Computational Commutative Algebra
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Lecture – 20
Spectrum – Part 2

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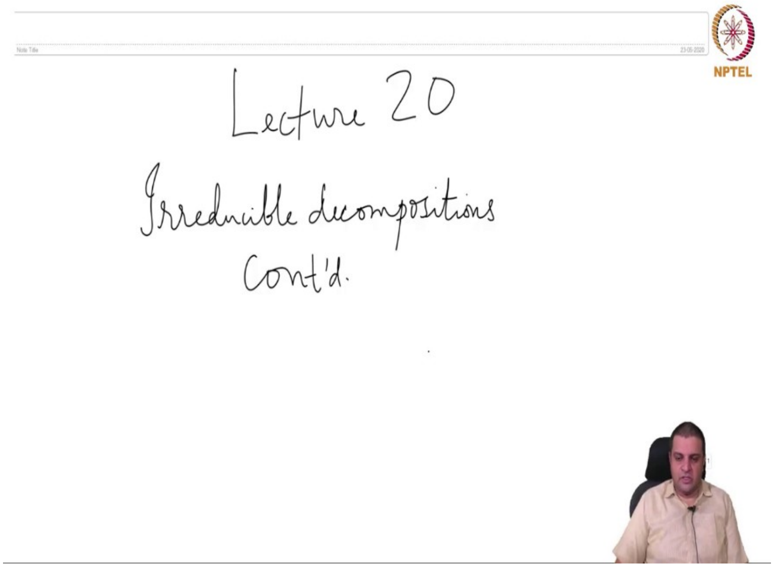

This is lecture 20. And, in the next four-five lectures, we will look at this idea of irreducible decompositions, primary decompositions, and associated primes. So, this is going to be more abstract than what we have done so far. But, after we discuss the basics behind these things we will come back to the computational aspects, but you will have to wait till we understand this; the reasoning the thinking behind these topics.

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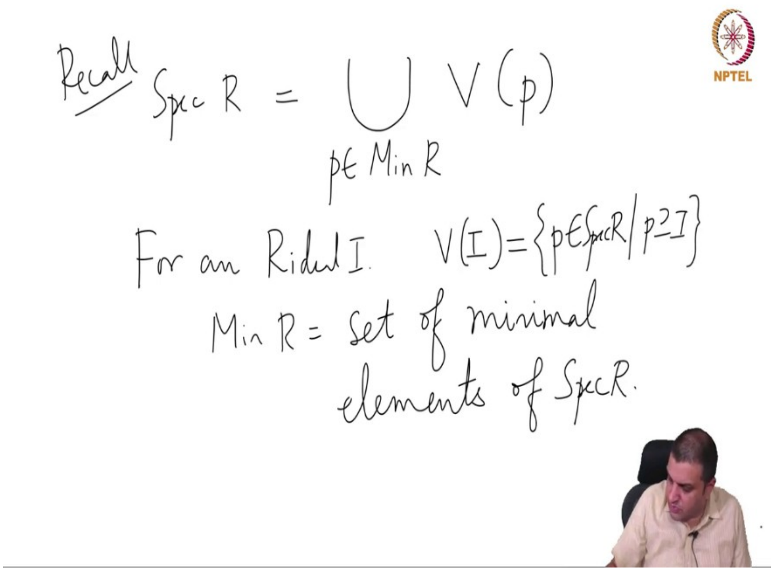

So, recall that in the last lecture we saw the following that irreducible that spec(R). So, recall $Spec\, R = . \bigcup_{p \in Min\, R} V(p)$. So, what are these things, for an R-ideal I , $V(I) = \{p \in Spec(R) : p \supseteq I\}$. And, this disagrees with our earlier notation of points at which elements of f vanish, but we will see today that these are not very different.

So, and algebraically this is the object that we would want to continue using. And, then we saw that so this is V(I) and Min(R) is the set of minimal elements of spec R. So, we would want to understand these things.

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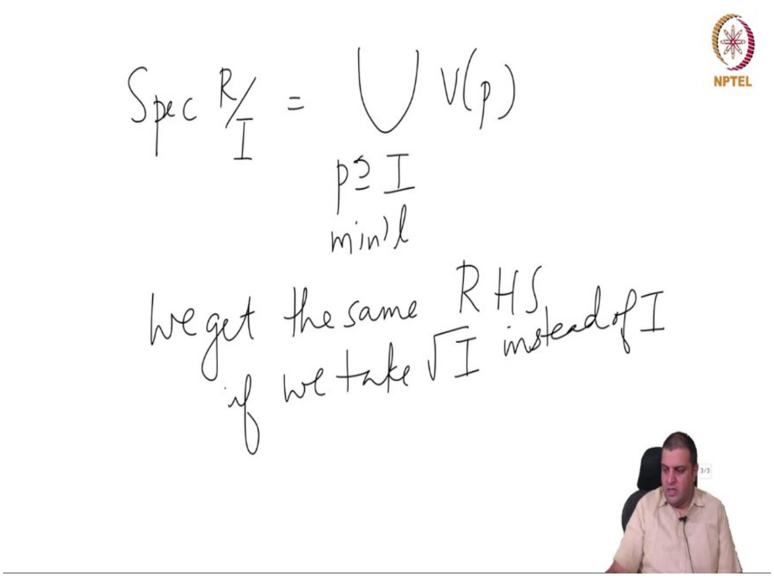

And, so in particular, we know that if we take $spec\left(\frac{R}{I}\right) =. \bigcup_{I \subseteq p, minimal} V(p)$

, where p contains I and is minimal with respect to that property. And, this is independent of whether you take I or the $\sqrt{I}$.

So, we get the same thing we get the same right hand side, if we take $\sqrt{I}$ instead of I. So, this is the, this is an important, this is in observation. So, there must be a relation between; so there must be a relation between the radical of I and the primes containing I.

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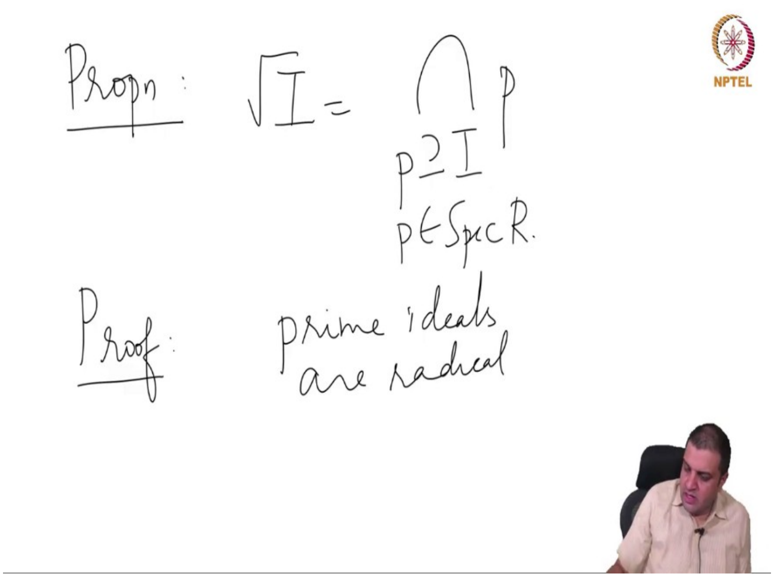

And the relation is the following proposition. $\sqrt{I} = . \bigcap_{I \subseteq p,\, p \in Spec(R)} V(p)$ is the intersection of primes p containing I where p ∈ spec(R). So, we already observed that radical of I would be inside. So, let us go over there that part. So, I should say that one could have proved these this result little earlier also.

But I felt that, it is the proof is really easy to state once we understand localization and which is the reason why I postponed it till this stage. One did not need to know localization to prove the statement. Notice the following things, primes are radical ideals, prime ideals are radical.

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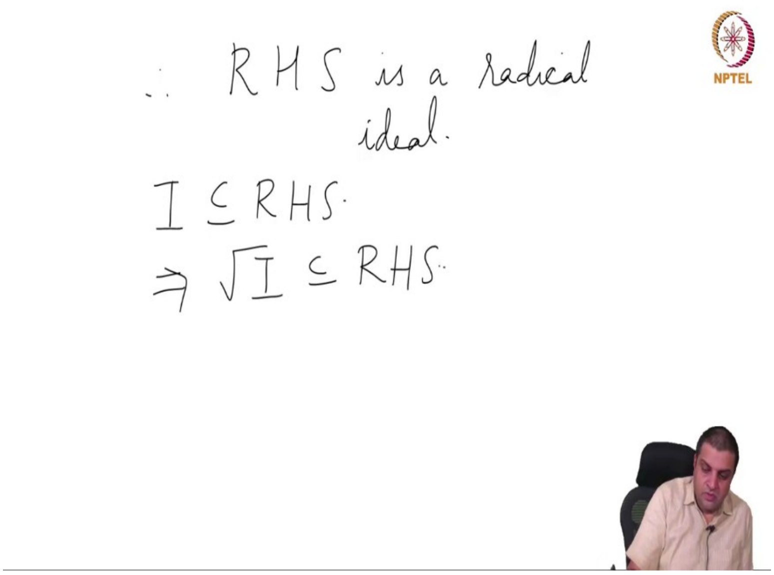

And, intersection of prime ideals is radical, therefore the right hand side is radical ideal; is a radical ideal. So, I is in the right hand side , which means that, the radical of I is in the right hand side, So, this is the so right hand side is the intersection. So, this is the one containment. So, again this we already proved; I am just recalling the statement this can side of the argument.

But, it is the other one that is very easy to state once we understand what localization is. So, to prove the other direction let us just. So, we need to so we know that, so we have established this. So, let us just kill I in R, and then we want to prove that the right hand side is 0 , or let us kill I.

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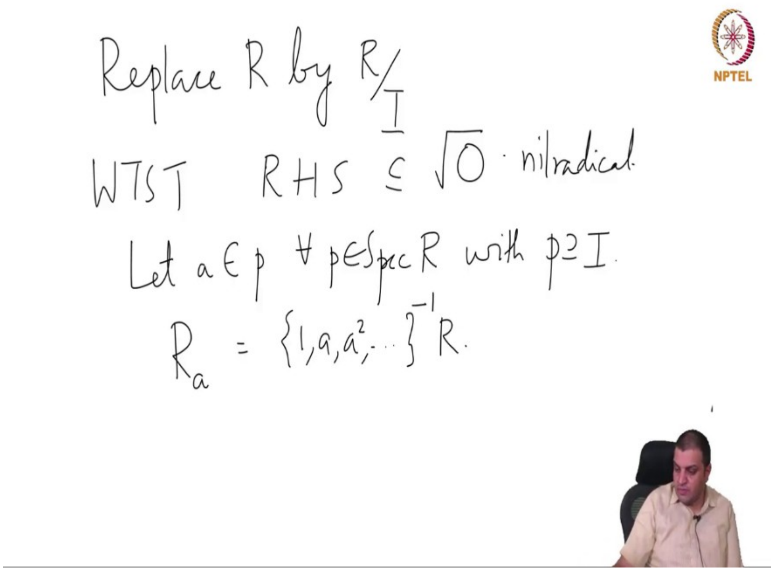

So, replace R by $\frac{R}{I}$. So, then we want to show that, we want to show that, the right hand side now replaced. So, right hand side is inside the nil radical. So, I replace the right hand side by the appropriate object, primes containing I are precisely the primes of $\frac{R}{I}$, and the left hand side becomes a $\sqrt{(0)}$, the nil radical.

So, let us take; so, let a belongs to p, for all p ∈ Spec(R) with $p \supseteq (0)$. Now, let us consider the ring R localized at a ,$R_a = \{1, a, a^2, \ldots\}^{-1} R$. So, this is a multiplicatively closed set and that we are inverting. Now, we know what the primes of a localization are.

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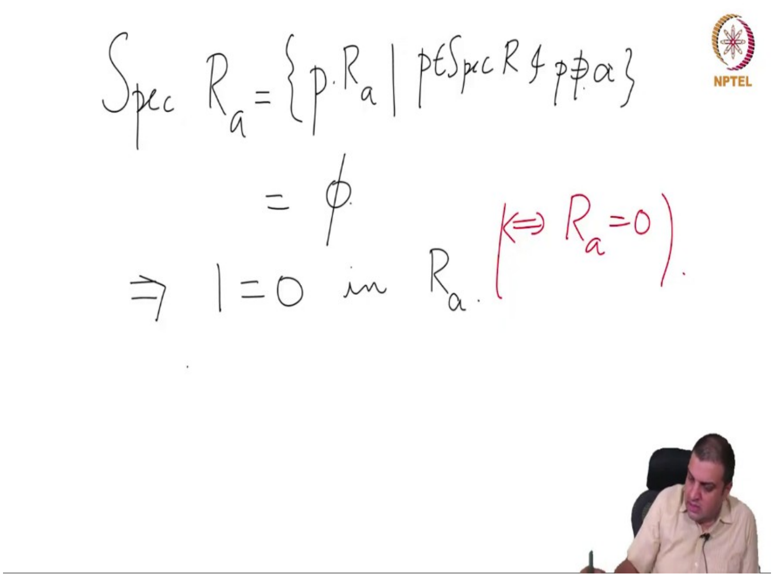

$Spec(R_a)=\{pR_a : p\in spec(R), a\notin p\}$. What it says is p does not intersect this set, but if p intersects this set, it would contain some power of a, and hence it will also contain a. So, it should not do that. So, that is the p does not contain a. So, this is the condition for this. But, what are the prime ideals that satisfy this condition, because a is in every prime.

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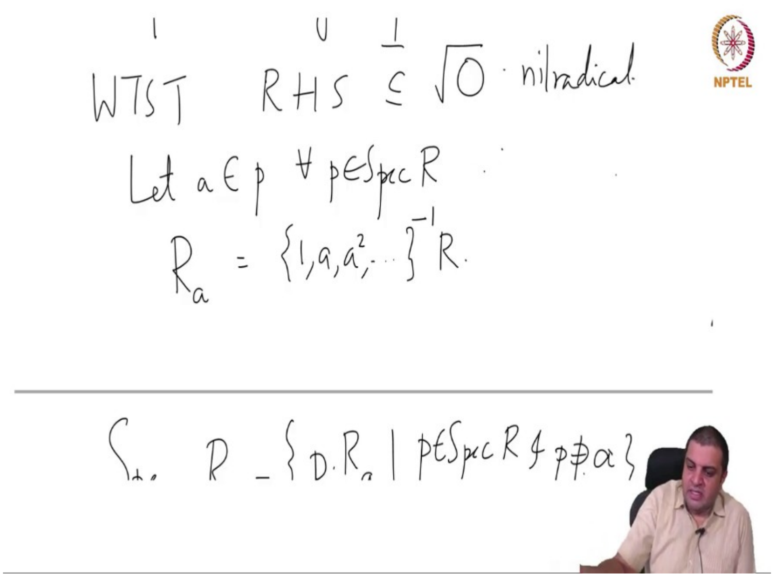

So, p contain 0 , so which now implies that this is the empty set. Now, what sort of rings; what sort of ring would have a property that spec is empty.

The only ring with that property is a ring in which 1 is equal to 0. So, the; this implies that 1 = 0 in R_a. We have we had not ruled out such rings. We had never said that 1 is the multiplicative identity, 0 is the additive identity, and we had never ruled out that this is these are this, whenever ruled out the case this could be the same. In fact, the ring is 0, if and only if this is so maybe let us had there is a remark here this is if and only if $R_a = 0$.

So, only ring with an empty spec is the zero ring, in which the multiplicative and additive identities are the same. You, one could any ring in which 1 is different from 0, then it would have a maximal ideal, and maximal ideal is prime. So, spec is non empty. So, now let us I mean that statement is proven using Zorn's lemma . So, therefore, 1 is equal, so but when can $1=0$in R_a. So, this means that, so let us go back to this statement.

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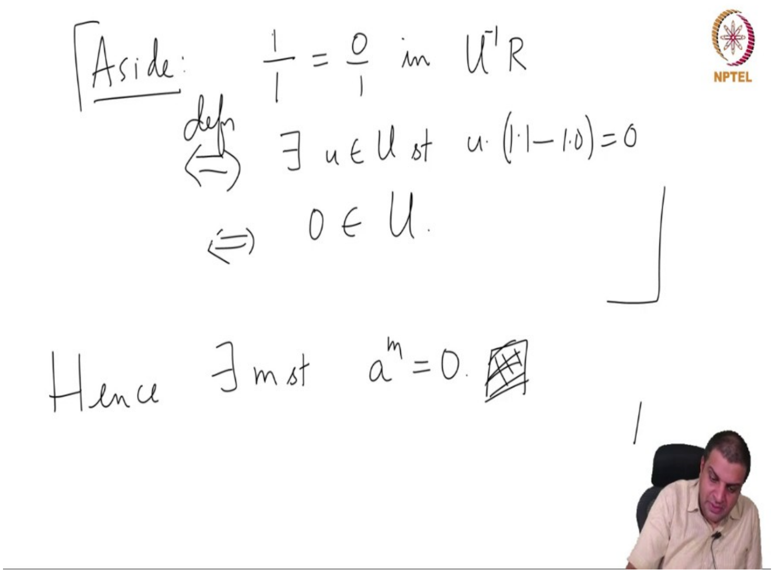

So, let us, so this is an aside just to think about what this means. So, 1/1 = 0/1 in U^{-1}R. So, in the equivalence relation (1 ,1) = (0 ,1) if and only if this is by definition. I mean this is this follows it directly from equivalent to the definition.

There exist some $u \in U$ such that, u (1.1 – 1.0) = 0 .And, this is the same thing as saying there exists a u such that u. 1 = 0. So, in other words, $0 \in U$. So, if you invert multiplicatively closed set the ring becomes 0 if and only if 0 is in the multiplicatively closed set.

So, now, let us go back here. So, this now means that, hence in our question, there exist some m such that $a^m = 0$. There are multiplicatively closed set is a elements of this form and 0 is there means this is the property, and this is exactly a is important. So, this is the proof of this proposition. So, one could have written this proof without really using localization, but it is much easier to visualize the proof once one understands not even full generalities of the localization, just such multiplicatively close sets, and this observation.

So, now so that is just an aside, which we could have done which I wanted to do, I mean this whole statement was it is not immediately related to irreducible decomposition, but I just want this to be people to know this observation. So, now we want to discuss, what is the relation between, so now we have back to the problem in the geometric version of the problem.

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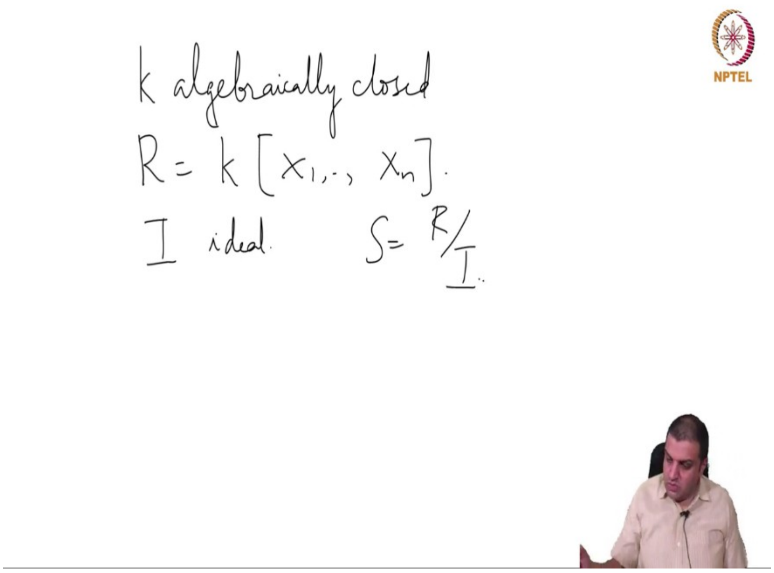

Which is that k is algebraically closed; $R=k\left[X_{1,},\ldots,X_n\right]$, and I is an ideal, and $S=\dfrac{R}{I}$.

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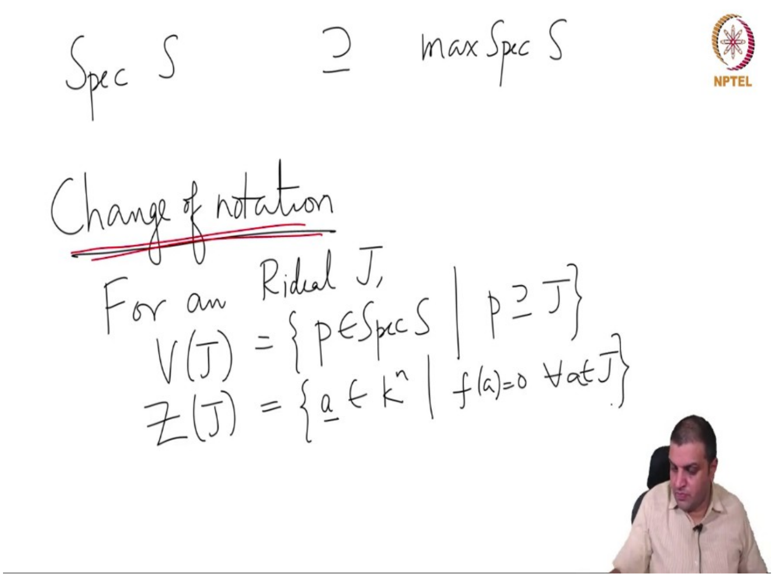

Now, we defined two things called $maxSpec(S) \subseteq spec(S)$. So, maxSpec(S) is the set of all primes; this is a set of maximal ideals.

And, we know what in this particular case what this one looks like; so I, want to change the notation from last time a little bit what from earlier lectures a little bit. So, this is change of notation I. So, please keep this in mind in this immediately immediate future lectures.

So, by for an R - ideal J , $V(J) = \{p \in Spec(S) : I \subseteq p\}$. And, by $Z(J) = \{a \in k^n : f(a) = 0 \; for\, all\, a \in J\}$. So, this is what we had called V(J) earlier when we were discussing Nullstellensatz, but I want to call it Z (J) and well one justification why it is like 0's of J. So, we will just call it Z.

So, this is what we, so I want to keep these two sets different. And we will see. So, I want to keep these two sets different at this stage, and we will keep them different, but we will see at the end of this discussion, that it is sort of immaterial I mean there are different sets, but irreducibility of one is same as irreducibility of the other etc. which is what I want to here to do now.

So, just again; this is the set of primes containing J all of them, and this is the set of points at which every element of f; every element f of J vanishes sorry this is for av f in J vanishes, . And therefore, we know what max spec; so, let us just let me just repeat this line here.

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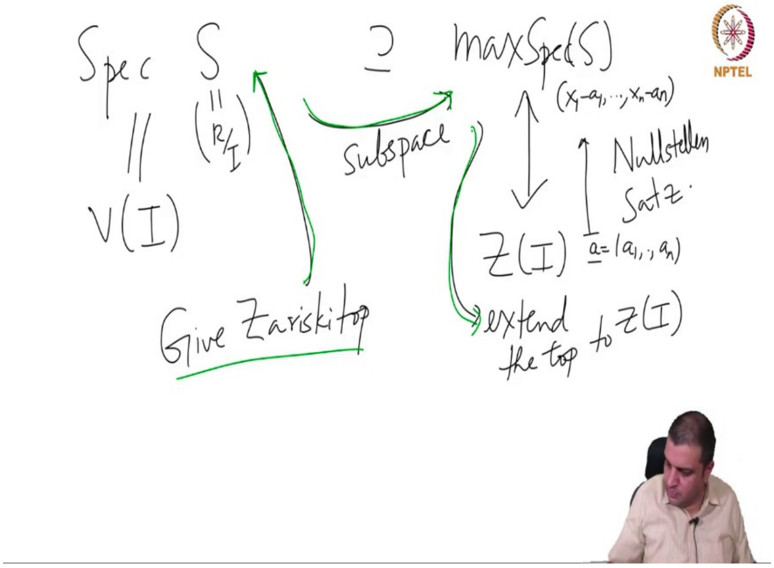

So, this is we know that $spec(S)=V(I)$, since $S=\frac{R}{I}$. So, you can think of this as V(I) inside spec(S). And, this one contains max spec(S) the maximal spectrum of S. And, what Nullstellensatz told us the the description of points in V(I), this is Z(I) that this is in one to one correspondence with Z(I).

And, what is the correspondence? A point $\underline{a}$ here goes to the ideal generated by $(x_1 - a_1, \ldots, x_n - a_n)$, so that is in one direction, that an n tuple, gives that maximal ideal. And, every maximal ideal looks like that, that was (Refer Time: 17:10) one of the versions of Nullstellensatz.

So, I just write Nullstellensatz, that $Z(I) \subseteq k^n$ is precisely in bijective correspondence with max spec . We had given Zariski topology to this, we can induce Zariski topology on the subset, and we can identify give a Zariski topology on Z(I) using this . So, give Zariski topology here, then induce this is subspace topology and using that is extended to here extend the topology here.

So, I hope that what we have doing is clear. We have given Zariski topology to spec(S) by saying V (J) where J is an ideal those have a closed sets, and this is a subset. So, then we can induce topology on that, the subspace topology, and then this is a sets they are the same. So, just use this bijection to induce a topology on this set. And, we can talk about irreducible subset here and irreducible subset there and so that is what we want to explore now.

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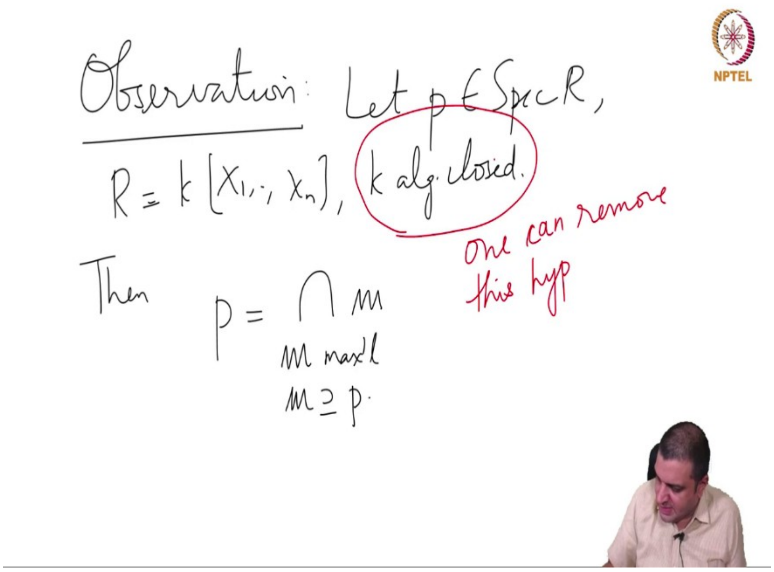

So, now so we want to observe one point here which is relevant to this question. Let p be a prime in R, where $R = k[X_1, \ldots, X_n]$ and k is algebraically closed. Then, $p = \cdot \bigcap_{m\, mxml,\, p \subseteq m} m$. so this is a property about polynomial rings over fields. One can remove this hypothesis.

So, the way we will prove this, remove this hypothesis, but the way we will prove this is we will write this as a corollary of Nullstellensatz. But, one can remove this and still this true, but poof of where as far more involved, and in fact that thing is that statement is so general that Nullstellensatz can be derived from it.

So, this is somehow, this statement is closely related to the Nullstellensatz. As it is written with this hypothesis this can be derived from Nullstellensatz. If you remove this it is still true, but that is stronger than Nullstellensatz, and one can prove Nullstellensatz from there; although, that is not the approach that we are going to use. So, but anyway let us just prove this special case.

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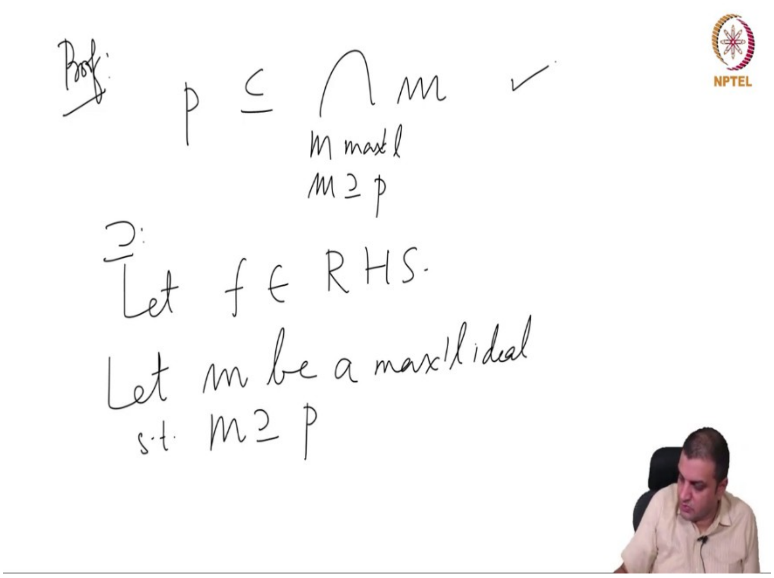

So, this is what we want to show. So, $p \subseteq . \bigcap_{m\, mxml,\, p \subseteq m} m$. So, this is and this is the proof, proof of the earlier observation p is clearly inside that Note, so now, let us take an element inside here. So, let f be inside the right hand side.

So, this is the proof of this inclusion pertaining. What does that say? That says that $f(\underline{a})$, so what are maximal ideals of this, they are of the form $(x_1 - a_{1,\ldots}, x_n - a_n)$, so one can, so then $f(\underline{a}) = 0$. So, we sorry before I stay there let me just explain. So, let m be a maximal ideal containing p.

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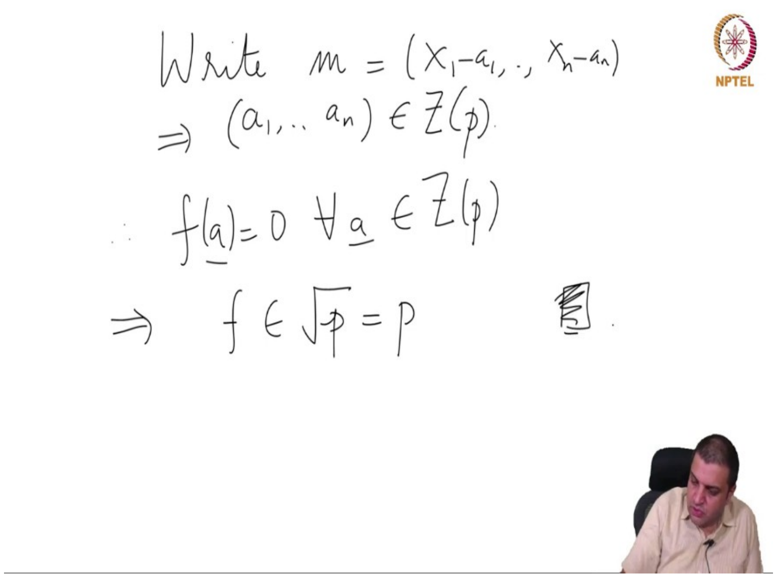

so write $m=(x_1-a_{1,\ldots},x_n-a_n)$, this implies that the point $(a_{1,\ldots},a_n)\in Z(p)$,. Because, if you take p is inside m and if you reduce all of elements of m and if you evaluate an element of p, it would go to 0 because every element of m itself will go to 0 when you evaluate at this at this point. So, this is what it means.

So, now so what does this say if we take an f which is in the right hand side, then we have taken every maximal ideal containing M, so which means we have taken every point inside V(p). So, f vanishes at every point inside V(p). Therefore, $f(\underline{a})=0$ for all $\underline{a}\in Z(p)$.

So, now let us what does this mean. So, Nullstellensatz says one of the versions in Nullstellensatz that we proved says that $f\in\sqrt{p}$, but p is a prime ideal it is its own radical is what we wanted to prove. So, this proves the other direction. So, it is an important property of polynomial rings I mean we have seen it only for algebraically closed, that every prime is an intersection of the maximal ideals containing that prime.

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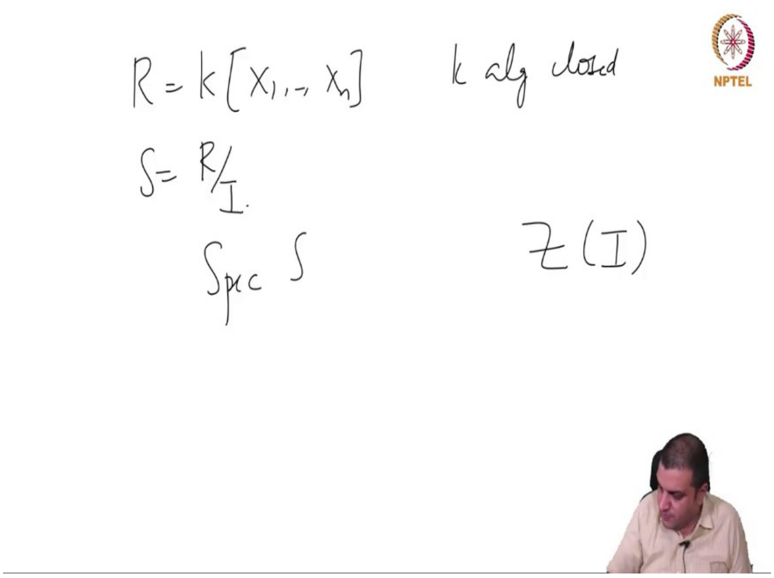

So, now let us continue with our discussion about irreducible decompositions of spec S. So, now so again we have now S, $R=k\left[X_{1,}\ldots,X_n\right]$, k algebraically closed , $S=\frac{R}{I}$ is some ideal and we wanted so we have spec S, and we have Z(I) which is identified with max spec and is now subspace of this. Now, let us take an irreducible subset of this.

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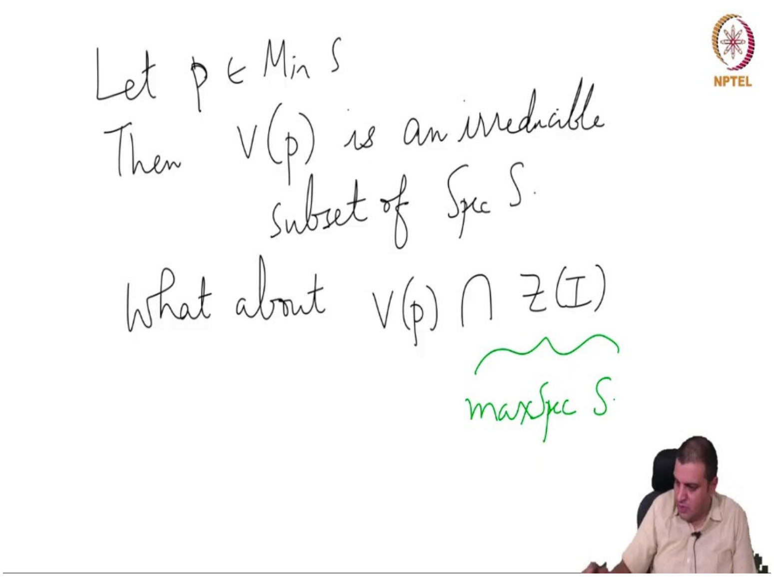

So, let so we know that in irreducible subset of spec (S) is V(p), let $p \in Min(S)$; then, V(p) is an irreducible subset of spec(S), this we proved in the last lecture. So, now we ask what about V(p) ∩ Z (I), this is just remember this is just max spec. So, we have a larger topological space spec S, inside that there is a smaller topological space; I mean fewer sets fewer elements possibly and if we intersect V(p) with the smaller set is it irreducible, so that is what we would like to understand.

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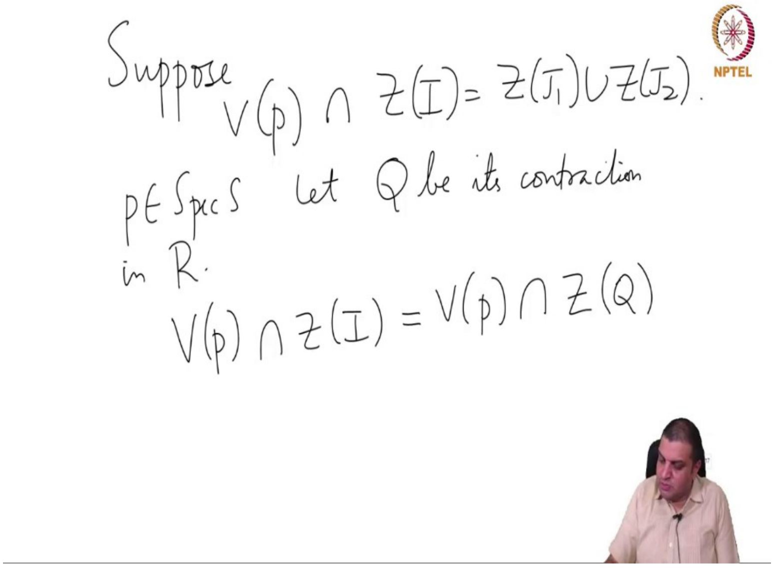

So, $V(p) \cap Z(I) = Z(J_1) \cup Z(J_2)$So, this suppose this is the case, but see we can simplify this a little bit; V(p) is some subset of this, it intersect Z(I) which is the maximum spectrum of S. is we can just take Z(p) itself or technically we $p \in spec\, S$, let Q be its contraction in R, that is also prime ideal and notice that $V(p) \cap Z(I)$.

So, here p is an ideal of S, so Q contains I; so this is same thing as $V(p) \cap Z(Q)$itself. Any maximal ideal containing I, and here because of this condition containing p will just contain q it is , so this will be the statement. So, I am sorry; why did we reduce it like this.

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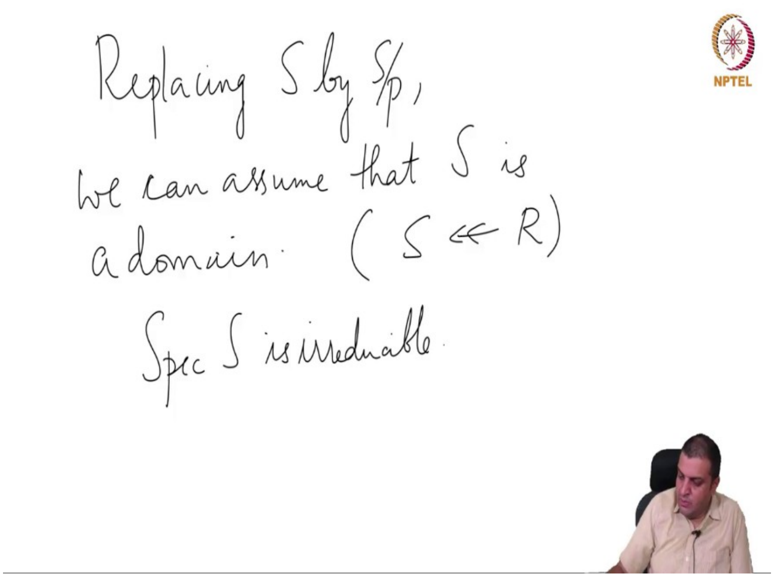

So, replacing S by $\frac{S}{p}$, we can assume that S is a domain. And then, the question is so S remember, S is still; S is still a quotient of R; it is still a quotient not arbitrary S. So, now we have a polynomial ring in n variables over an algebraically closed field this R and a quotient ring which is a domain and then so spec(S) is a irreducible , and we want to ask.

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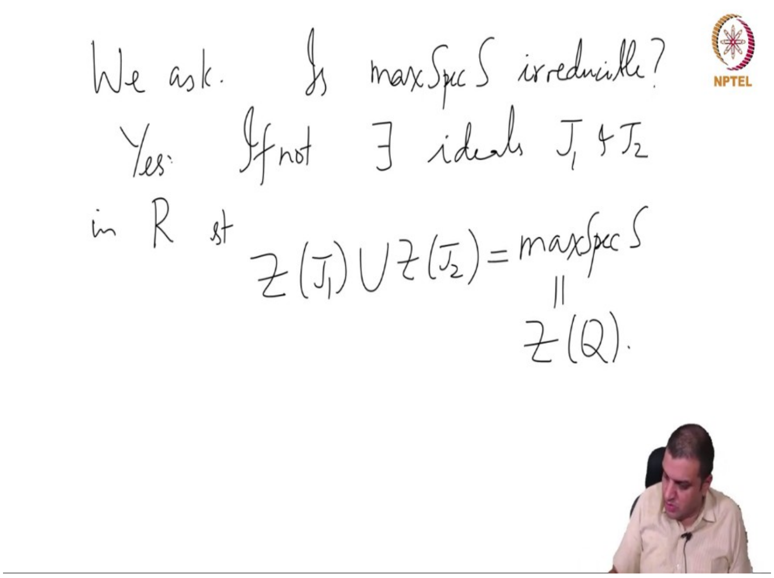

So, we ask is $max\,spec(S)$ irreducible?; so the answer is yes. If not, there exist ideals J_1 and J_2in R such that $Z(J_1) \cup Z(J_2) = maxspec(S) = Z(Q)$. So, what does this say?

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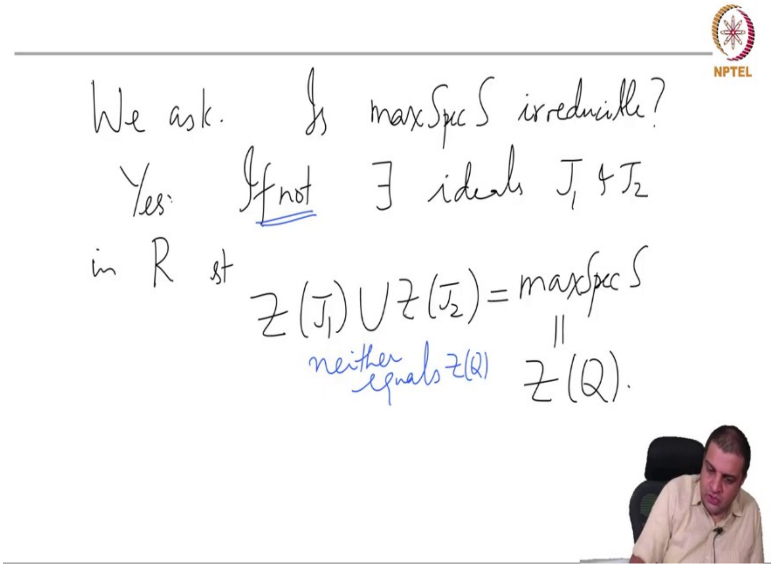

So, then notice $Z(J_1) \cup Z(J_2) = Z(J_1 J_2)$, this is this was one of the, it is also the Z of the intersection, and so also Z of the product, this is one that we had proved. We have been using the word V at that point, but now I want to think of them as 0 sets and call them Z, as we said we will discuss.

So, this implies that $J_1 J_2 \subseteq \sqrt{(Q)} = Q$, this is again by Nullstellensatz. The only way this is going to be the whole space is that this ideal is nilpotent or in other words, thinking of it in terms of R its radical of Q and which is 0 , So, we claim that J_1 or $J_2 \subseteq Q$.

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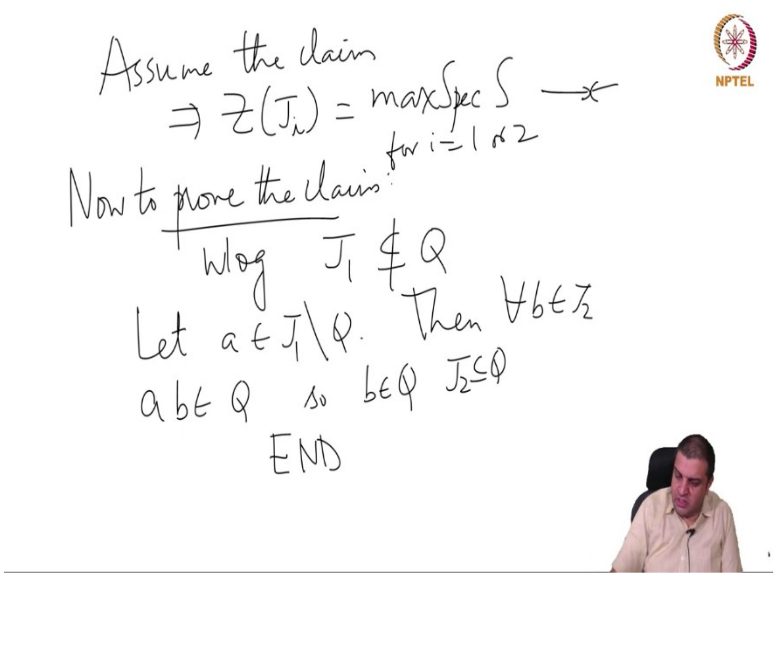

If they assume the claim for now and we will prove with claim in a minute, this implies that, $Z(J_1)=maxSpec(S)$, but then this is a contradiction since we assumed that this was a irreducible decompensate. I mean this way if, so when you say if not there exists ideals such that this is true; then I although I did not say it explicitly, we implicitly assume that neither of them is equal to this.

Otherwise, there is nothing to prove, the problem is I mean if not, the assumption if not assume, we implicitly assumes that these are distinct from Z(Q); although I did not; I did not write it here, so. So, neither of them is equal to Q, but here we are concluding that one of them is equal to Z(Q), so therefore this is the contradiction.

Now, to prove the claim; so this is just a general fact, if a product of ideals is inside a prime ideal, then one of the ideals is in it. Without logs of generality, we may assume that $J_1 \not\subseteq Q$; because if $J_1 \subseteq Q$, then there is nothing to prove,. So, let $a \in J_1 \setminus Q$; then for all $b \in J_2$, $ab \in Q$, so $b \in Q$ which implies that, $J_2 \subseteq Q$. So, this is just a general fact about product of ideals inside a prime ideal.

So, let us just go back and review what we just discussed now. So, this is the end of the; this is the end of the discussion and this is the end of the lecture this lecture. So, let me just go back and just quickly review what we did. So, we have an algebraically closed field and S is $\frac{R}{I}$, where R is a polynomial ring. And, we have two sets that we have discussed so far with topology, Zariski topology on this and the induced Zariski topology, on this; when this is identified with the maximum spectrum of S.

And, we ask if you have an irreducible subset of spec(S), thus its intersection with the maximum spec given a reducible subset of the maximum spec, so that is what we are asking here. $V(p) \cap Z(I)$ is it irreducible in, so the assumption here was that it is irreducible; is it irreducible here that is what we want to know.

So, let us assume that now it is that there is some like this, but the point is p contains I; so let us Q be the inverse image and$V(p) \cap Z(I)$ which means all the maximal ideal is containing p, maximal ideals of S containing p, but that is just the same thing as we can rewrite it like, this it is just maximal ideals containing Q.

So, then replacing S by S/p we can assume that S is a domain, so that spec(S) itself is irreducible, it just simplifies a notation proceed further. And, then we ask, under this assumption if spec S is irreducible is the max spec irreducible ok, the answer is yes; if not, so we going by contradiction. There exist ideals J_1 and J_2 in R such that their respective 0 sets are proper subsets of max spec Z(Q), but the union equals Z(Q), so such a thing exists.

So, then this is the product; so then there is a claim and we prove the claim, but the claim does lead to a contradiction, because J_1 is equal to Q or J_2 is equal to Q, so $Z(J_i)=maxspec(S)$ for i = 1 or 2. So, it then remains to prove a claim and the claim is quite straightforward, all that reduces that there is one element inside J_1not in Q, therefore every element of J_2should be inside Q. So, this is the end of this lecture.