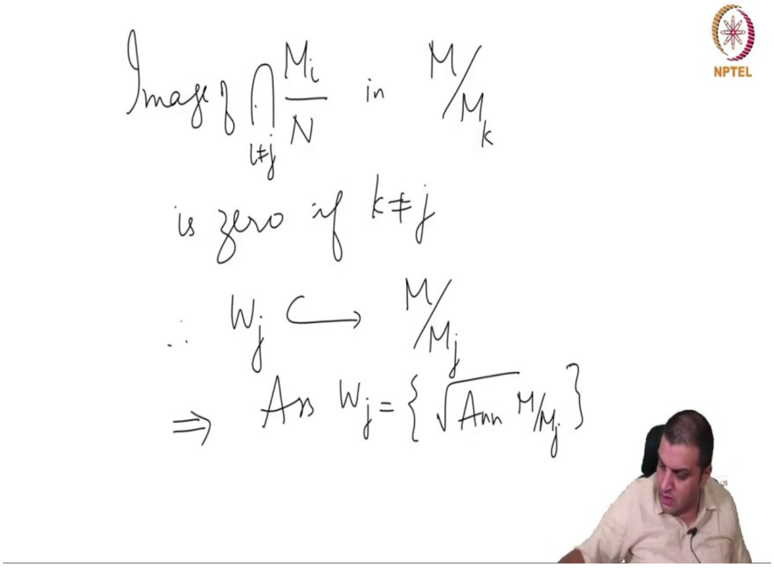

So, image of $\frac{\cap_{i \neq j} M_i}{N}$ in $\frac{M}{M_k}$ is 0 if k is different from j, so that is the observation that we have to make. And this is the intersection this is W_j, hence W_j to this map has to be injective on to the jth factor here, it has to be injective therefore W_j injects into $\frac{M}{M_j}$.

In other words, $Ass\, W_j = \{\sqrt{Ann \frac{M}{M_j}}\}$. So, this is again a primary ideal and its radical of the annihilator is some prime ideal therefore it is primary to that prime ideal. But what does that say, so it says that $Ass\, W_j$ is the associated prime of the jth factor here, but W_j also sits inside $\frac{M}{N}$.

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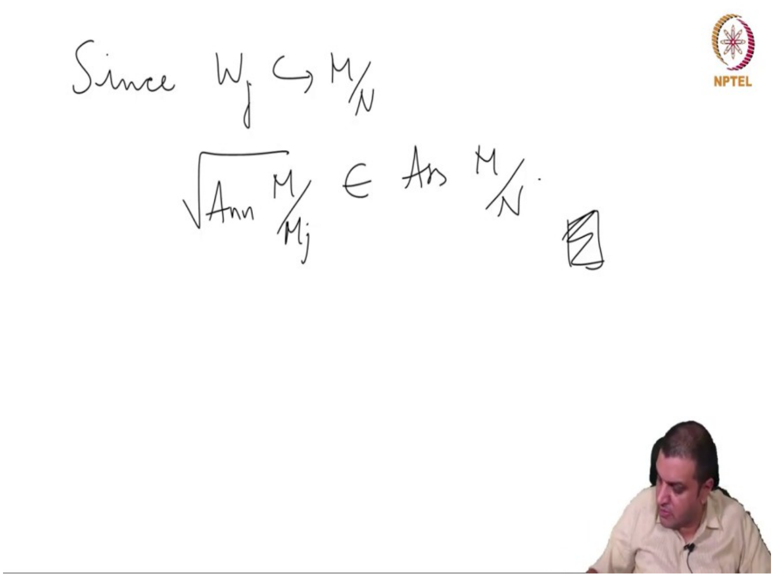

We conclude that $\sqrt{Ann \frac{M}{M_j}} \in Ass \frac{M}{N}$. So, this is the another proof, so this proposition tells us that not only do we have primary decomposition I mean, decomposition into primary sub module as an intersection of primary sub modules. One also gets that the associated primes corresponding to the primary sub modules that is exactly the list of associated primes of $\frac{M}{N}$.

And in the redundant case, they occur exactly once if there are m terms here; N has m distinct associated primes. So, now we would like to discuss little bit more about. So, this is one property of the primary decomposition that we have, we would like to discuss one another uniqueness property, so which is proposition.

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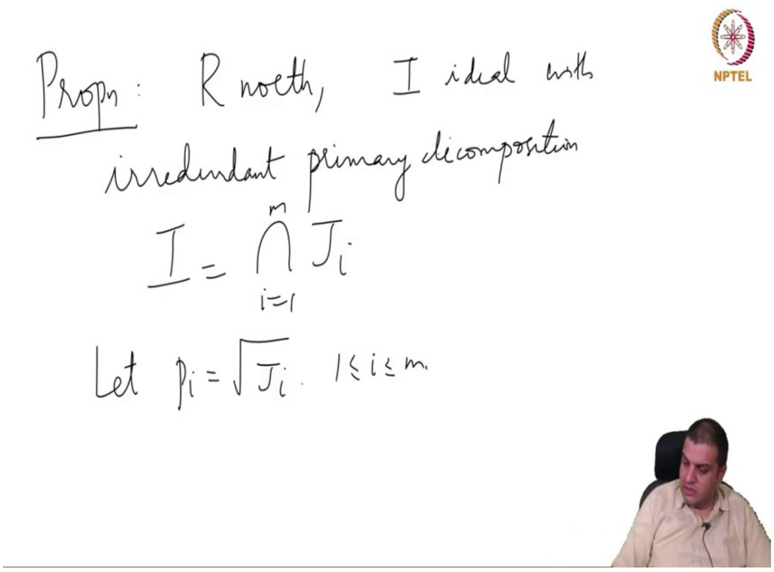

So, now we will restrict ourselves to rings and ideals and not work in the generality of modules. It is a little easier to visualize this picture than the one for modules although the proof is essentially the same, but we will restrict ourselves too. So, R noetherian I is an ideal, then with irredundant primary decomposition $I = \cap_i J_i$.

So, all the J_i's are strictly bigger than I; they $\frac{R}{J_i}$ have distinct associated primes, their primary and distinct associate primes and we saw in the previous proposition that associated primes of I is the union of the associated primes of the sets of associated primes of these individual components. Let $p_i = \sqrt{J_i}$ and this is what we have, so this is $1 \leq i \leq m$.

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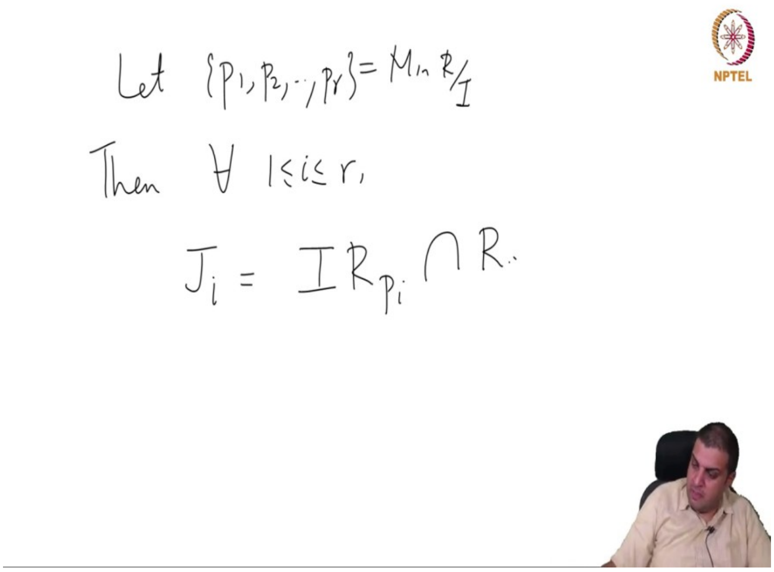

Let $\{p_1, p_2, \ldots, p_r\} = Min\frac{R}{I}$. So, we have observed earlier that the minimal primes are associated, so it will show up in the primary decomposition. So, like this exactly be the minimal prime ideals of $\frac{R}{I}$.

Then $\forall\, 1 \le i \le r$, $J_i = I R_{p_i} \cap R$. So this is what this proposition says.

So, if you have a primary decomposition, so first of all there is an analogous statement for modules which you omit, because it is this the statement is a little bit more clear. If you here is an ideal and it has an intersection so we write as an intersectional irredundant primary decomposition, now you just look at the components corresponding to the minimal primes they are uniquely determined that is what this. It is extend this ideal to localization and contract back, this is what we have proof.

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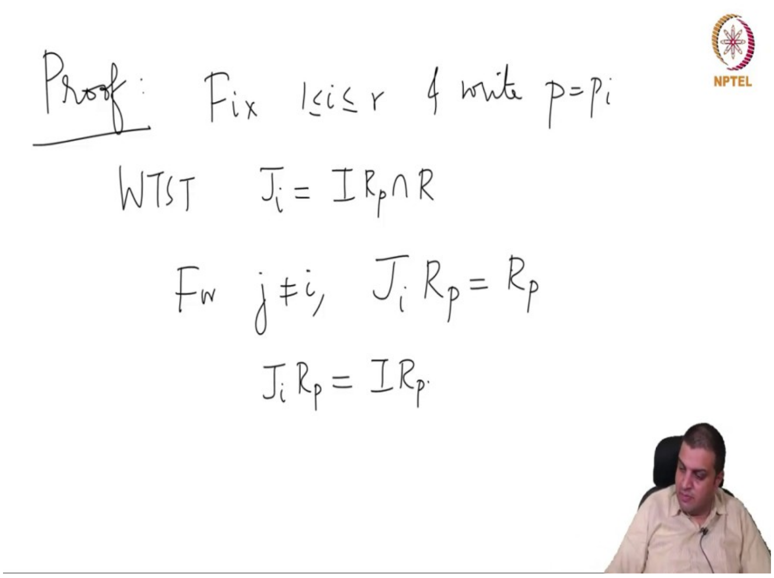

So, let us extend I to R_p. Now, this is the same so what is it let us fix some i in this range, where it is corresponds to a minimal prime and write p for p_i. So, we want to show that $J_i = IR_p \cap R$ that is what we want to show. So, now one can check the following.

For $j \neq i$, $J_i R_p = R_p$ why is that so remember p is a minimal prime, and J_i for this property means that its radical will contain an element which is not inside p. Therefore, it will contain an element which is not inside p, because powers complement of p is multiplicatively closed.

And therefore, when invert the complement of p something inside J_i would get inverted and it will become the whole ring. So, using this one can check that $J_i R_p$ is exactly $I R_p$, so that now when we contract it back we might get a bigger ideal.

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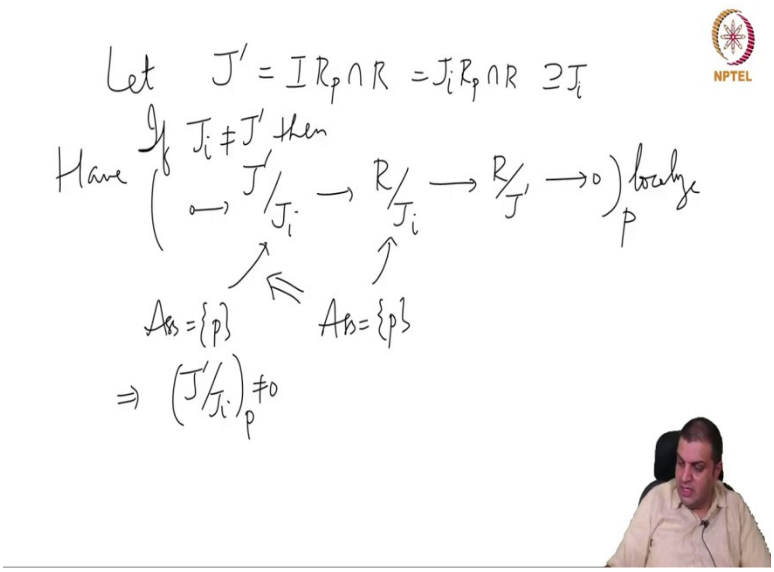

So, let $J^{'}=I\,R_p\cap R=J_i R_p\cap R$. Now, because of this, this could be bigger than J_i. But what kills this $J^{'}$; so notice that we have $0\rightarrow\frac{J^{'}}{J_i}\rightarrow\frac{R}{J_i}\rightarrow\frac{R}{J^{'}}\rightarrow 0$ we have this; now localize this at p.

So, first of all what do we know what are the associated primes of this, associated primes of this is just p. So, here associated primes is just p which means that we get that associated primes is also just p. So, we want to prove that $J=J_i=J^{'}$, all of this argument is necessary if it is not equal.

If J_i is not equal to $J^{'}$, then do this; so then this is a non-zero module and it has an associated prime and we localized at p. So this now implies that $\left(\frac{J^{'}}{J}\right)_p\neq 0$ because associated primes are inside the support, so this localize is non-zero.

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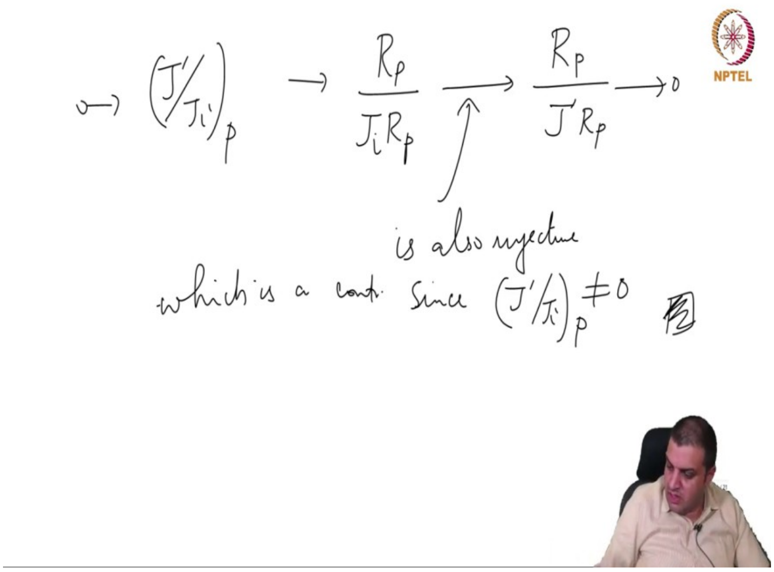

But on the other hand we get J from the exact sequence we get $0 \rightarrow \left(\frac{J^{'}}{J_i}\right)_p \rightarrow \frac{R_p}{J_i R_p} \rightarrow \frac{R_p}{J^{'} R_p} \rightarrow 0$; but these two ideals are equal, they may be different than R; but once you invert element outside P, they become equal which means that the subjective map is actually also injective.

So, this is also injective which is a contradiction since $\left(\frac{J^{'}}{J_i}\right)_p \neq 0$, so that proves that the minimal components are uniquely determined. The primary component corresponding to the minimal primes are uniquely determined, just one definition which we should.

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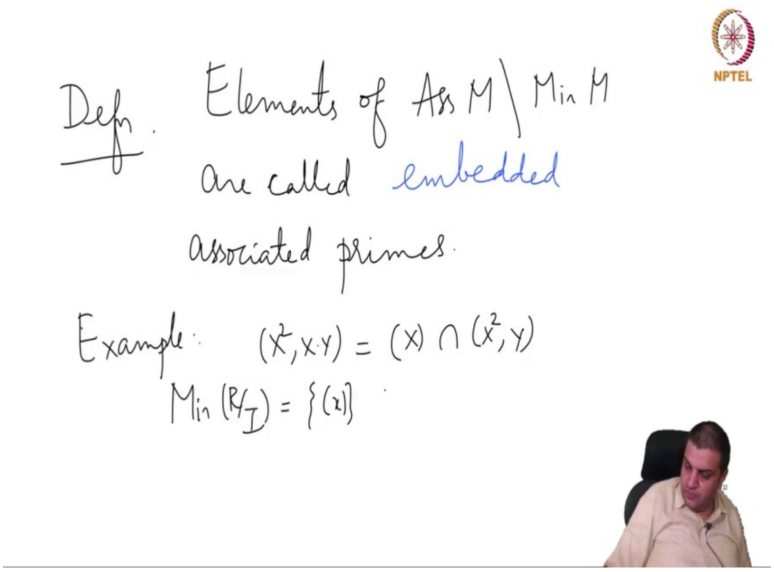

We say that p in associated prime, we (Refer Time: 31:46) elements of associated primes of M, which are not in the minimal primes of M are called embedded associated primes. So, in the example that we had seen earlier, in the ideal $\left(x^2, xy\right) = \left(x\right) \cap \left(x^2, y\right)$. This is minimal of $\frac{R}{I}$, this is the previous example is just this prime while associated primes are.

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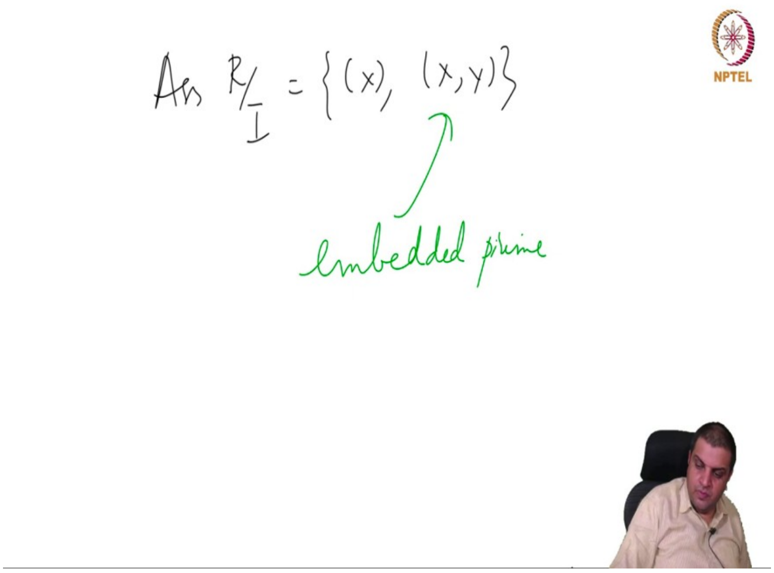

The (Refer Time: 33:01) prime ideal (x) and the prime ideal (x, y); and this is an embedded prime. So, we will end this lecture now. And in the next lecture, we will look at few more properties and some examples.