Computational Commutative Algebra

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Lecture - 19
Spectrum - Part 1

Welcome, this is lecture 19. We look at Spectrum of a ring, irreducible components of spectrum, and we will also look at couple of examples in Macaulay.

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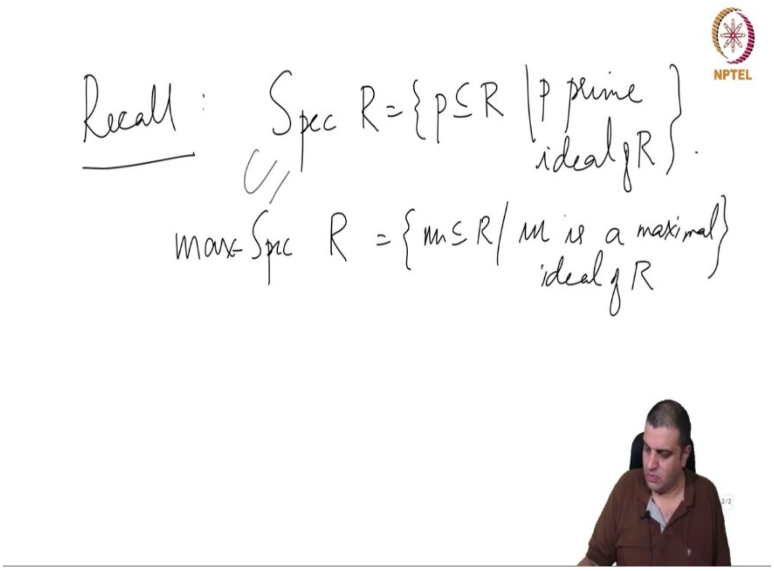

So, recall we defined the set $Spec(R)=\{p\subset R : p \; prime \; ideal \; of \; R\}$ and we can also define what is called the $maxSpec(R)=\{m\subset R : m \; maximal \; ideal \; of \; R\}$ which is a set of maximal ideals and this is in Spec(R) mean this is a subset of every maximal ideal is a prime ideal . We would like to give a topology to Spec R and whenever we talk about a topology of max spec, we mean the induced subspace topology .

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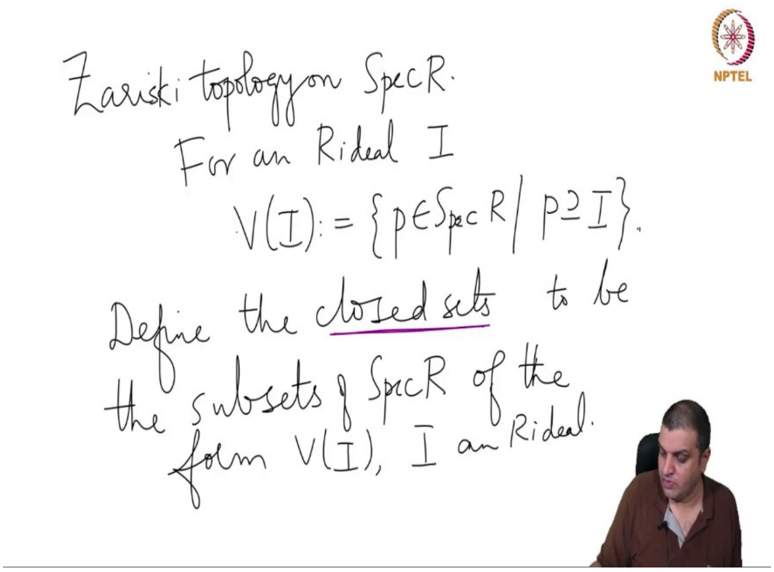

So, so this is called Zariski topology on Spec(R). So, there is there will be a little abuse of terminology or notation now . We will write for an R- ideal I, we will write V(I). So, this is an arbitrary ring. It has it is not the points do not look like points of k^n or anything . So, this is an over algebraically closed, no such restrictions on the ring, just a commutative ring. But we will still use the same notation to mean something which is slightly different. $V(I)=\{p\in Spec(R): I\subseteq p\}$.

We know in some sense, this definition analogous to the definition of what we had V(I)in the case of polynomial ring over a field, which is that V(I) would be the points at which elements of I vanish. Points correspond to maximal ideals in a way by nullstellensatz that we saw. So, therefore, there it would just say it would the description that we gave there or the definition that we gave there would be equivalent to saying V (I) is the maximal ideal is that contain I.

But that is not exactly enough for us to work in this in this generality. So, we need to define V(I) to be the subset of Spec R with this property and we can get an induced subspace topology or we can take $V(I)\cap maxSpec(R)$ and then, talk about max spec restrict ourselves to the earlier situation .

So, we can define so, this is this is just a definition, So, define the closed sets of the topology . So, this is important. We are defining the closed sets now to be the subsets of the form $\{V(I): I \, is \, any \, ideal\}$. So, these are the closed sets . Now, we need to check that this is fine.

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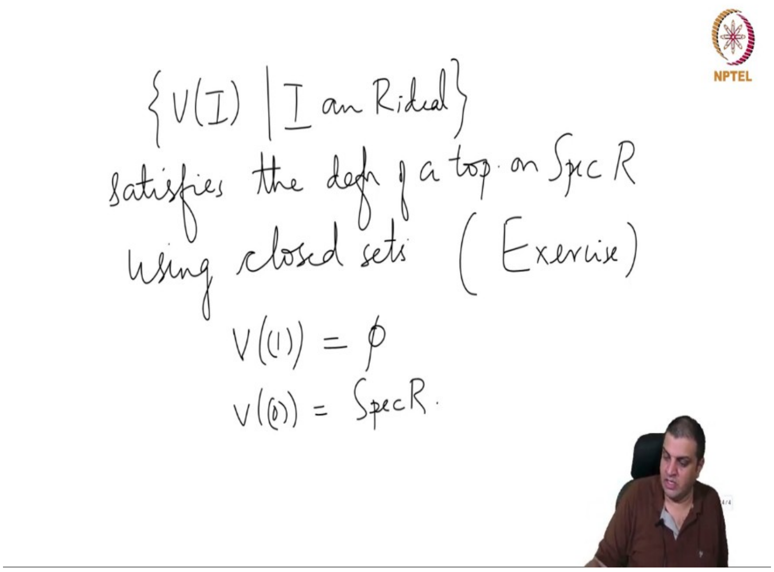

So, that will be an exercise that if you look at the collection of V(I) can satisfy satisfies the definition of a topology using closed sets . So, this is check. This is an exercise . Just I will explain with exercise what would have to be checked. So, the key point is that if $V((1))=\varnothing$.

If you take the V((0)) = Spec R and then, , if you take V(I) $\cup V(J)$, V of what is it and if you intersect a family, what is what are we doing if you if you take intersection of a family of closed sets, how does it happen on the ideals? So, this we will discuss in the exercises and then, we will show that this is indeed a topology on Spec R . This is called the Zariski topology, topology on Spec R.

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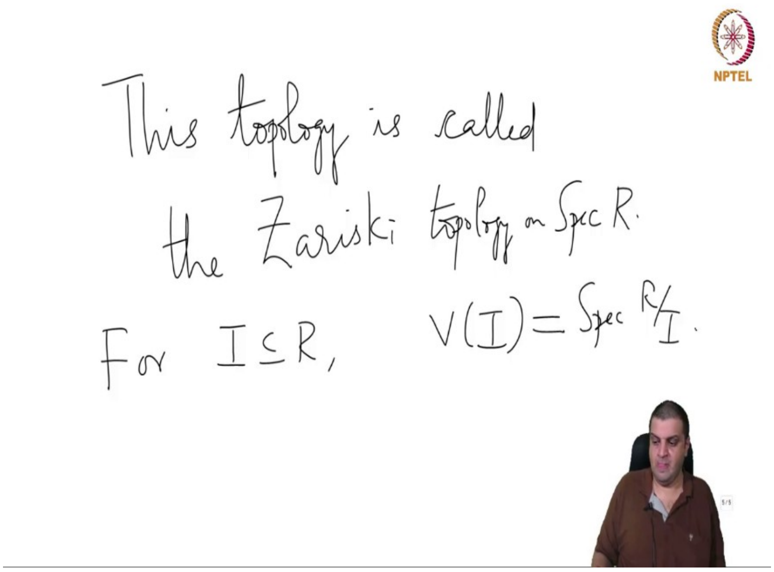

Now, one observation that we would like to make is that for $I \subseteq R$, the set V(I) is same as the set $spec\left(\frac{R}{I}\right)$.

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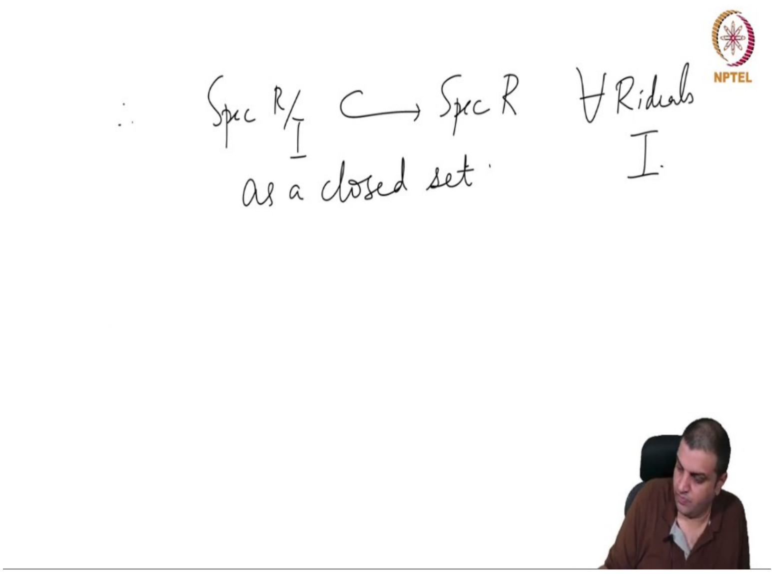

So, in other words, there is a map therefore, spec(R/I) injects into spec(R) as a closed set, the image is a closed set of spec(R)for every ideal I. We will discuss more properties about spectrum in the exercises, we will see some specific open subsets in this in the setup etc. So, that I postponed to the exercise.

What I want to do is to motivate the discussion towards studying primary decomposition and associated primes. So, what we are trying to understand is the topology; in topology there topological spaces, there is a notion called irreducibility and the idea of primary decomposition and primary decomposition generalizes that to I mean puts that in the context of Zariski topology, but one has to worry little bit more than just irreducible components and this will become clear in the next few lectures, but for now let us discuss irreducibility

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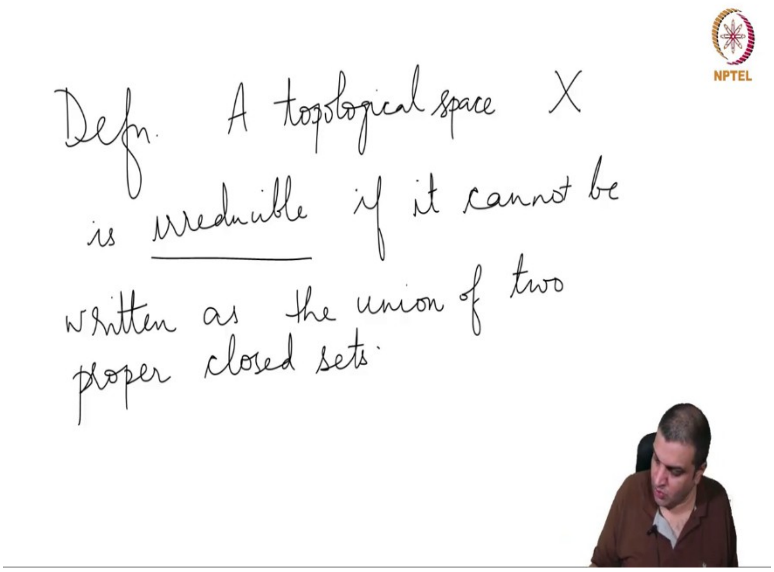

So, definition a topological space X is irreducible, if it cannot be written as the union of two proper closed sets .So, this is not something that we come across lot in when we study basic topology course because most of the spaces are do not fit this description. So, it is not a question that we would typically worry about. Let us say an undergraduate course in topology. But here this is a crucial point, I mean this is an important point for Spec(R), also for varieties that we were discussing in earlier lectures .

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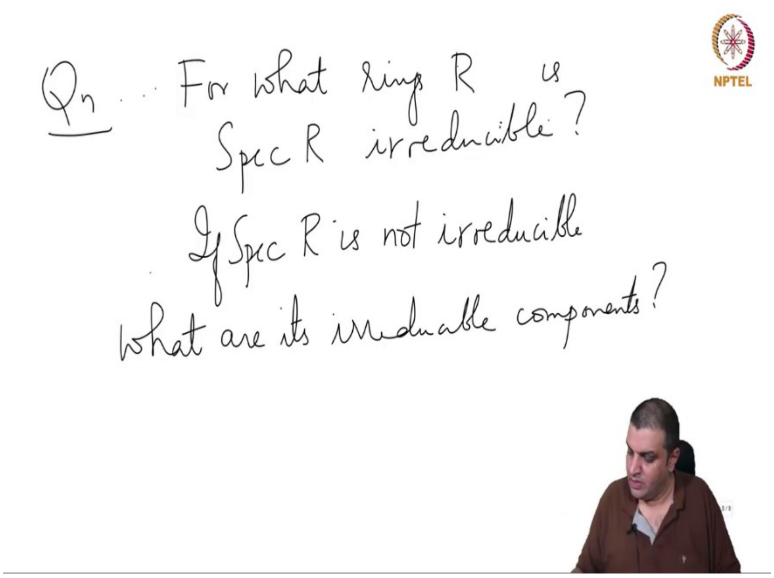

So, now we would like to understand the following question: For what rings R, Spec(R) is irreducible ? So, this is one question that we would like to understand and in general, if Spec (R) is not irreducible ; what are its irreducible components ? So, this is what we want to understand.

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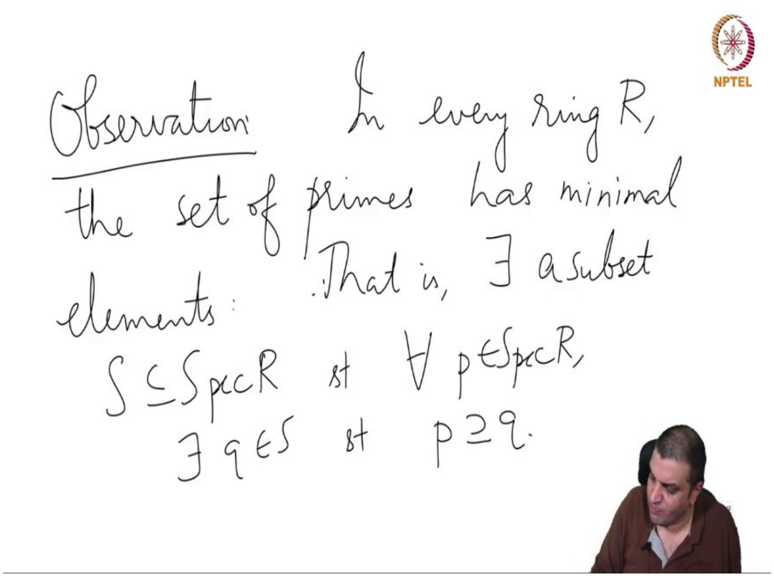

To understand these questions, we need a small observation. In any ring, so we want a small observation here. In every ring R, the set of primes have minimal elements . What does that mean? For every prime q, there exist a subset S inside Spec(R) such that for every p in Spec (R), there exist some $q \in S$ such that $q \subseteq p$. That is what we mean by minimal elements .

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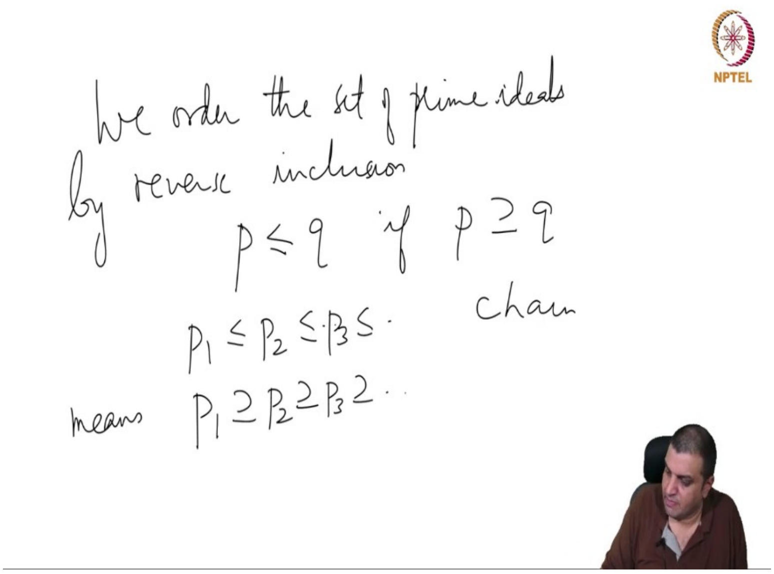

The proof is not very difficult. It uses Zorn's lemma. So, what is the point that we would need to do? So, we order the set of prime ideals by reverse inclusion . So, we say that p $\leq$q, if p $\supseteq$ q.

So, this is not the typical way we order, but this for the sake of for this proof, let us order I mean like this and then, observe that ; so, let us say $p_1 \leq p_2 \leq p_3 \leq \ldots$this is a chain. So, in other words, what does it mean? It means. So, $p_1 \supseteq p_2 \supseteq p_3 \supseteq \cdots$

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Now, intersect them . We want to show that this is a prime . So, then so. So, let $ab \in \cap p_i$; $a \notin \cap p_i$; . So, this means that there exist some j such that $a \notin p_j$; , that is why it, but this is a descending family. So, this implies that $a \notin p_k$, for all k $\leq$j.

So, but these are prime ideals. So, now, this implies that $b \in p_k$for all $k \geq j$and that means that $b \in \cap p_i$. It is a descending family. So, this is the proof of the claim.

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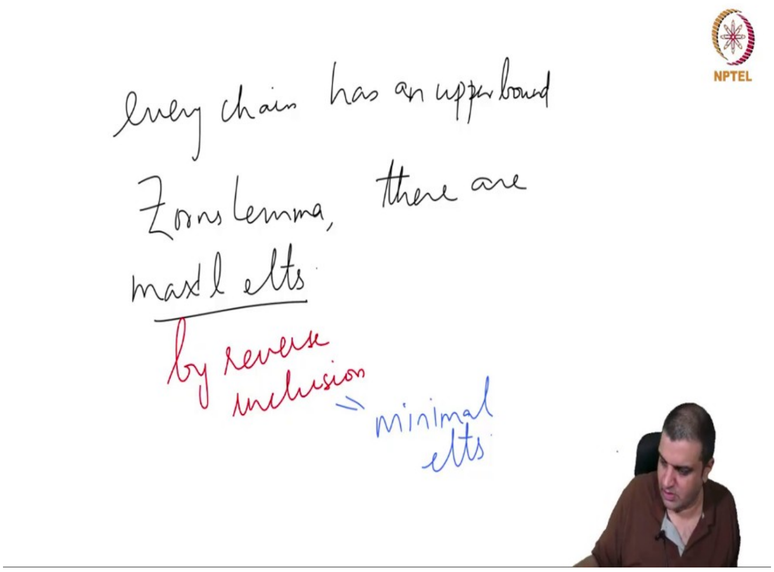

So, so, the this proves the claim and what we just what we just argued is that every chain, in this collection in the poste that we are considering has an upper bound. Therefore, by Zorn's Lemma, there are maximal elements and this is what we I mean maximal element in by reverse inclusion. So, this is by reverse inclusion. So, which really means minimal elements . So, this is the proof of this observation that there are minimal elements.

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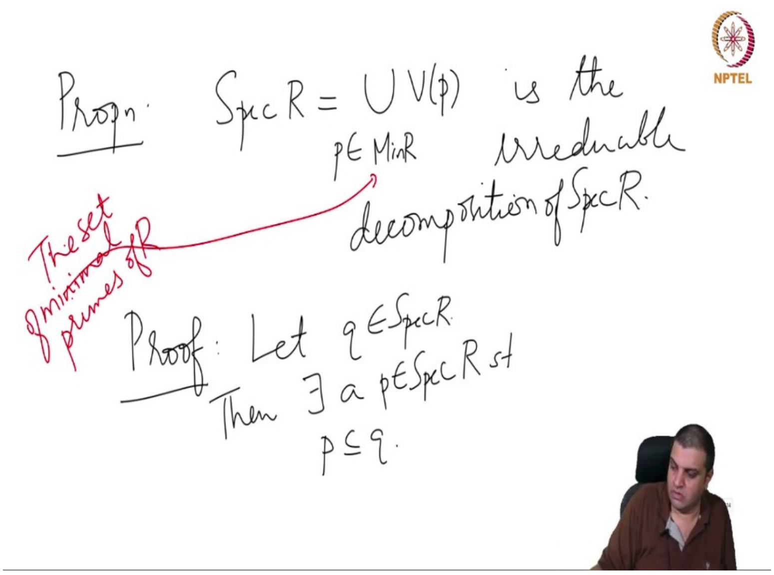

So, now we can write the proposition : $Spec\,R = .\bigcup_{p \in Min\,R} V(p)$ where Min(R) is the the notation for the set of minimal primes of R. So, $\cdot\bigcup_{p \in Min\,R} V(p)$ is the irreducible decomposition of Spec(R).

So, we need to do one thing Spec (R) is indeed the union and that these are irreducible, that is what we would need to do.

Proof; let $q \in Spec(R)$ then there exist a minimal prime p such that $p \subset q$.

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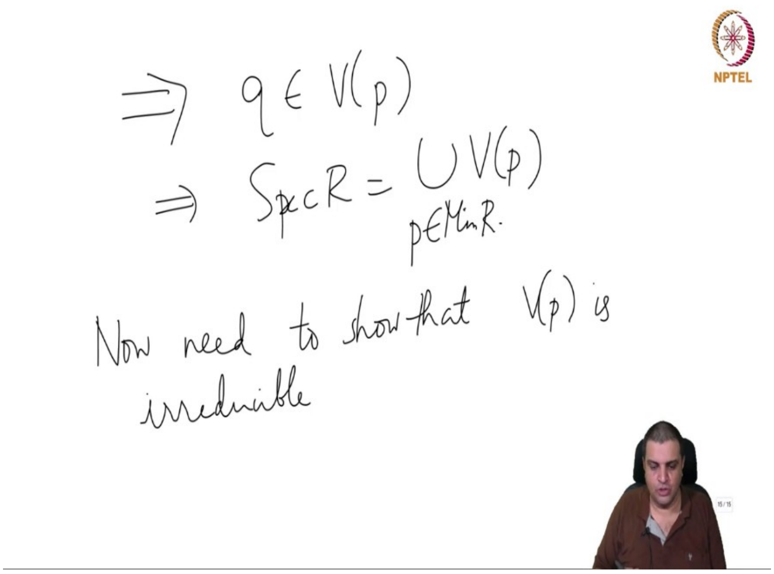

This implies that $q \in V(p)$ which proves that $Spec(R) = .\bigcup_{MinR} V(p)$. Notice that the right side $V(p) \subset Spec\,R$, this is the union is always in the right side. What we need to prove is the other direction, which we prove like this. This is the proof . So, we now need to show that V(p)is irreducible. So, $V(p) \subseteq Spec(R)$ and its topology is the induced topology.

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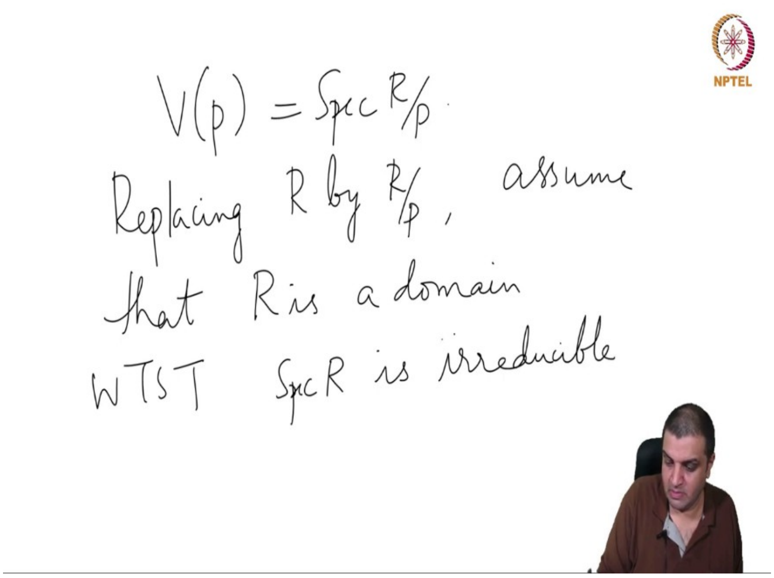

So, notice that the subspace topology of V (p) is same as spec(R/p). So, replacing R by R /p, we want to show that we assume that that R is a domain and we want to show that Spec R is irreducible.

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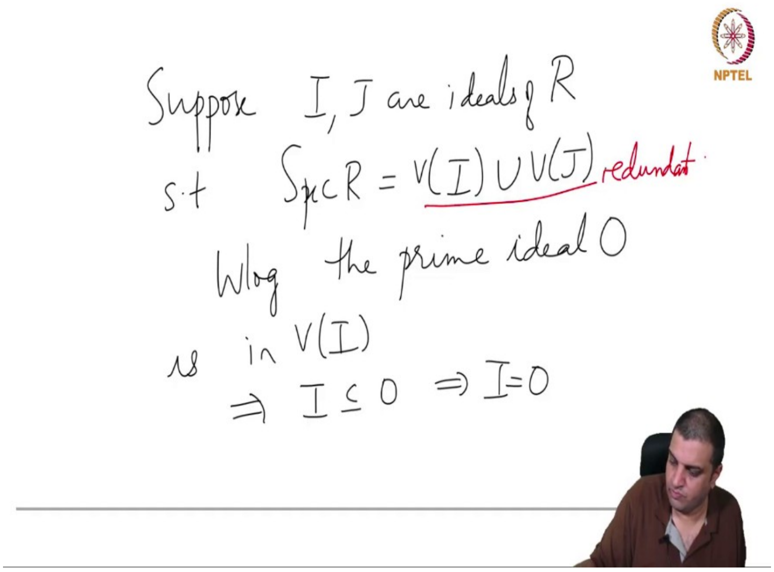

So, suppose, I and J are ideals of R such that $Spec(R) = V(I) \cup V(J)$. So, R is a domain, that is we have reduced to that case $V(I) \cup V(J)$. So, this is what we mean by union of two closed sets. So, without loss of generality, the prime ideal 0 is in V (I). The prime ideal 0 is a point here, it must be in one of these subsets. So, it must be in V (I) but what does that mean? That means, that I is contained in prime Ideal 0; in other words, I = 0.

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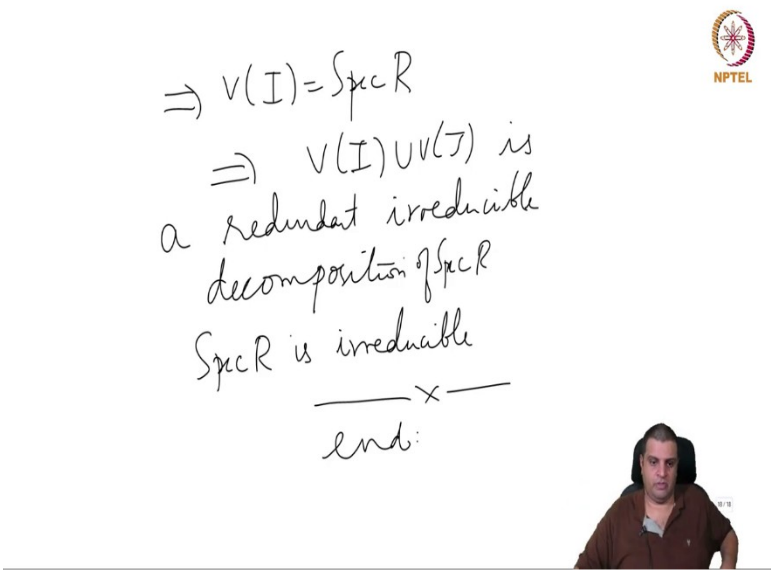

But that would mean that V(I) = Spec(R) . Therefore, that the decomposition that we saw here is not redundant already the right. This itself is Spec(R), this does not do anything . So, this is the point here is that this is redundant . So, this now means that $V(I) \cup V(J)$ is a redundant, irreducible decomposition. I mean this is not doing anything of Spec(R) .In other words, so Spec(R) is too irreducible.

So, this is a brief discussion about irreducible components and we will now try to see that the Spec R = $\cdot \bigcup_{p \in Min\, R} V(p)$. So, we will try to take some ; so we will now do an example in Macaulay.

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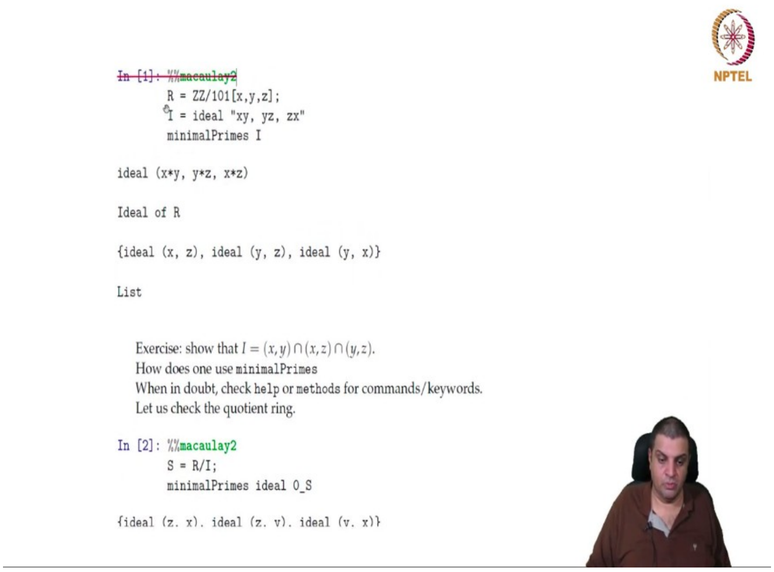

So, here again, let us just we ignore this part altogether and this is this is the input line first input line. So, this offset code is the input code R is this polynomial ring, then we give this ideal to Macaulay. So, it says so this is I this is we saw such an example earlier.

We can put it like a string without the exponents and thing provided the variables do not have subscripts. So, this is just xy, yz and zx. So, we have this ideal, then we ask for this we use this thing called minimal primes I. It will produce the primes over I, primes containing I that are minimal with their property.

So, in the proposition, we proved we the proposition that we proved says that spec(R/ I) has minimal primes or in other words, V(I) is minimal primes; but it does not tell us that it is a finite. The proposition did not say it is a finite, but and it is not true in general; it may not be true in general. But here we are working with Noetherian rings and there we will prove that the set of minimal primes is finite. We will prove in the course in the course of next few lectures.

So, it just gives us a list of the primes that are minimal over this and it is. So, you it is an exercise to for you to show by hand that these primes indeed contain this. These are indeed minimal primes and in fact, I is the intersection of these three primes. And how does ah; so, the question is how does one use minimal primes? So, whenever you are in doubt, check the help command for minimal help or the methods, maybe start with help use also methods for you can figure this out.

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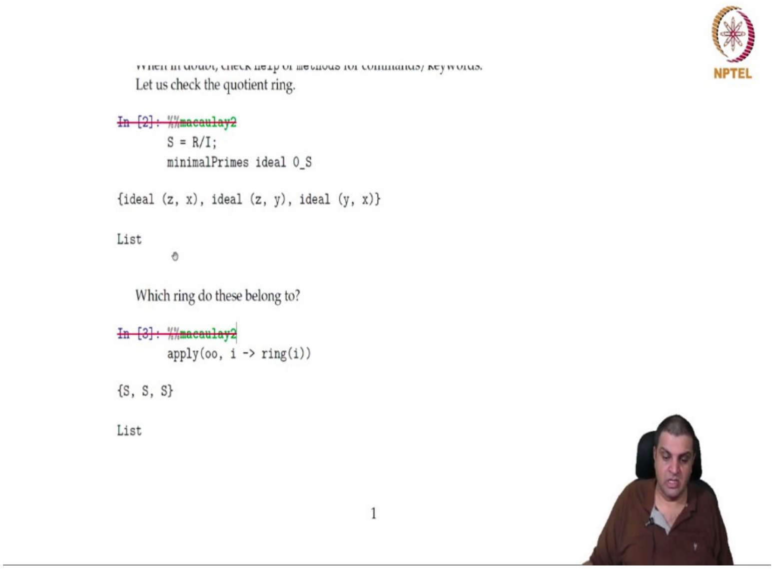

Now, let us try to do this for the quotient ring, let us write R/S to be, S as R/I and then, we ask for the minimal primeS of the ideal generated by 0 of S. So, we need if you put 0, it will think of as an integer, that is not what we want. We wanted to think of it as a 0 of S. So, we write 0_S , 0_S and so, minimal primes we ask and then, it produces the same list. But, but it knows that it is these are not elements of R, these are elements of S. So, we can ask that question which rings do these belong to?

So, again, as I said this is not relevant for the say that is not part of the code. The code is only these lines, they the offset lines and we get. So, we run this command called oo, oo is the previous output which is this list. Now, we have seen this command called select which selects elements of a list according to whether it satisfies some true or false condition.

So, here this is not; what this does is to apply this function to the elements of this list. So, this is a function which takes some i to ring(i). So, in this case, i is an ideal and ring of an ideal tells you what is the ambient ring in which that ideal is defined and here, it shows that these three things are ideals of S which is what we had expected.

So, this is the end of this lecture and in the next few lectures, we will look at we will look more at the idea of associated primes and primary decomposition.