Computational Commutative Algebra
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Lecture – 22
Primary Decomposition

This is lecture 22, in this lecture we continue our discussion about associated primes and I mean eventually or this is going to be used to clarify the idea of an irreducible decomposition and in fact, to refine it and thereafter we will forget about irreducible decompositions and we will know only what we will learn.

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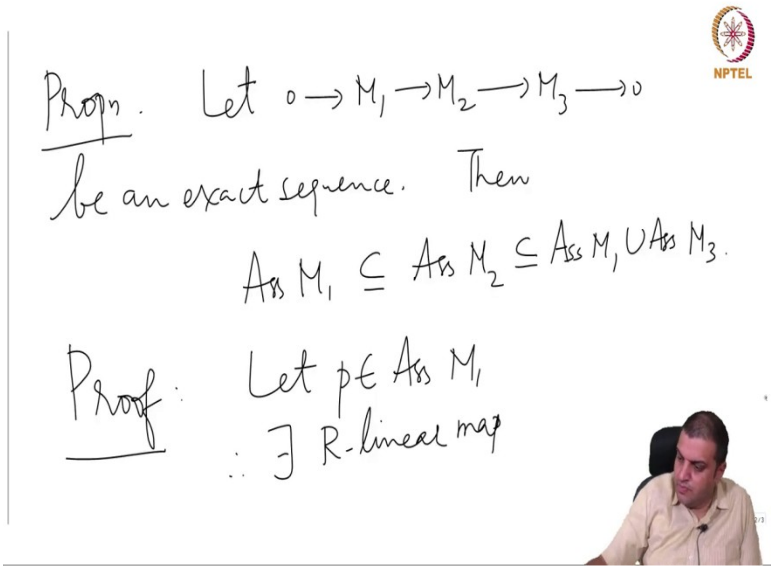

So, as a proposition about associated primes, proposition. Let $0 \to M_1 \to M_2 \to M_3 \to 0$ be an exact sequence. Then $Ass\, M_1 \subseteq Ass\, M_2 \subseteq Ass\, M_1 \cup Ass\, M_3$.

So, the associated prime of the left one of any module is a subset of the associated primes of a larger module and the associated primes of this middle module is a subset of the union of the associated primes of the sub module and the corresponding quotient; so this is what the proposition says.

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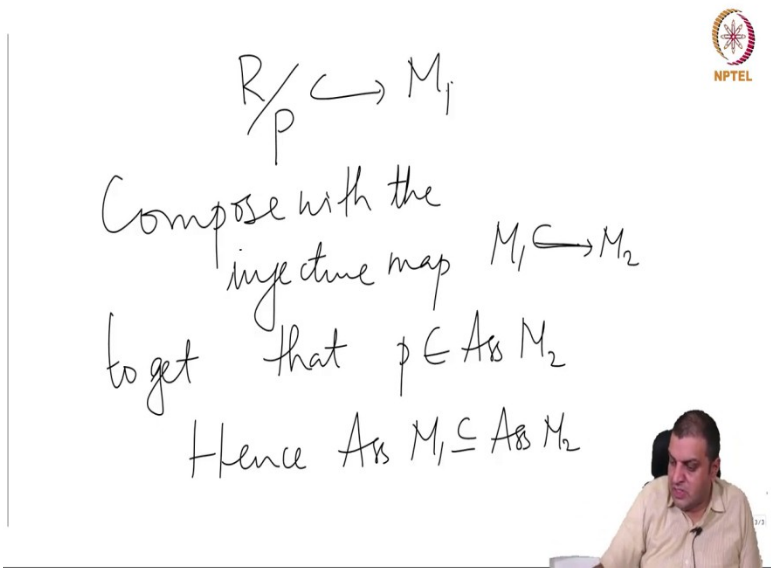

Proof. Let $P \in Ass\, M_1$, then there exist an R linear map $\frac{R}{P} \to M_1$. This is we mentioned earlier in the previous lecture, although we said associated prime means an annihilator of an prime ideal that is annihilator of an element; very often we just think of it in terms of this such a injective map. Compose with the injective map $M_1 \to M_2$ to see that; so all both these are R linear to get that p is associated to M_2.

Hence $Ass\, M_1 \subseteq Ass\, M_2$; so, this is the first inclusion here. For the other, let us look at the other one we were to work a little bit with element wise.

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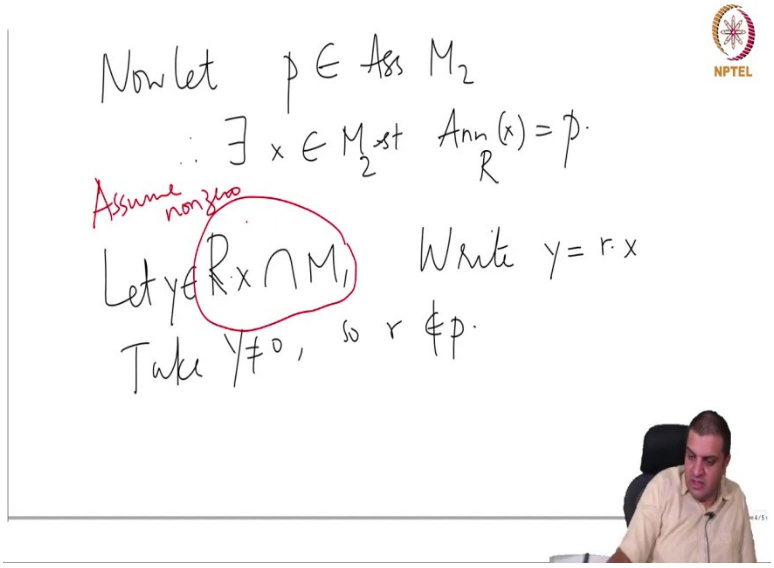

Let $P \in Ass\, M_2$. So, therefore, there exist some $x \in M_2$ such that its annihilator is P. Now, we consider the sub module Rx, so $Rx \cap M_1$ is a sub module; Rx is a sub module of M_2, it is called a cyclic sub module because it has just one generator like the cyclic group; so it is called a cyclic sub module and that intersect M_1. So, this is a sub module of M_1.

So, let $y \in Rx \cap M_1$, write $y = rx$ and r has to be outside p because p kills x. Take y non zero, so $r \notin P$. So, now we would like to understand what the annihilator of this y is.

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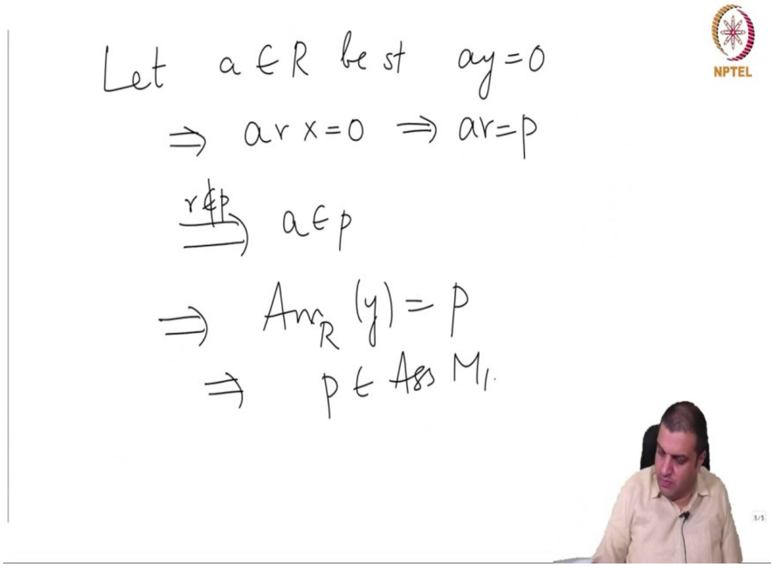

Let $a \in R$ be such that $ay = 0$; so we are looking for the annihilator of y. So, this now implies that $arx = 0$, this is happening inside M, this implies that $ar \in P$ because that is the annihilator of x, but $r \notin P$; therefore, $a \in P$.

So, sorry I should have been not careful; assume this is non zero. It is not necessary that two sub modules of M_2, namely Rx and M_1 need intersect at anything, the intersection need to contain a nonzero element. So, first assume non zero; take a non zero element then we go this calculation and then we are done; we conclude that $a \in P$.

So, in other words; so what is this calculation say? This says that $Ann_R(y) = P$ and that just implies now that $P \in Ass\, M_1$. So, let us just go over this proof; what does this half the proof say? It says that if p is associated to $M_2$and x is an element whose annihilator is P. If this intersection is nonzero, then P is also associated to M_1 that is what this argument says.

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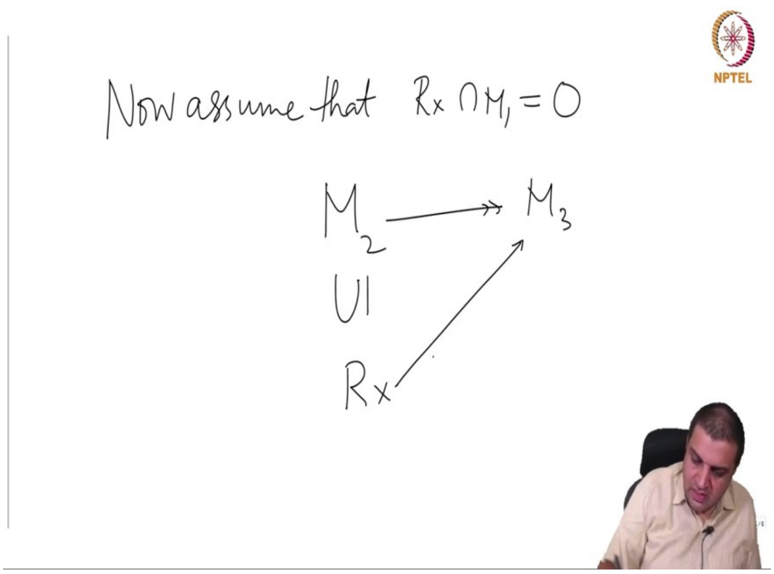

So, now assume that $Rx \cap M_1 = 0$. So, we have a surjective map from $M_2 \to M_3$ whose kernel is M_1; inside here is an sub module Rx. Now, this composite map gives a map here; not necessarily surjective, but a map. What is this map?

I mean; what is the nature of this map? Well, this is R linear, this is R linear; so we get an R linear map; from $Rx \to M_3$ and this is a composite.

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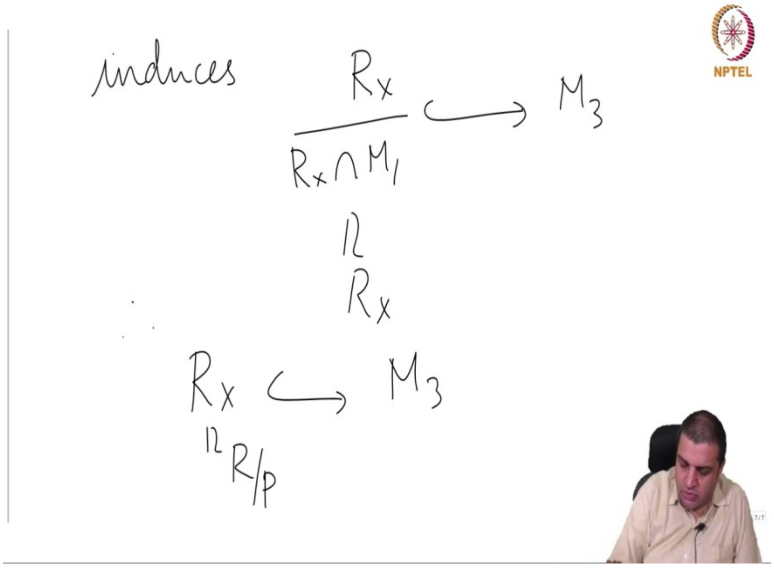

This induces a map which is Rx. So, remember kernel of this map is M_1. So, $\dfrac{Rx}{Rx \cap M_1}$ injects into M_3; that is we get this map, but this is 0. So, this is isomorphic to Rx itself that is because we are going modulo 0 sub module.

So, in other words Rx is a sub module of M_3. So, let me just quickly go over what we just said; let us assume this. So this the natural surjective map from $M_2 \to M_3$; that induces a map by first composing with this inclusion map from Rx.

So, we get a map like this, not necessarily surjective; we would like to prove that it is actually injective, what is the kernel of this map; of this diagonal map here it is precisely this $Rx \cap M_1$ but that is 0. So, therefore, the diagonal map is injective.

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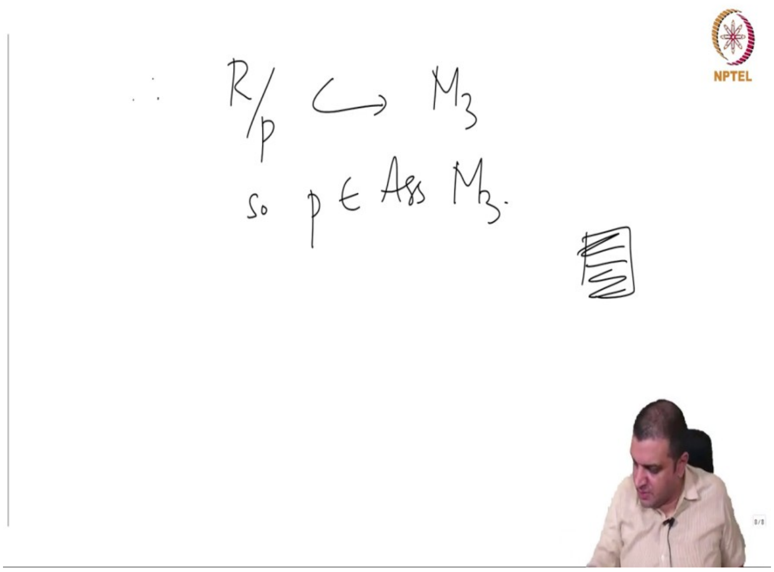

But this is isomorphic to $\frac{R}{P}$. So, therefore, $\frac{R}{P}$ injects into M_3. So, to prove that this is isomorphic to $\frac{R}{P}$; it just follows from the fact that, if you take a map the kernel is P.

And therefore, $\frac{R}{P}$ injects into M; with 1 going to x and if we have a injective map, then it is an isomorphism onto its image, not to the isomorphism to the larger module, but onto its image, but image is precisely this Rx; so, this is an isomorphism. So, therefore, Rx is the $\frac{R}{P}$ injects into M_3. So $P \in Ass\, M_3$.

This is what we wanted to show, P is either associated to M_1; that is if this set is nonzero and if this set is 0, then it is associated to M_3. So, these are some elementary property for associated primes. We will study more. We will restrict ourselves to noetherian rings, but we will proceed in as we go along.

So, next we need to generalize the idea of irreducible ideal. So, remember we started with irreducible subsets of maximal spectrum or $Z(I)$ then we said let us do it for ideals because if we wanted to do generality with all; I mean in this algebraic set up, one needs to do it for ideals, but then it is actually much easier to do for modules and we will do it so in that context. So, first we need to extend the notion of irreducibility to modules; so definition.

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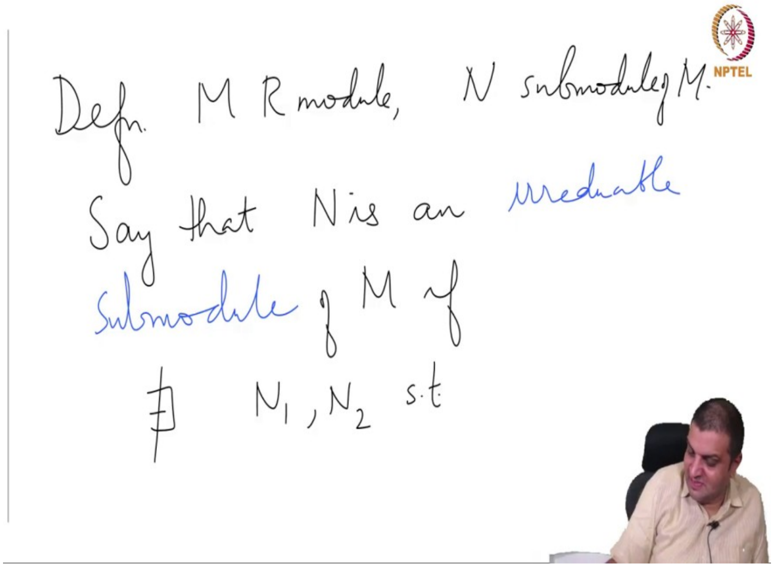

So, again R is a commutative ring, say M an R module, N sub module of M. Say that N is an irreducible sub module of M, if $\nexists N_1, N_2$ such that $N \neq N_1$ and $N \neq N_2$ and $N = N_1 \cap N_2$.

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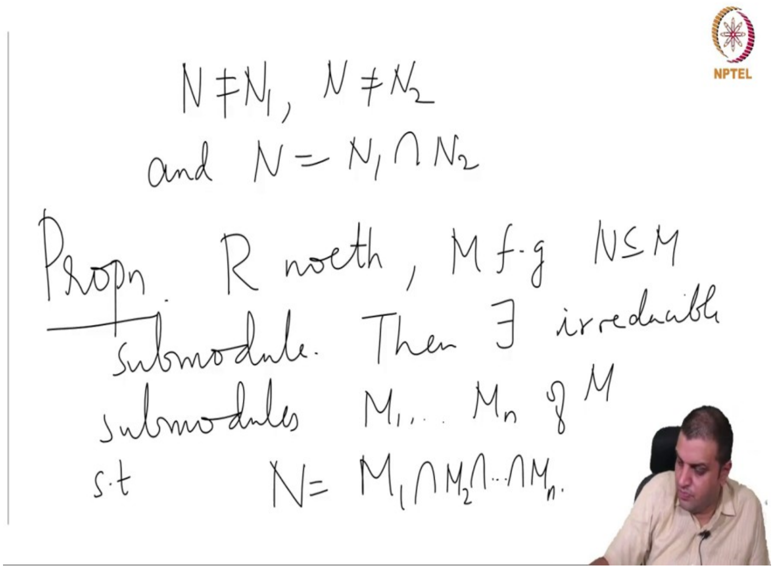

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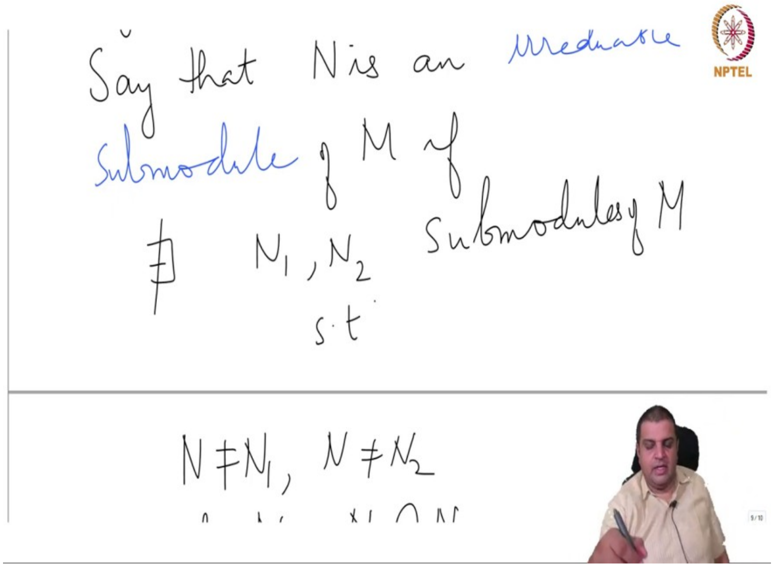

The point is all of this calculation, union, intersection is done inside some large module which is M. The same way we were working over ideals where there was an ambient ring in which all the calculations were being done.

So, this is we say it is an irreducible; so propositions. Let us assume R noetherian, M finitely generated, $N \subseteq M$ sub module. Then, there exist irreducible sub modules $M_1, M_2, \ldots, M_n$ such that $N = M_1 \cap M_2 \cap \ldots \cap M_n$.

So, this is a proposition; if you have a noetherian ring; a finitely generated module and a sub module that the sub module has an decomposition as an intersection of irreducible sub modules. This is a finitely many; so that is the thing and proof I will skip because it is just the repeating the argument for what we did for ideals, there is nothing new.

So, just if it is not true; then there will be an N_1 which is strictly bigger than N and then and so on, it will go on building a chain, but such an infinite ascending chain is not possible because we are in certain noetherian module. Therefore, it must admit such a decomposition.

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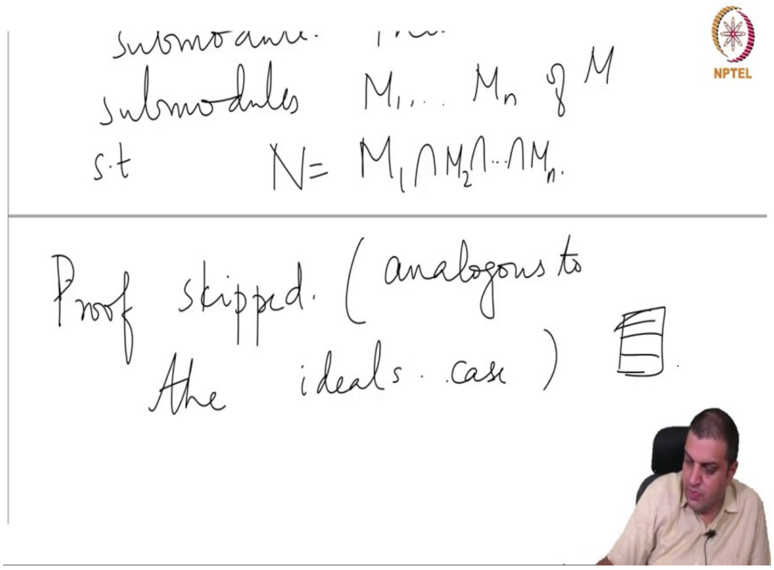

So, proof is skipped. So this is analogous to the ideals case, there is nothing new here.

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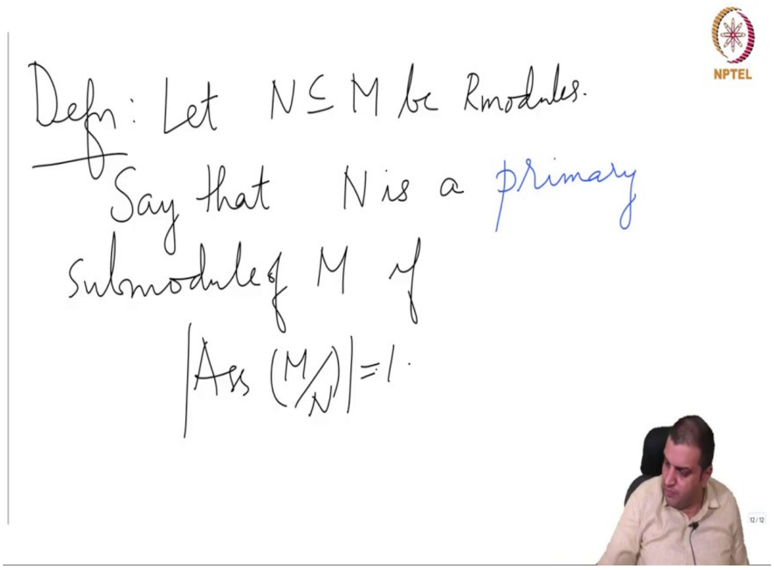

So, now we are going to begin putting extra structure or understand more what this irreducible decomposition is able to give us. So, let $N \subseteq M$ be R modules, meaning sub module not just a subset. Say that N is a primary sub module of M if the set of associated primes of $\frac{M}{N}$ is a singleton set. It is a singleton set that is this is exactly one associated prime of $\frac{M}{N}$.

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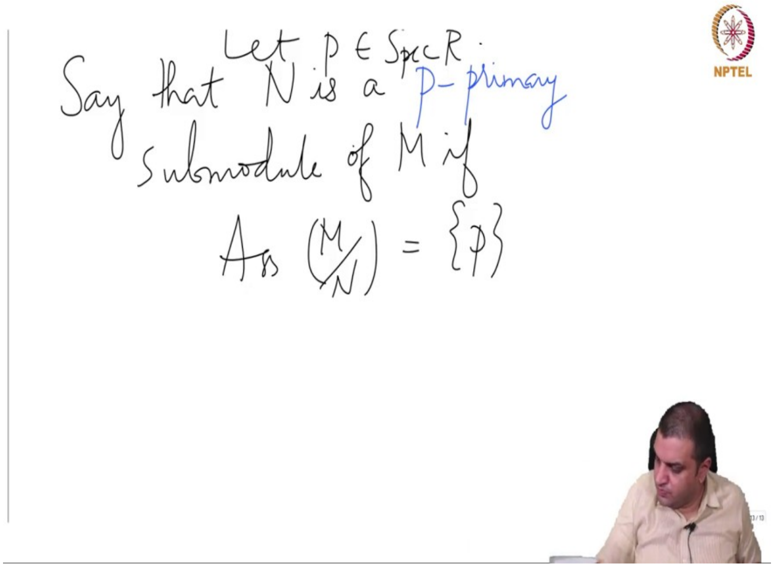

Let P be a prime ideal of R; say that N is a P primary sub module of M; if $Ass\left(\frac{M}{N}\right) = \{P\}$, let us just be precise here if it is a singleton P. So, this is just the previous definition, with the extra information; what is that associated prime. So, now they are the new thing about irreducible decomposition.

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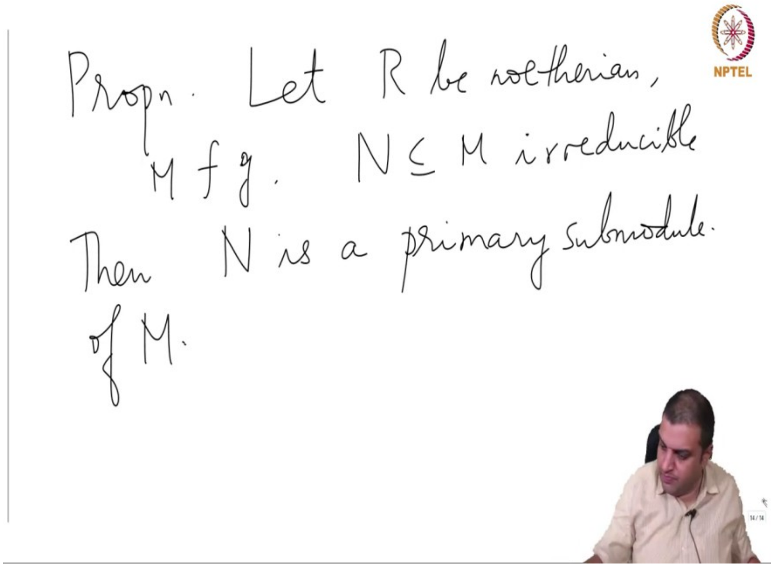

Proposition, let R be noetherian; $N \subseteq M$ irreducible, M finitely generated; then N is a primary sub module of M.

So, irreducibility as such was some minimal structure; it was just some not equal to intersection of two ideals; each of which is bigger. There was not much extra structure to that, but now we are saying that you know that is not true; in the case of where most people are interested in which whether ring and modules are all noetherian. Irreducibility does say that the module is primary and we will try to understand when, what does this exactly mean.

So, what is primary mean what extra structure that have; just one more point goes back to the definition. This is a statement not about M and N independently, it is a statement about the quotient. So, just one remark N is a primary sub module of M if and only if 0 is a primary sub module of $\frac{M}{N}$.

And similarly N is a P primary sub module of M if and only if 0 is a primary sub module of $\frac{M}{N}$. The statement is always about $\frac{M}{N}$ and not about M and N separately and we will see after this proposition, we will see what it means to say that for a module 0 is P primary.

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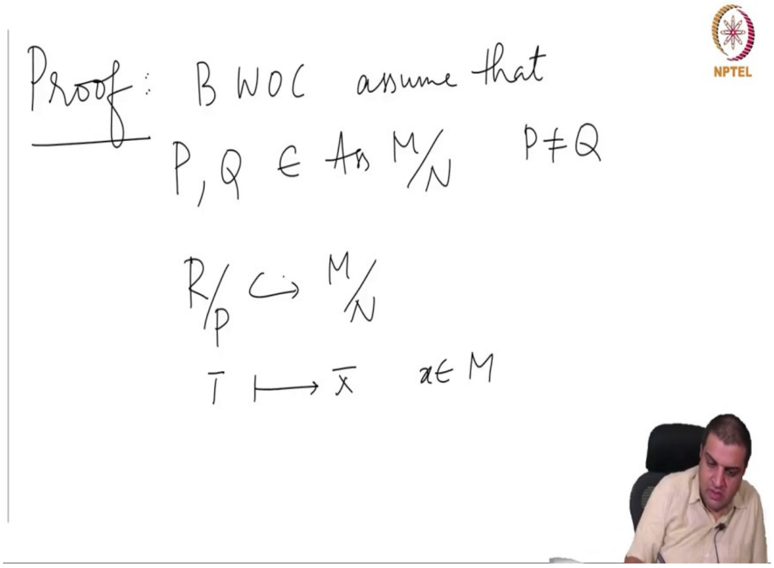

So, let us prove this. So, by way of contradiction; assume that there is $P, Q \in Ass(\frac{M}{N})$ and $P \neq Q$; in any case this set is non empty. So, to say this is not primary is to say this is not primary; N is not primary inside M is to say that it has at least two elements, call them P and Q.

So, then $\frac{R}{P}$ injects into $\frac{M}{N}$ and let say that the map is $\bar{1}$ goes to some $\bar{x}$ where $x \in M$ but not in N only then M.

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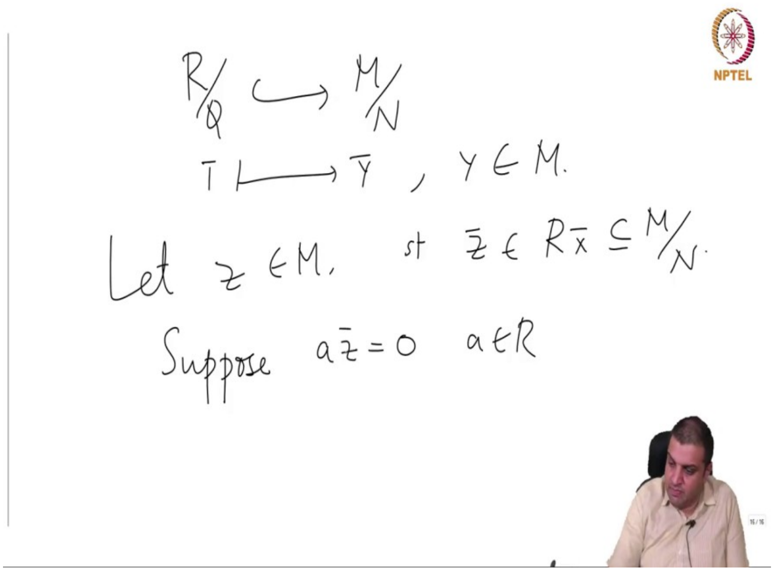

And similarly $\frac{R}{Q}$ injects into $\frac{M}{N}$ and in which $\bar{1}$ goes to some $\bar{y}$ and $y \in M$. So, we have two such elements x and y with these properties.

So, let z be also an element of M such that $\bar{z} \in R\bar{x} \subseteq \frac{M}{n}$. So we would now argue that every element in the annihilator of $\bar{z}$ is P. So, suppose $a\bar{z}=0$; so we are looking for annihilator. So, what does that say?

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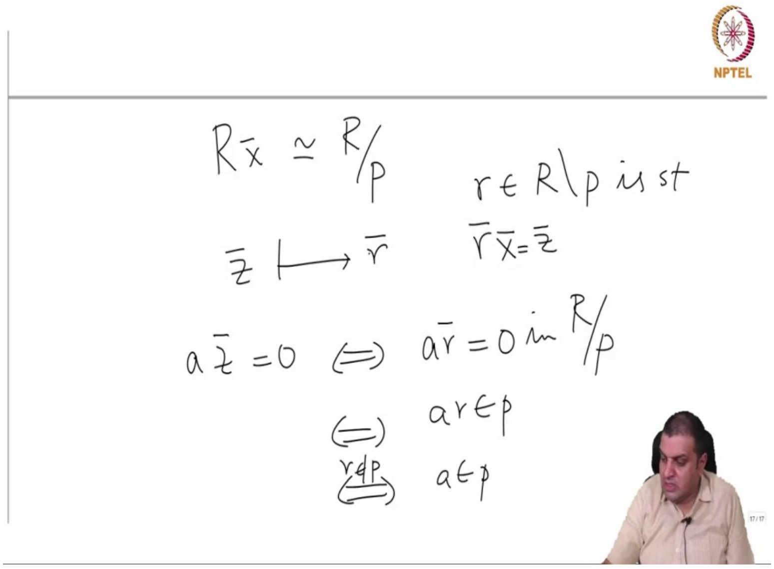

So, notice the following $R\bar{x}$ is isomorphic to $\frac{R}{P}$ because P is annihilator of this and if $\bar{z}$ inside here, under this map to some $\bar{r}$ where $r \in R$, but not in P is such that $\bar{r}\bar{x} = \bar{z}$, such a thing exist because that is what we chose.

So we will get a map like this, so then $a\bar{z}=0$ if and only if; so they we doing this calculation on this side, but the same thing as doing it over this side, $a\bar{r}=0 \in \frac{R}{P}$. And that is the same thing as saying $ar \in P$ and which is because $r \notin P$, this is the same thing as saying $a \in P$.

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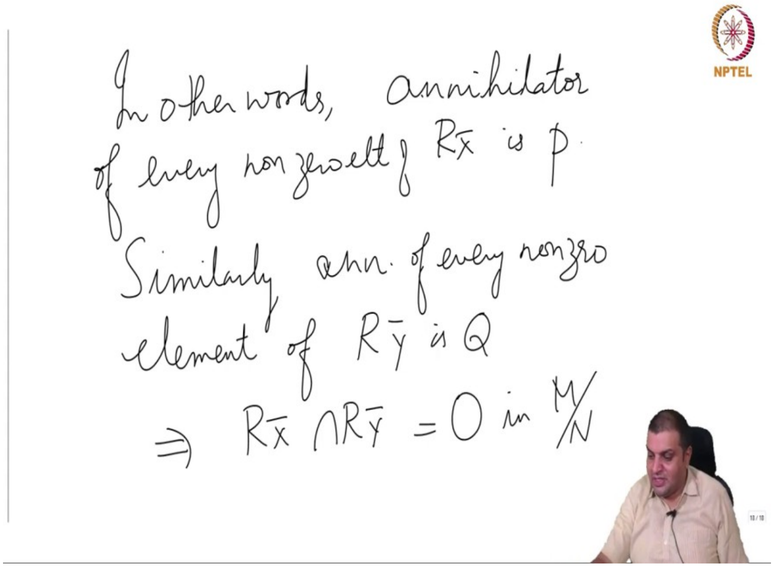

So, in other words; annihilator of every nonzero element of $R\bar{x}$ is P. Similarly, the annihilator of every nonzero element of $R\bar{y}$ is Q. And these are distinct primes, so this cannot intersect.

So, this now implies that $R\bar{x} \cap R\bar{y} = 0$ in $\frac{M}{N}$ because it cannot contain a nonzero element in which case that I mean there will be a contradiction.

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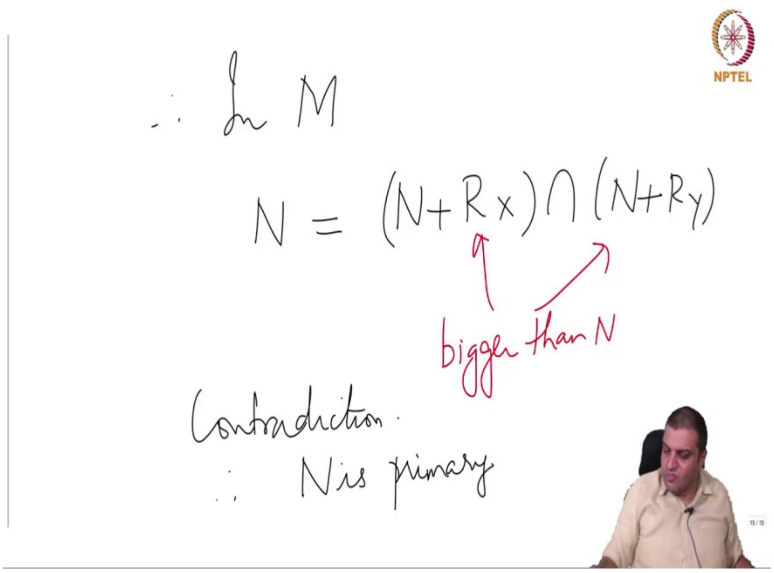

What does that say? It says that in M, $N = (N + Rx) \cap (N + Ry)$; both of these are bigger than N because we chose $\bar{x}, \bar{y}$ to be nonzero elements and that is a contradiction.

And the contradiction came by assuming that associated primes of $\frac{M}{N}$ contains at least two elements. So, it is non empty and we assume it has at least two elements, then we have a contradiction; so it contains exactly one; so therefore, N is primary.

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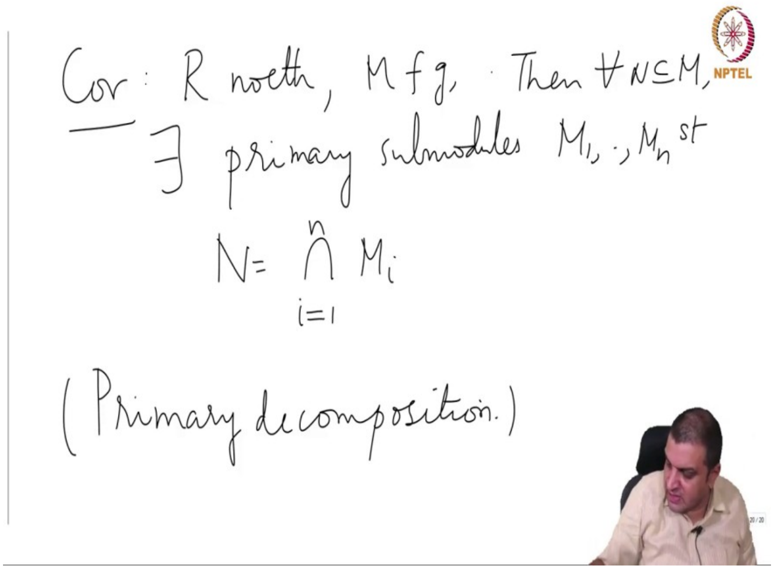

Let us just summarize these results. Corollary; R noetherian, M finitely generated and then for all $N \subseteq M$, there exist primary sub modules, finitely many; $M_1, \ldots, M_n$ such that $N = \cap M_i$.

So this is called primary decomposition. We started from the idea of irreducible decomposition proving; using noetherian hypothesis that every ideal or every sub module has a irreducible decomposition, prove that they are primary. So, we still have not understood; explained what primary means, but that we will come to and so one more proposition, just to refine this a little bit of proposition.

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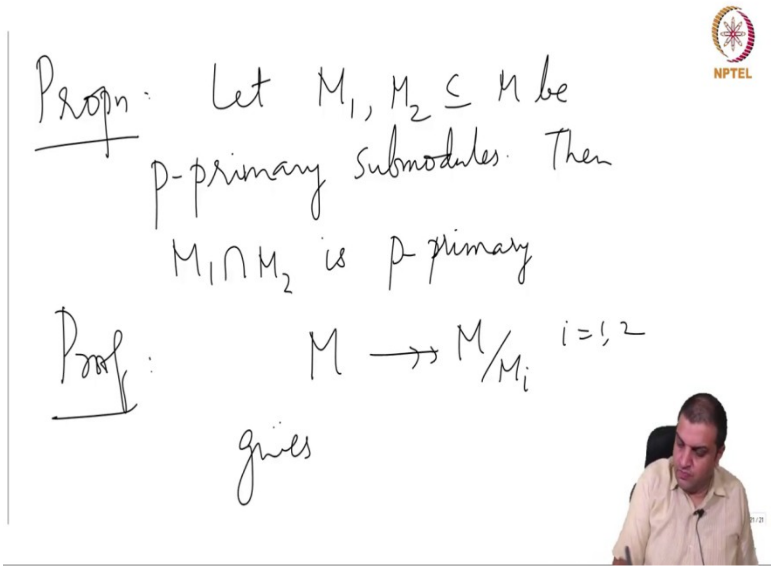

Let $M_1, M_2 \subseteq M$ be P primary submodules, then $M_1 \cap M_2$ is P primary. So, what does this mean? So this is one of the usual statements or an exact see or a map that we see we use quite often in these when we discuss these things.

So, now we have a map from $M \to M_i$; $i=1,2$.

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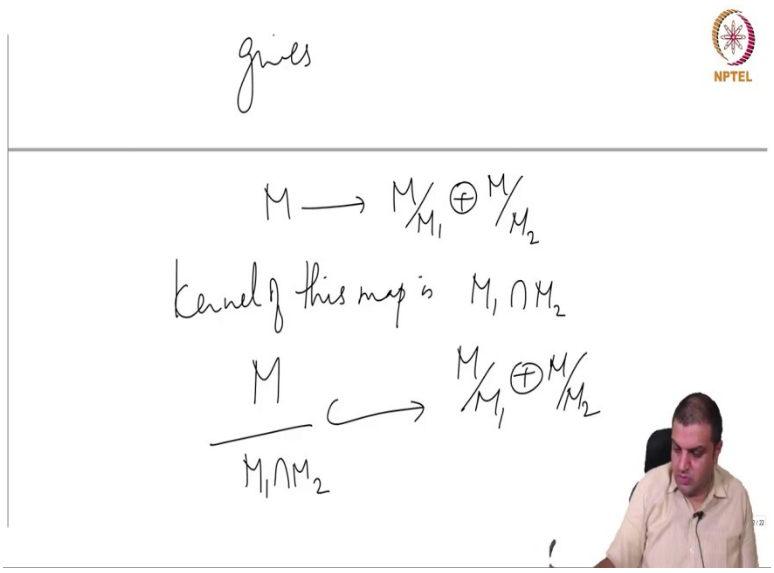

So, this gives a map $M \to \frac{M}{M_1} \oplus \frac{M}{M_2}$, in this case it is easy to visualize; whatever x here, goes to x mod M_1, x mod M_2. But we did not discuss, direct sum have certain category theory properties and then it is from there it is automatic that there is such a map. But what is the kernel? Kernel of this map is $M_1 \cap M_2$.

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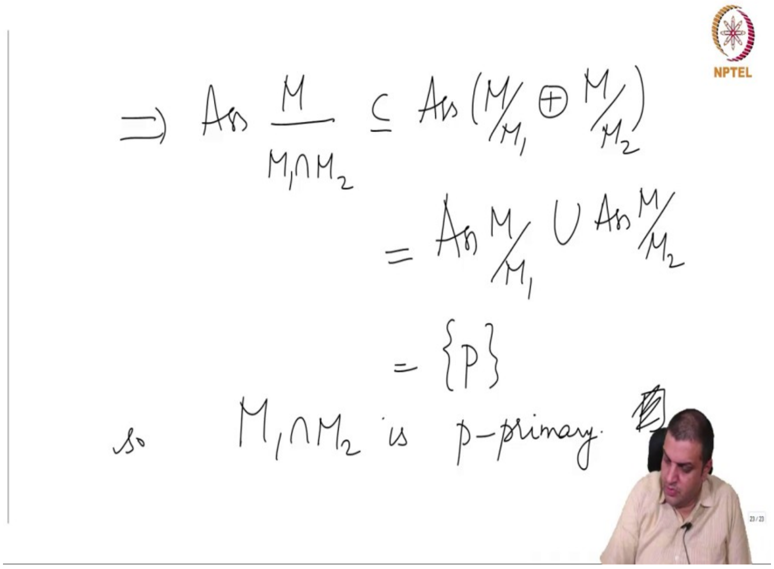

So, therefore we have $\frac{M}{M_1 \cap M_2}$ sitting inside this direct sum. Which now means that $Ass\left(\frac{M}{M_1 \cap M_2}\right) \subseteq Ass\left(\frac{M}{M_1} \oplus \frac{M}{M_2}\right)$. And for a direct sum associated primes one can just take unions.

So, this is related to the proposition that we saw immediately I mean after above the short exact sequence. But a direct sum like this can be put into a short exact sequence which splits and so it can be written in this direction and as well as in the opposite direction and from which we can conclude the statement. So, this is true, but what is this? Both these are primary to the same P; so this is P.

So, $M_1 \cap M_2$ is P primary, this is what we wanted to do.

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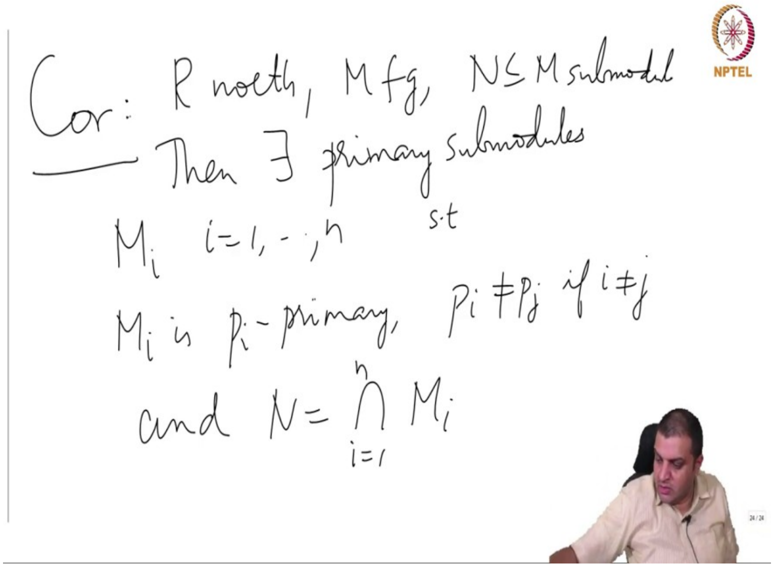

So, now we can strengthen the earlier corollary a little bit; R noetherian, M finitely generated and $N \subseteq M$ sub module. Then, there exist primary sub modules M_i, $i=1,\ldots,n$ such that M_i is P_i primary, P_i is different from P_j, if i is different from j and $N = \cap M_i$.

In other words, in the earlier corollary where we saw this intersection, club all the ones with the same P together and call that thing M_i. So, now we can just get this to be distinct. So, we will stop here and we will continue discussing this further, in the next lecture.