Computational Commutative Algebra

Prof. Manoj Kummini
Department of Mathematics
Chennai Mathematical Institute,
Indian Institute of Technology, Madras

Lecture – 23
Support of a module

So, now we continue our discussion about primary decomposition, first proposition which give some light on primary ideals.

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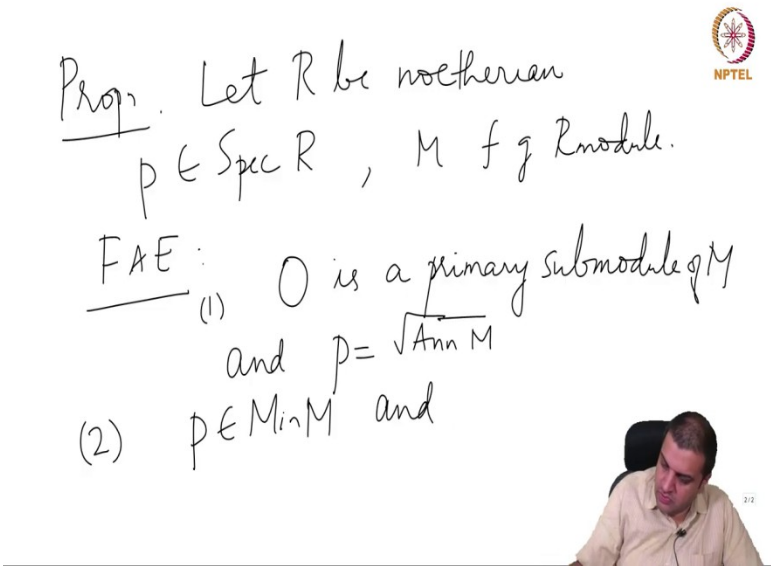

So, proposition let R be noetherian, P a prime ideal and M a finitely generated R module and then the following are equivalent. We want to understand what it means for M to be P primary.

One, 0 is a primary sub module of M and $P=\sqrt{Ann\,M}$. We are not saying what M is P primary we are trying to understand the condition when is 0 a primary sub module. This is the same thing as what we were discussing in the previous lecture we were discussing N is a primary sub module of M that is a relay condition about M mod 0 inside M mod N. So, we are just restricting ourselves to 0 sub module.

Second condition, P is a minimal prime of M. I will define this immediately after I state.

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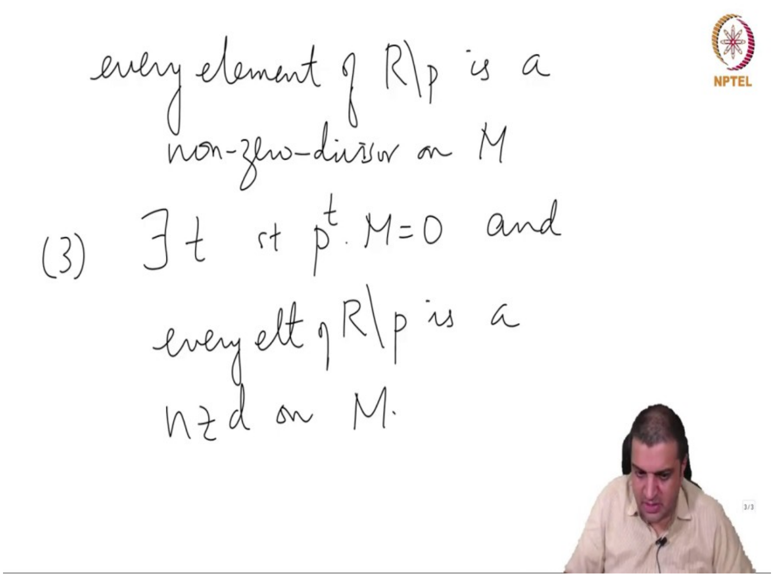

And every element of R¿is a non-zero-divisor on M. And three, there exist some t such that $P^t M=0$ and every element of R¿ is a non-zero divisor on M.

So, sorry I missed two definitions here which I should have done before I got to this. So, let us define those things.

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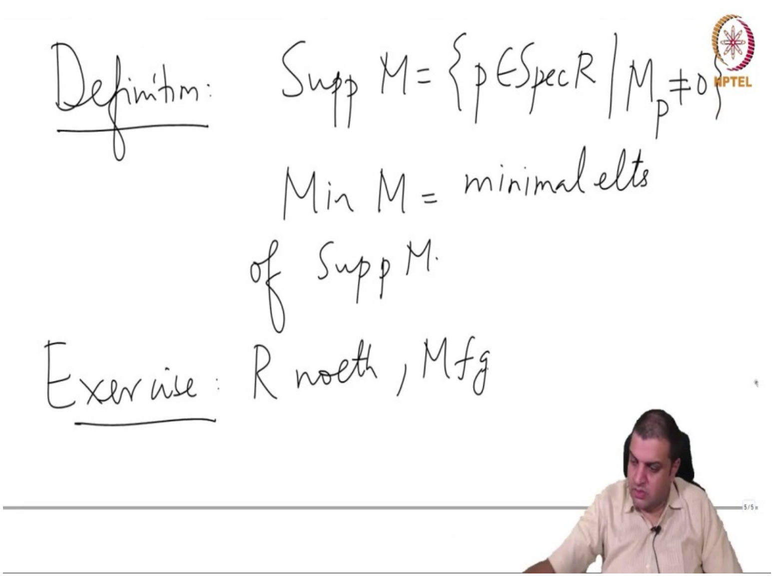

Definition: support of M is the set $Supp\, M = \{p \in Spec\, R | M_p \neq 0\}$. In particular, $Ann_R M \subseteq P$ we have inverted elements outside p and in this case we have not inverted anything that annihilates M. So, this is what support of M is.

And min M is the minimal elements of support of M. So, this agrees with the definition of min of a ring that we that we used in earlier lectures which is just to say support of a ring is all of spec R. So, min is minimum elements of that. So, this is just a definition and here is an exercise R noetherian M finitely generated then $Supp\, M = V(Ann_R M) = Spec \frac{R}{Ann_R M}$.

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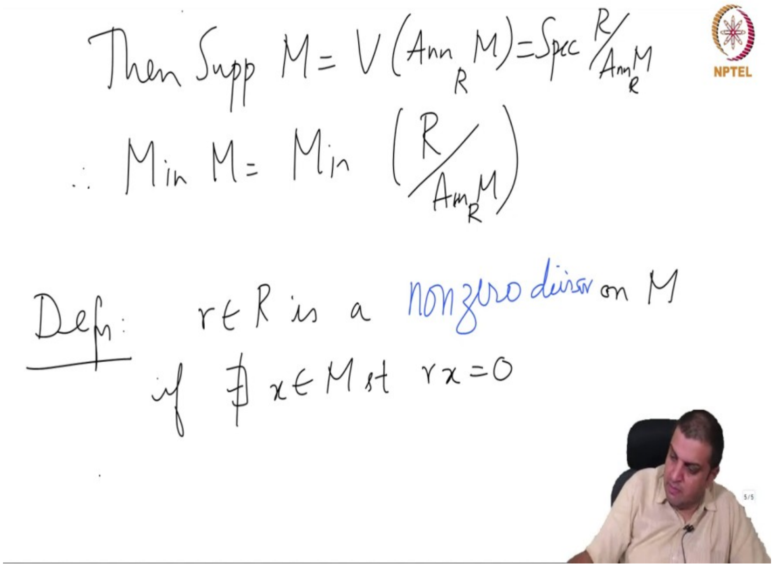

Annihilator of M is the elements of R that kill all elements of M. So, this is an exercise which you should do. So, this is what support is. So, therefore, in conclusion from this is Min M is minimal elements I mean in this context R noetherian M finitely generated this is same thing as $Min(\frac{R}{Ann_R M})$. Support of M is V of this and its minimal elements are precisely the minimal elements of the spec of this ring. We often views notation like this technically this is a subset of spec R this is some other set which maps into this as a closed subset, but we will often not worry about these differences unless there is something to be done in the (Refer Time: 06:15) ok.

So, this is one and another definition was non zero divisor $r \in R$ is a non zero divisor if there does not exist nonzero $x \in M$ such that $rx = 0$.

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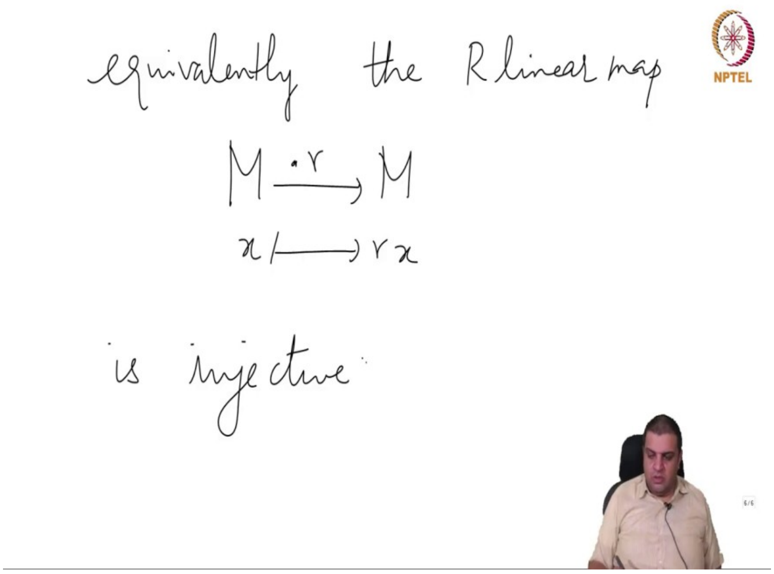

The map $M \rightarrow M$ given by multiplication by r which is some x goes to rx, the R linear map is injective. So this is what will be a non-zero divisor. Now, let us go revisit the proposition which we need to prove. So, we are in a noetherian ring finitely generated module and we ask the question when is 0 a primary sub module with this property. So, 1 implies 2 this is proof of the proposition 1 implies 2.

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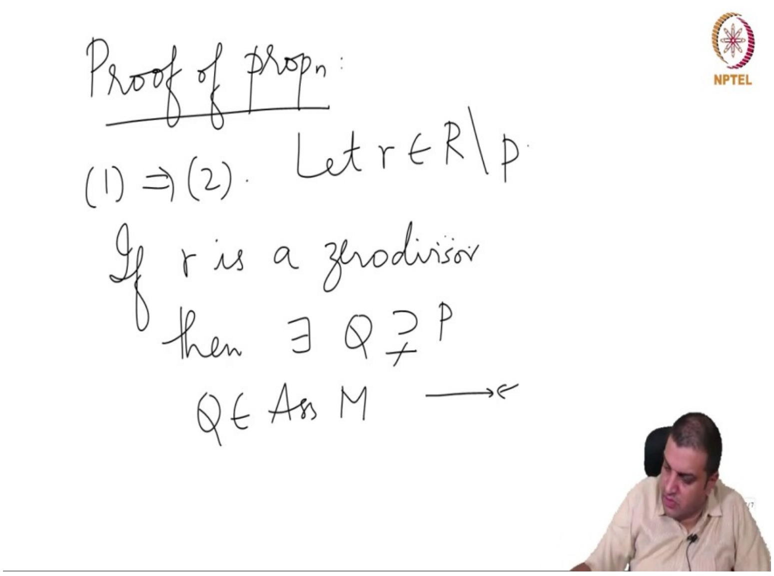

Let $r \in R$¿, if r is a zero divisor, then there exist a Q which is strictly bigger than P and $Q \in Ass\, M$. This is a contradiction.

If P is the radical of the annihilator of M then every element inside M will be killed by some power of P and hence among the associated primes P will be there, but because 0 is primary among the associated primes of M, P will be there.

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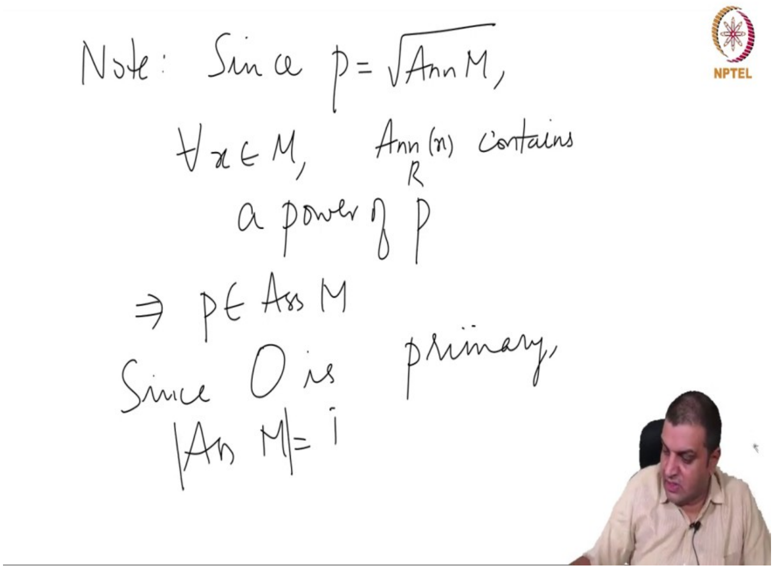

Since P is the radical of the annihilator of M for every $x \in M$ the annihilator of x contains a power of P and which now implies that P is associated to M; however, since 0 is P primary, 0 is associated primes of M is a singleton set, but P is already there and that is the contradiction here.

You cannot have a Q different from P which is associated to M.

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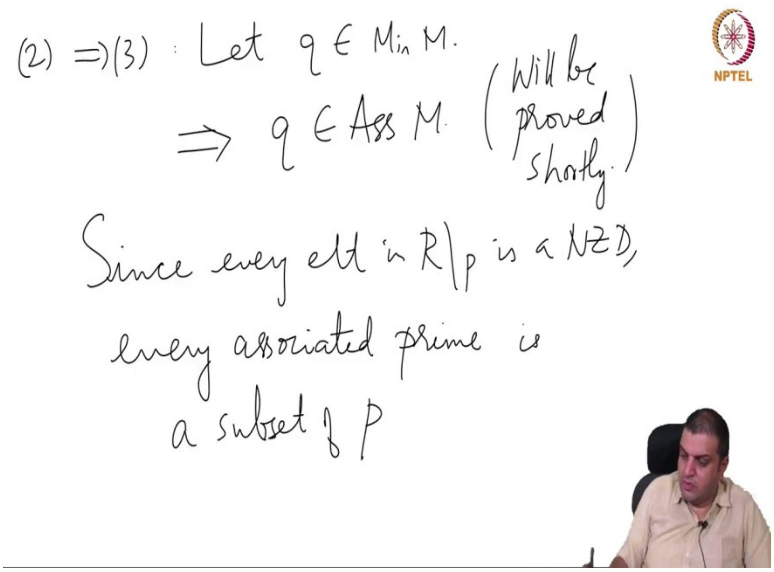

Now, 2 implies 3. So, let us look at 2. So, part of 2 and 3 are the same which is about some things being nonzero divisor. So, two says p is a minimal prime and every element of $\frac{R}{p}$ is a non zero divisor. From this we have to conclude this part and the rest is just the same. So, let us do that not from all of two we will assume that $p^t M=0$ and then the rest is just rewriting.

Let q be a minimal prime of M. So, p is a minimal prime of M and every element of R¿ is a non zero divisor on M which means that one cannot have associated primes outside of p meaning every element outside this is a non zero divisor and remember that associated primes are maximal associated primes contain 0 divisor because if q is an associated prime q must kill something inside M.

So, if q is a minimal prime. So, I will have to use a fact that minimal primes are associated which I will prove in a little while. So, this will be proved I mean in a little while shortly, but we just observed that every element outside p is a non zero divisor. So, therefore, a prime ideal not contained in p cannot be an associated prime. Since every element in $\frac{R}{p}$ is a non zero divisor every associated prime is a subset of p.

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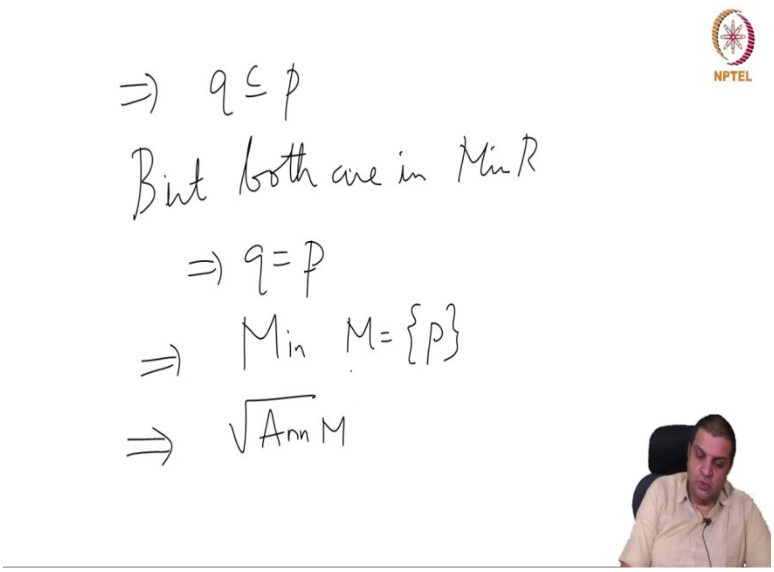

In other words, $q \subseteq p$, but both are in $Min\,R$ which means that$q = p$. So, that is what we have. So, what does it say. So, this one says that $Min\,M$ is a singleton set p. So, from these two statements together that p is a minimal prime and every element outside p is a non zero divisor what we concluded that the minimal primes over M is a singleton set containing just p.

So, which now implies that the radical of the annihilator of M. Remember we proved a while ago that radical of an ideal is the intersection of prime ideals containing it and this is the support of I mean the prime ideals containing this is precisely the support of M, the minimal elements of those are precise p. So, therefore, when you intersect we just have to intersect the minimal primes and this is equal to p and now we can observe the following.

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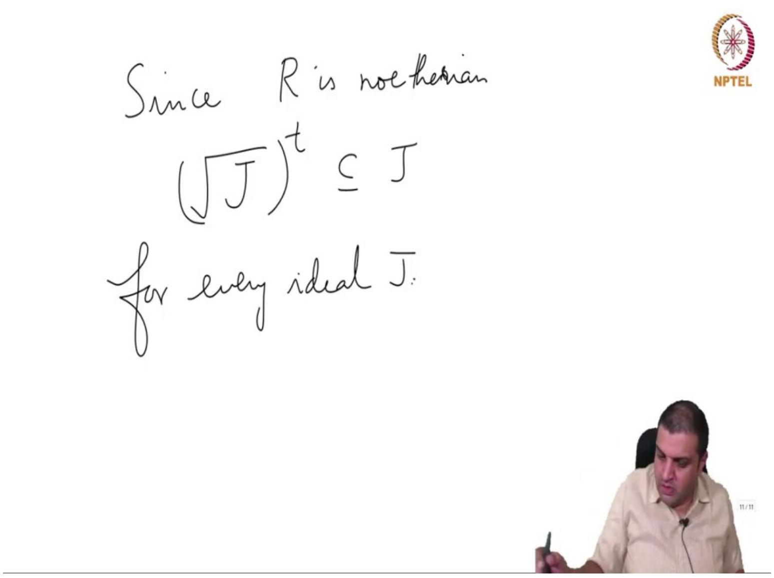

Since R is noetherian $(\sqrt{J})^t \subseteq J$ for every ideal J what one needs is that this ideal is finitely generated remember this is the radical. So, if this is finitely generated by let say $x_1, \ldots, x_n$ then $x_1^t, x_2^t, \ldots, x_n^t \in J$ and then we can just take some very large power and then arbitrarily linear combination of the x's will also be inside J and all that is used here is that this ideal is finitely generated.

So, R is noetherian therefore, for every ideal J this is true. So, this prove gets 3 from 2.

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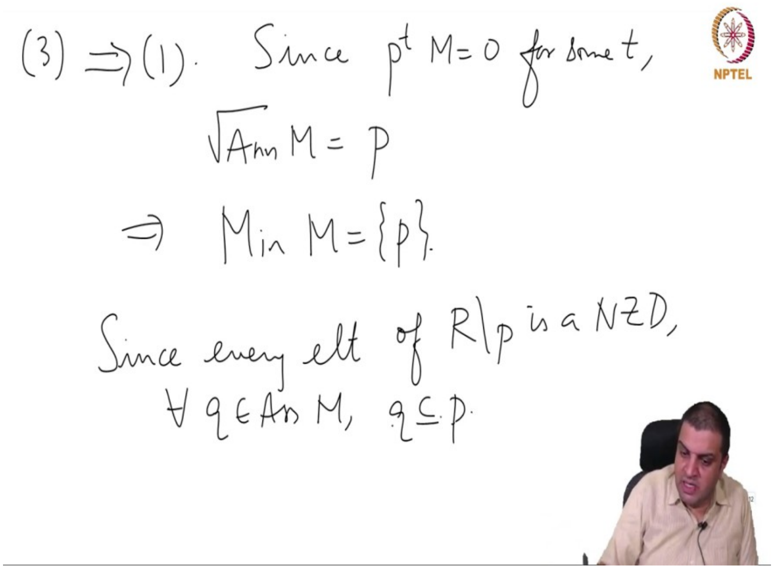

3 implies 1. So, 3 says that some power of p annihilates M and every element of this is a non zero divisor on M. Since $p^t M = 0$ for some t the radical of the annihilator of M is p. In other words $Min\, M = \{p\}$. I still would have to use the fact there are no other associated primes.

So, now we need to show that. Since every element of $\frac{R}{p}$ is a non zero divisor. For every associated prime q must be a subset of p because we discussed this little earlier because if q is an associated prime q has to kill a nonzero element of M which means that elements of q are themselves 0 divisors on M therefore, it cannot belong to this set, but p is a minimal prime.

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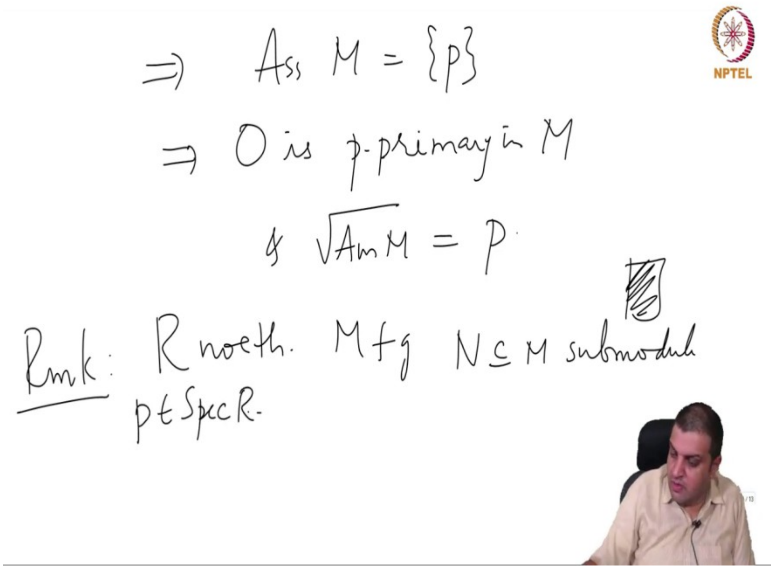

So, which means that associated primes of M is also this one, but what does that say this now says that 0 is p primary in M and the radical of the annihilator of M is p ok. So, this proves a proposition.

So, one can think of it in the following way, what do elements inside R look like on M. So, remark. So, this is what a primary sub module means. So, R noetherian I mean sorry not that this is what it means, but it has the following property, M finitely generated and $N \subseteq M$ sub module p is a prime ideal.

So, I am just rewriting the previous proposition in a slightly more convenient or familiar way.

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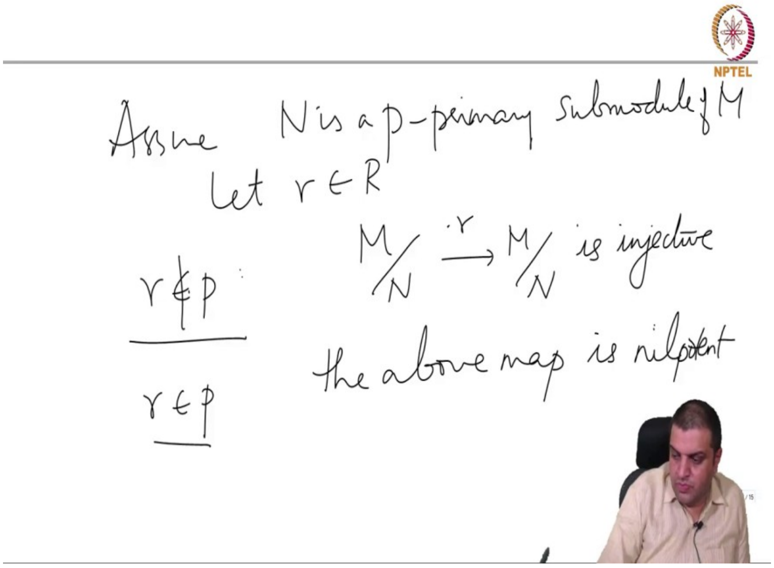

Then if assume that N is a p primary sub module of M. Then every element of $r \in R$ can act only in two different ways on $\frac{M}{N}$. If $r \notin p$ then the map $\frac{M}{N} \to \frac{M}{N}$ is injective and if $r \in p$ then the map is nilpotent.

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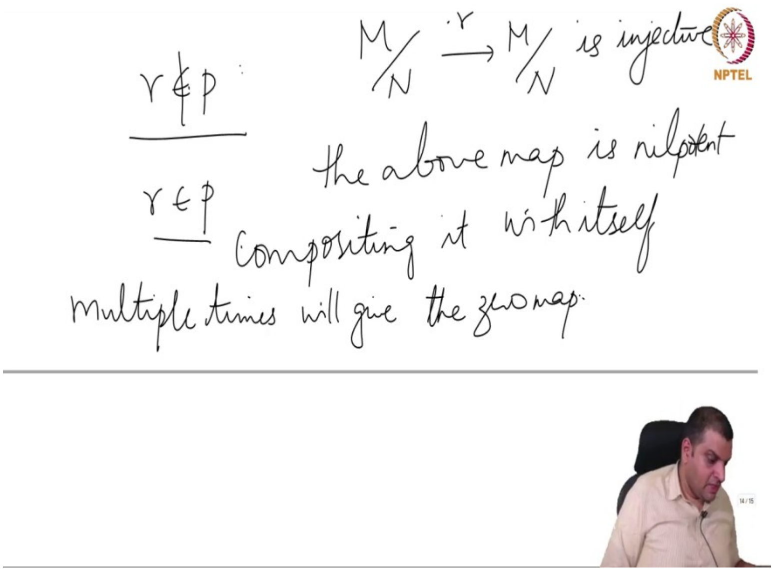

In other words, if you compose at map enough times it becomes a zero map. So, that is composing it with itself multiple times will give the zero map. We just prove it that elements of r act either as non zero divisors or as nilpotent elements.

This is a special property about primary sub modules that an arbitrary sub module will not have some examples and these irreducible and primary components. So, in this proof we observe we use the following that minimal primes are associated. So, that requires a proof and we will prove now, but before we prove that we need to make an observation about associated primes and localization.

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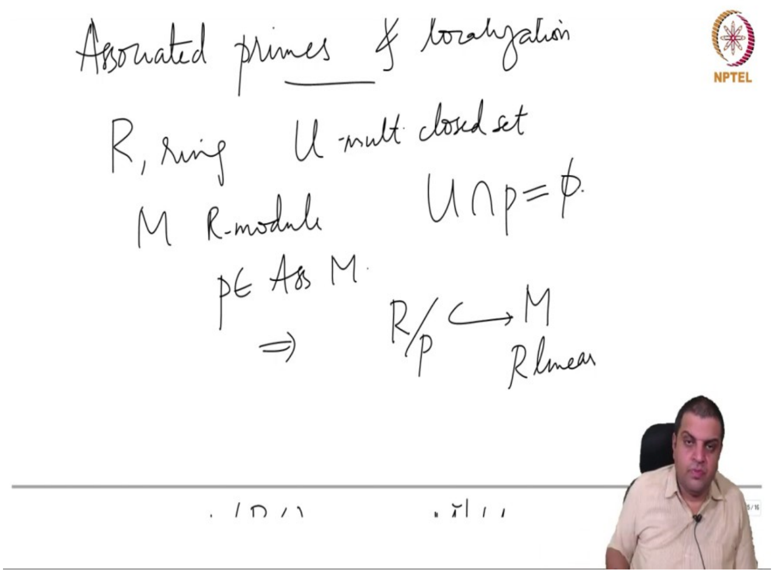

So, say associated primes and localization. So, this is the situation R is a ring, U is a multiplicatively closed set and M is an R module suppose that p is associated to M, then what does this say? This says that there is an R linear injective map from $\frac{R}{p} \to M$. So, this implies this is R linear.

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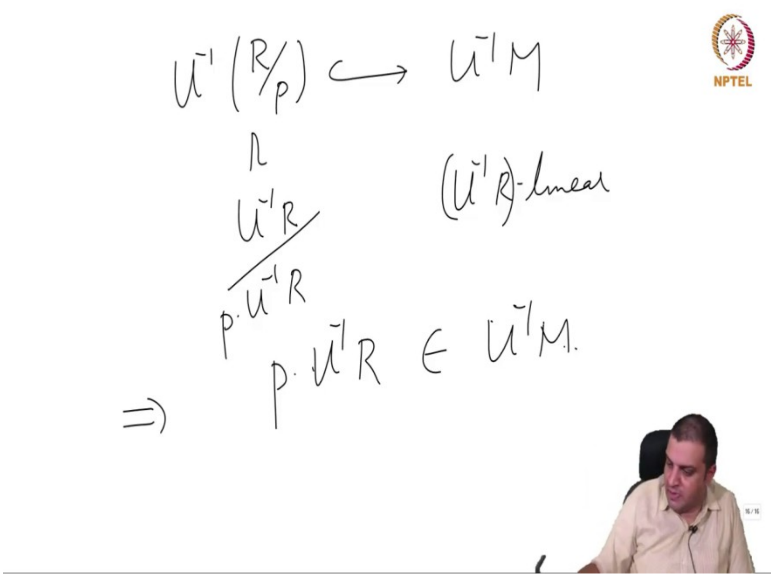

Now, we could just localize this whole map, this gives a map from $U^{-1}\left(\frac{R}{p}\right) \to U^{-1}$, localizing preserves injectivity. But what is this? This is isomorphic to $\frac{U^{-1}R}{pU^{-1}R}$ extended to that $U^{-1}R$. So, this is $U^{-1}R$ linear. So, in other words. So, we could assume in this situation U does not intersect P. So, that P extended to this is something non trivial. So, this is not the whole zero ring.

So, an associated prime remains an associated prime after localizing provided the prime itself did not become the whole ring. So, we inverted something in the complement of that prime. So, this is one statement.

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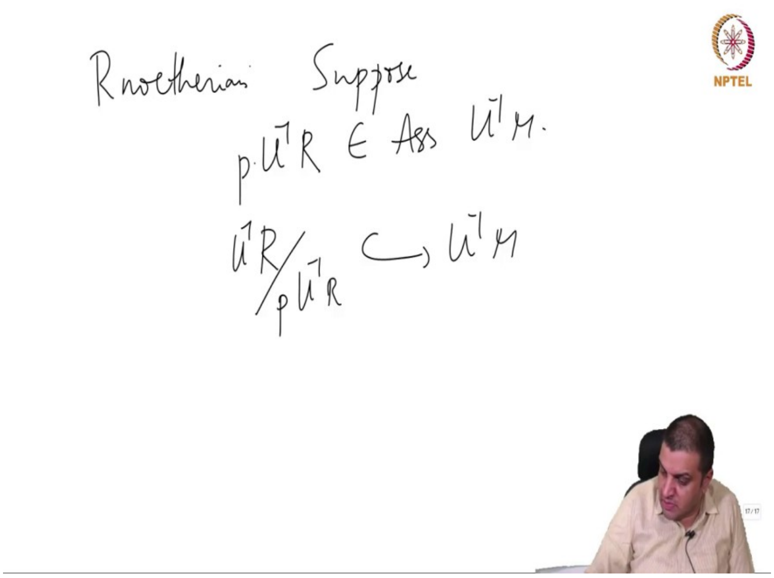

So, now suppose R is noetherian in addition to the previous things and assume that suppose that $U^{-1}p$ extended to $U^{-1}R$ is associated to $U^{-1}M$. So, therefore, there is an R linear injective map from $\frac{U^{-1}R}{pU^{-1}R} \to U^{-1}M$, now the fact that U is finitely generated in this context gives us the following.

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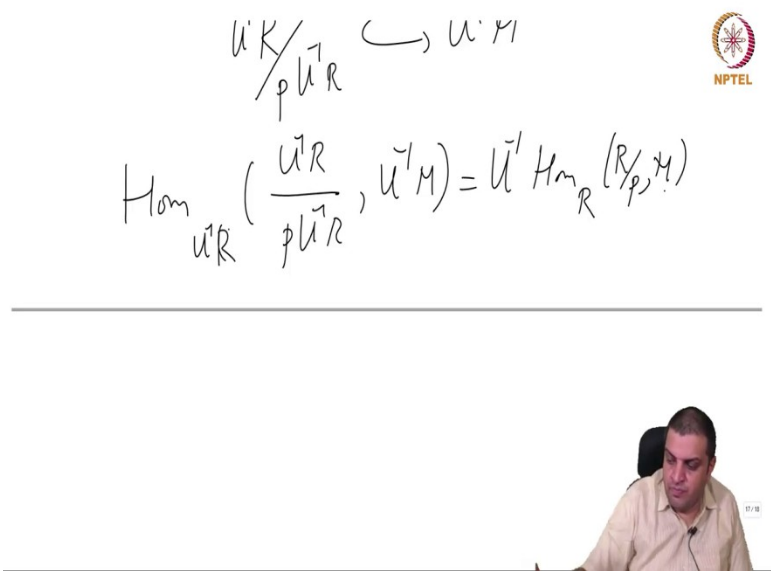

$$Hom_{U^{-1}R}\left(\frac{U^{-1}R}{pU^{-1}R}, U^{-1}M\right) = U^{-1}Hom_R\left(\frac{R}{p}, M\right)$$

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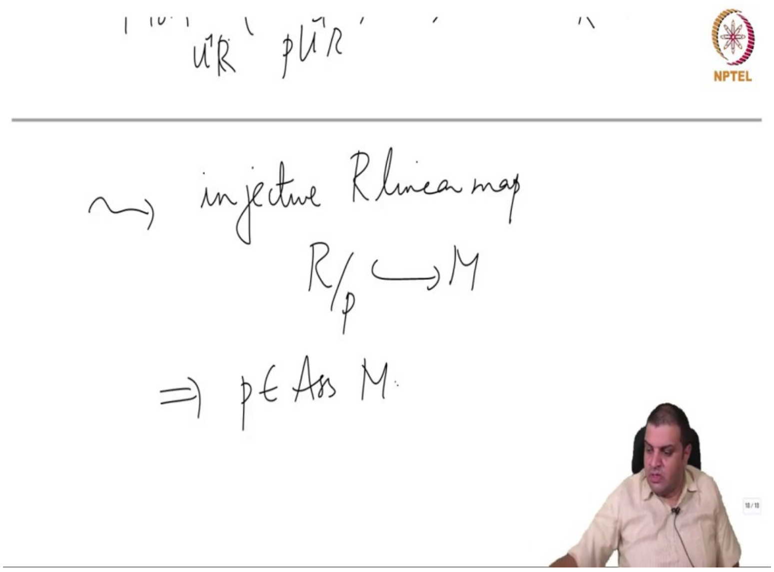

And this will give an injective R linear map from R mod P to M and this now implies that P is associated to M ok. So, this is ok. So, therefore, the conclusion is that if R is.

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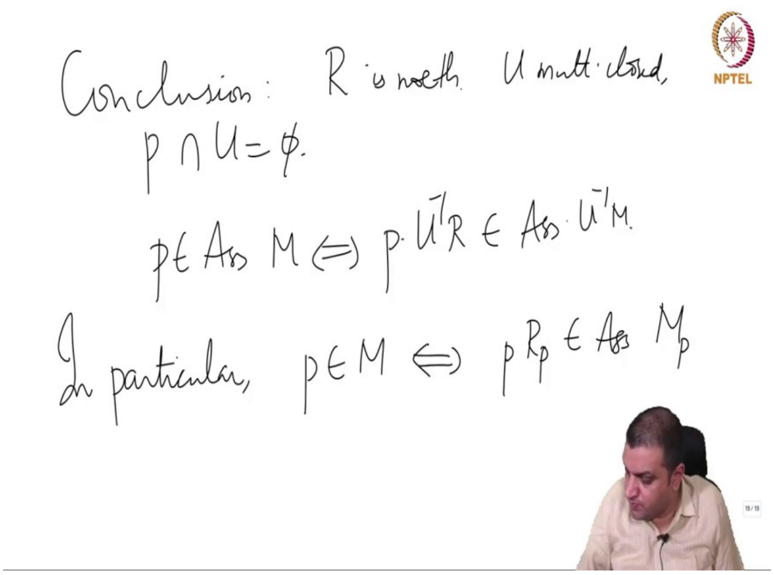

So, conclusion that if R is noetherian U multiplicatively closed $p \cap U = \phi$ then p is associated to M if and only if p extended to the localization is associated to localized M. So, this is associated to M and here we talk about R modules and here we are talking about $U^{-1}R$ modules phenomenally this one. In particular, $p \in Ass\, m$ if and only if $p R_p \in Ass\, M_p$. So, this is what we would want to use.

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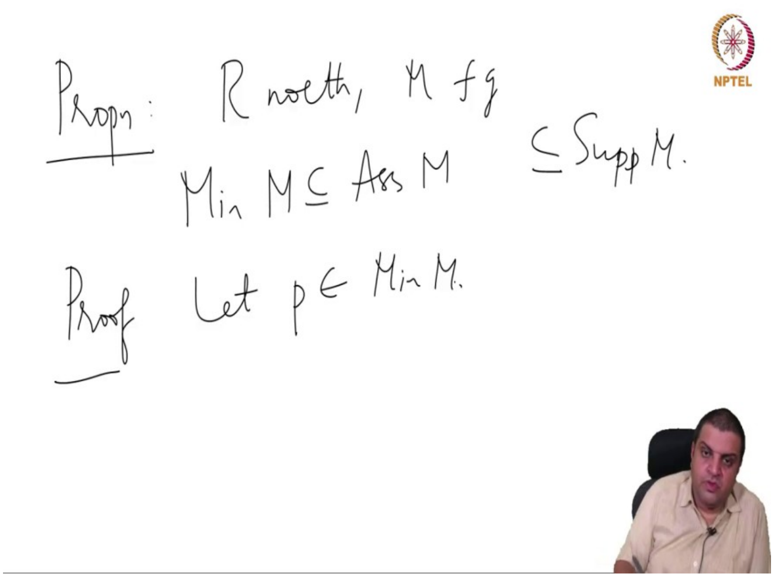

So, now let us prove the following proposition which we had used in the previous proposition. So, R noetherian M finitely generated, then $Min\, M \subseteq Ass\, M$ and all of these are actually subsets of $Supp\, M$. So, why? So, let us prove these things. So, minimal inside support is by definiton these are the minimal elements of that. What we want to show is that minimal primes are associated to the minimal to M.

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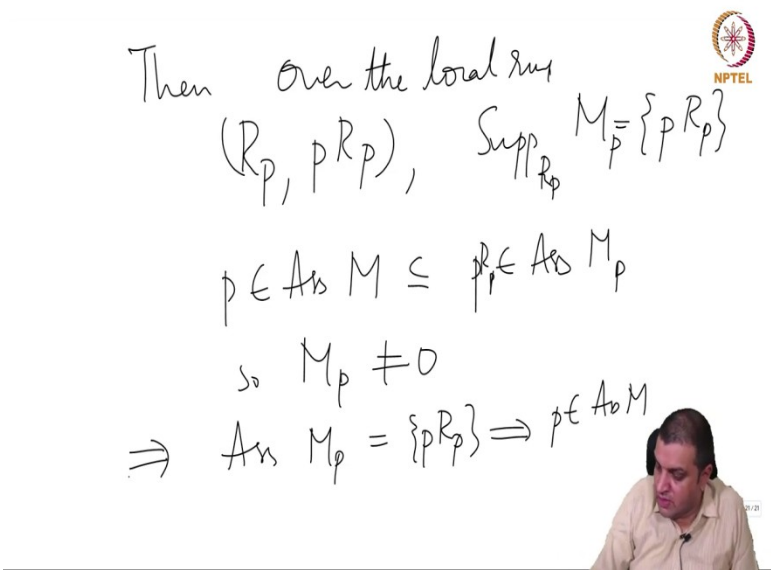

So, let $p \in Min\, M$ then over the local ring R_p its maximal ideal pR_p. Now $Supp\, M_p = \{pR_p\}$ because after we localize the only primes that would remain are primes that are inside p, but M is not supported at any prime which is inside p other than p itself. So, this is what it is.

So, $p \in Ass\, M$ implies that $pR_p \in Ass\, M_p$. So $M_p \neq 0$. So, if support is all of this and the associated prime was nonzero is non empty then it can only be one value which is this.

Which means that associated primes of M_p as an R_p module is pR_p this is where minimality is used of this one to conclude this and then use this fact to do this and this now implies that $p \in Ass\, M$ that was what the one can use localization both ways because its noetherian.

So, now just. So, this proves that minimal primes are associated and we just observed in the proof of the previous step itself, but let me just clarify this.

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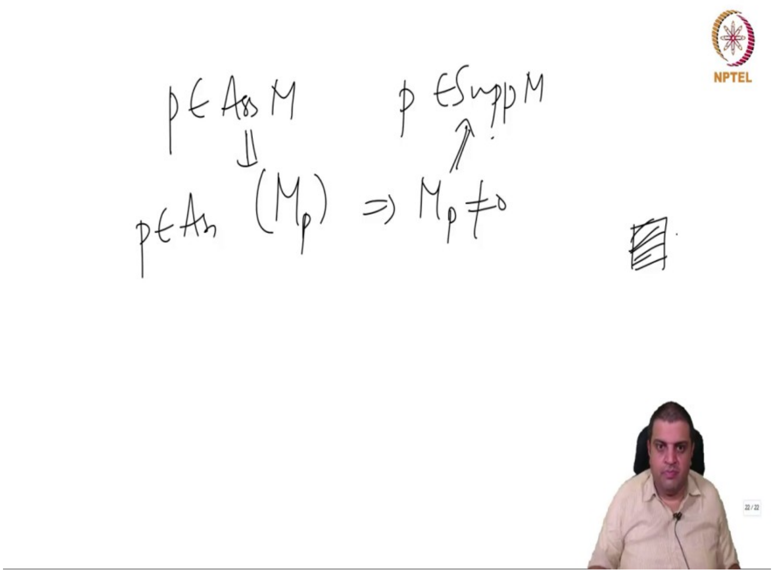

If p is associated to M implies that p is in the support of M in other words p does not kill every element inside M.

So, that is because M_p is nonzero. Then we have this. So, this is the end of this lecture and in the next lecture onwards we will continue with exploring further properties of these primarily decomposition.