Computational Commutative Algebra
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Lecture – 17
Localisation

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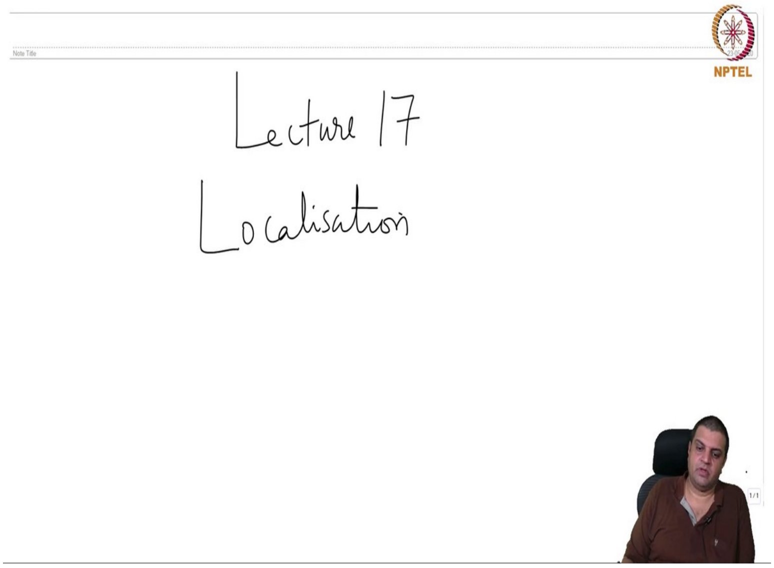

Welcome to lecture 17, in this lecture we discuss basics of about Localisation and this is the topic that we will come to again and again. So, we will see as and when we need we will discuss this as much as possible. So, what exactly do we mean?.

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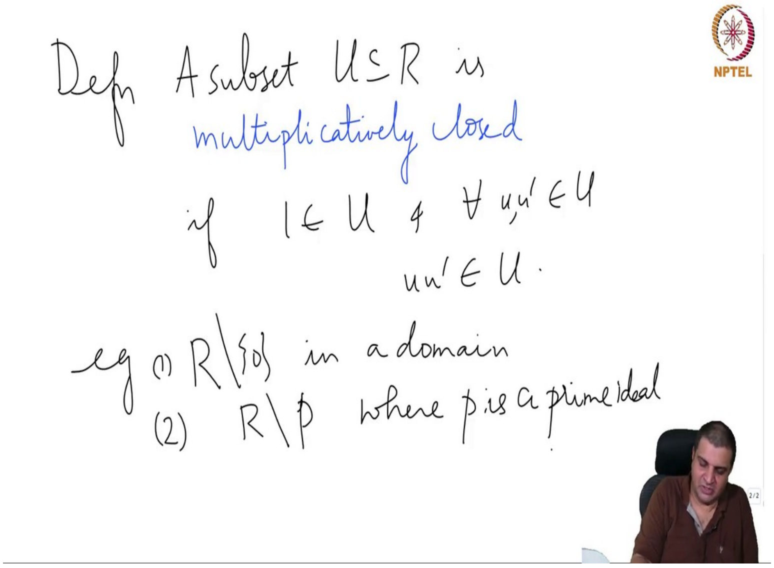

So, definition: a $U \subset R$, R is a ring is called is multiplicatively closed, if $1 \in u$ and for all $u, u' \in U, uu' \in U$. So, for example,1) the nonzero elements in a domain, 2) $R \setminus P$, where P is a prime ideal. By definition of P being a prime ideal, it is clear that if u and u prime are not in P, the product is not in P and 1 is definitely not in P.

So, really the first one is an example of the second one and, but there is a new example.

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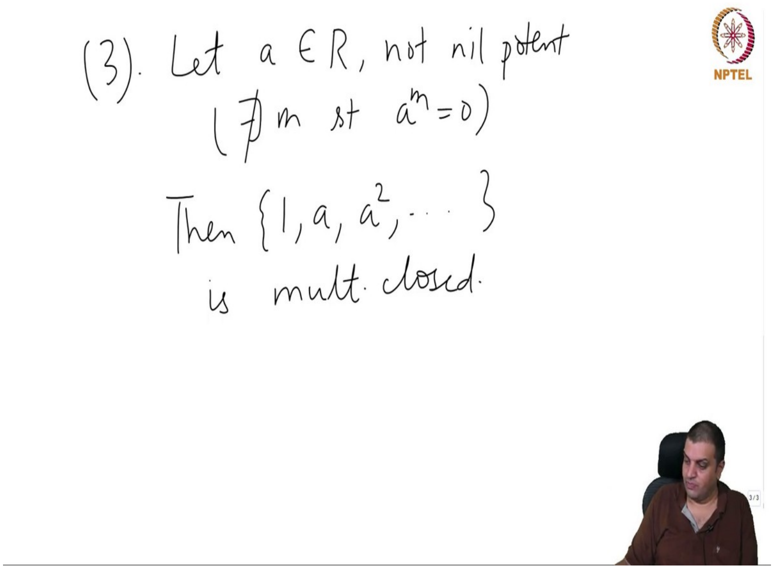

3) let $a \in R$ not nilpotent, In other words, there does not exist m such that $a^m = 0$ ok. Then the set $\{1, a, a^2, ...\}$ is multiplicatively closed. So, this is what we mean these are examples of what we mean by a multiplicatively closed set. There might be others also for example, you could take a union of two prime ideals and then take the complement and so on.

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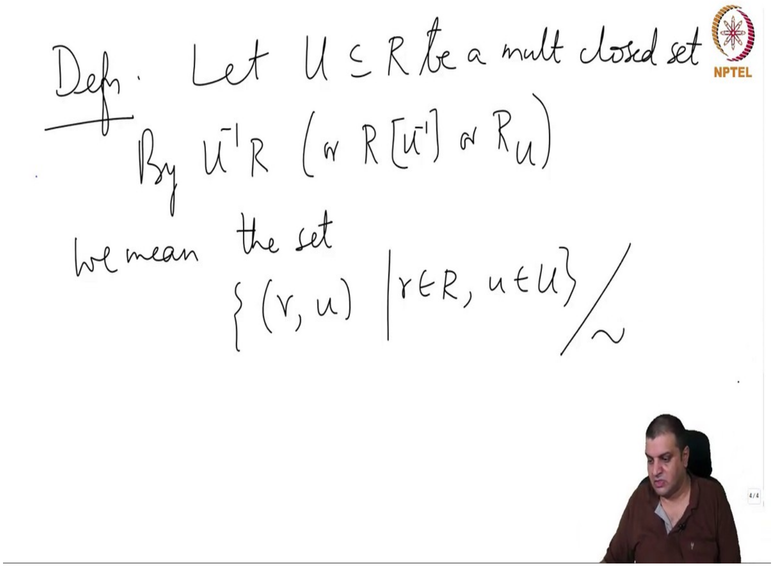

Let $U \subset R$ be a multiplicatively closed set. By $U^{-1}R$ (or $R[U^{-1}]$ or R_U) various notational convention used by various authors, we mean the set $\{(r,u) : r \in R, u \in U\}$ under an equivalence relation.

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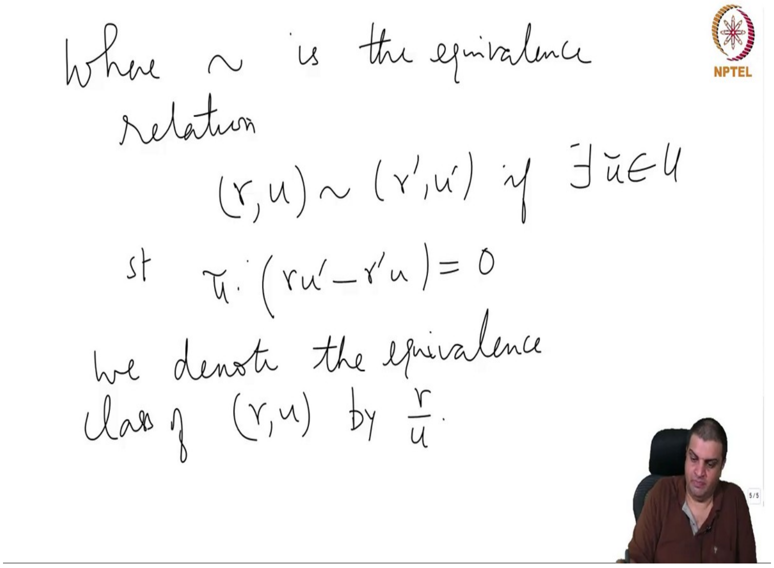

where this is the equivalence relation $(r,u) \sim (r',u')$ if there exists some $u \in U$ such that $u(ru' - r'u) = 0$.

We denote the equivalence class of (r, u) by the fraction $\frac{r}{u}$. So, really we are saying we are thinking about fractions, but of course, as in Q, $\frac{1}{2} = \frac{2}{4}$. So, it is really there. So, this is really that statement, how do we identify two fractions and if we are working in general not just rationals one needs to put this as the definition.

So one define such a set and few.

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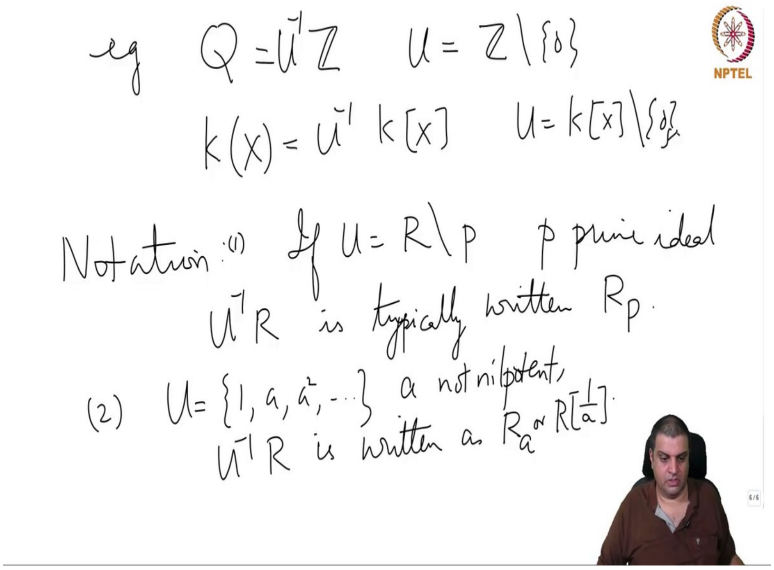

So, for example, $Q = U^{-1}Z$, where $U = Z \setminus \{0\}$. Or the rational function field in one variable is $k(x) = U^{-1}k[x]$, where U is the non-zero polynomials. So, it is these constructions that actually the that construction $U^{-1}R$ generalizes.

So, what is so, just a little bit of notation. 1) If $U = R \setminus P$, where P is a pime ideal then $U^{-1}R$ is typically written as R_P,

2), if $U = \{1, a, a^2, \ldots\}$, a not nilpotent, then $U^{-1}R$ is typically written as R_a or $R[\frac{1}{a}]$.

So, this is just variations of a notation that one would keep using or comes across but why are we studying this.

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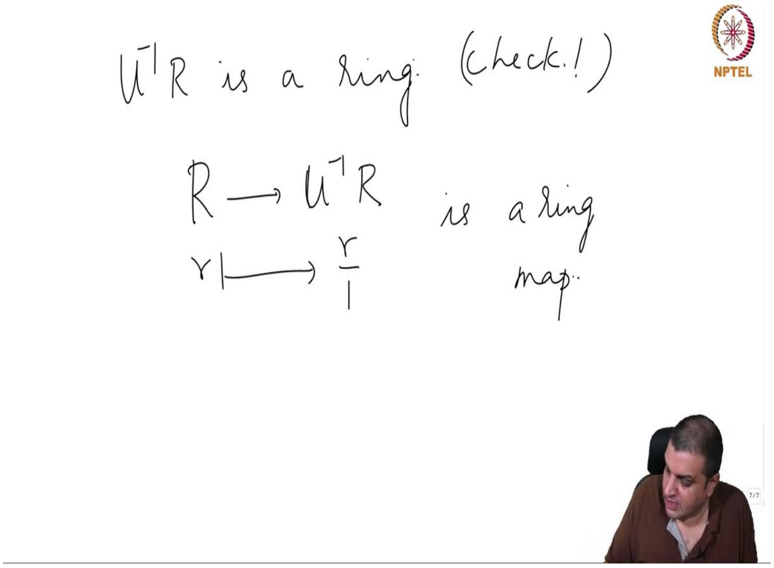

$U^{-1}R$ is a ring, you should check not very difficult at all. How I mean as how one should add fractions. So, just keep that is all that you have to keep in mind.

And $R \to U^{-1}R$ in which a ring element $r \to \frac{r}{1}$ is a ring map. So, $Z \to Q$ is injective, $k[x] \to k(x)$ is injective, but in general they it need not have such properties. So, it is injective.

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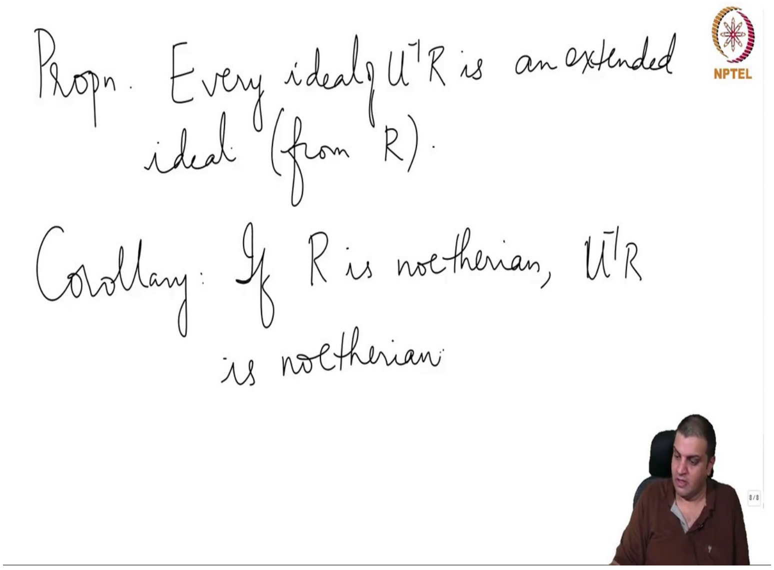

What property does this ring have?. Proposition: every ideal of $U^{-1}R$ is an extended ideal (from R). .

There is an ideal in R whose extension gives J, but this might sound like a curiosity, but it has an important application we will prove the proposition just. This is the first time we are coming across this notion of working with fractions etcetera I will do it. So, that it become we become familiar with that. So, it says an important application. If R is noetherian, $U^{-1}R$ is noetherian.

That is because everything is an extended ideal. So, whatever generating set generate set in I will generate it as a $U^{-1}R$ also, but it is a finite generating set here. So, there it is finite generating set there. So, this corollary is important.

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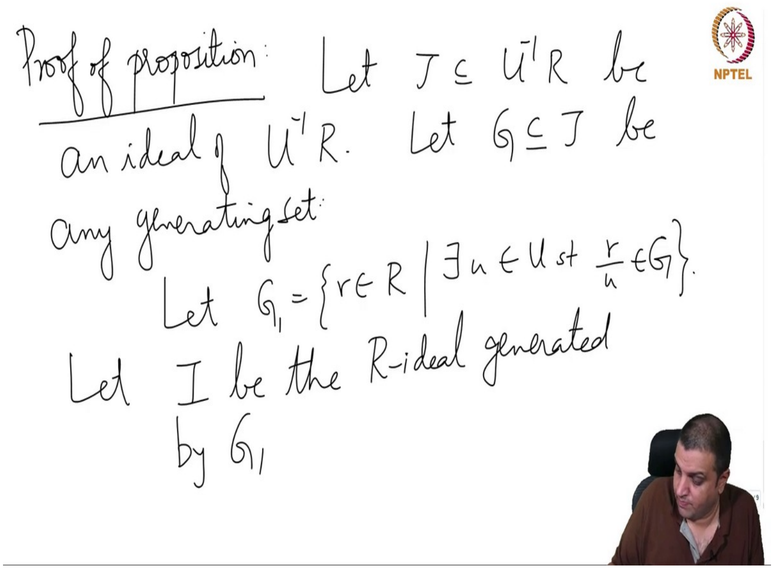

So, now let us prove the proposition. So, let J be an ideal of $U^{-1}R$. Let take a generating set for it , $G \subset J$ be a generating set.

So just we keep saying generating set there is no significance to that statement there is no substantial significance to the set (Refer Time :11:16) proof J itself is a generating set for J. So, one could have just taken J itself, but what I want to say that the argument works for every generating set.

Now let $G_1 = \{ r \in R : there\ exist\ u \in U\ st\ \frac{r}{u}\ is \in G \}$. So, these are the numerators that appear in the fractions inside G_1. Let I be the R_ideal generated by G_1. The claim is that J is the extension of I let us write that.

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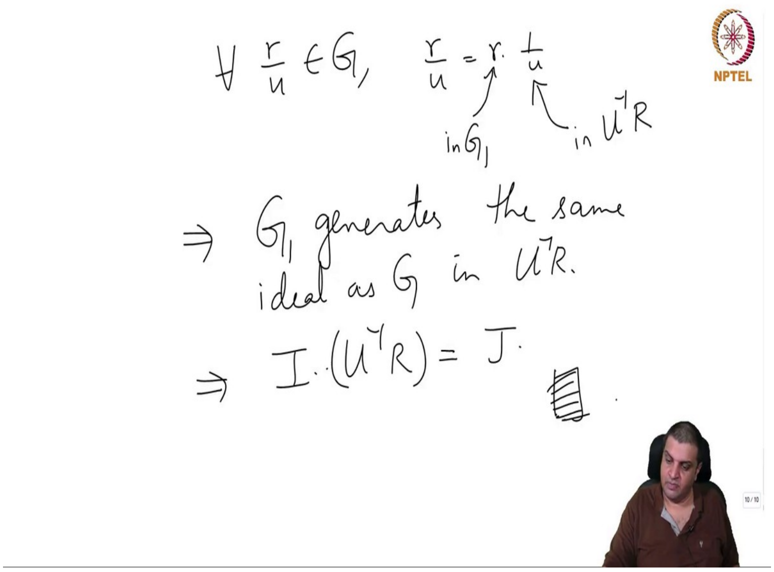

So, notice that for every $\frac{r}{u} \in G, \frac{r}{u}$ can be written as $r\frac{1}{u}$. So, $r \in G_1$ and $\frac{1}{u} \in U^{-1}R$. So, in other words G_1 generates the same ideal as G in $U^{-1}R$.

After localisation, G does not belong to R. So, this statement make sense only in $U^{-1}R$ and from then we can conclude that I extended. This implies $I(U^{-1}R) = J$

Notice that I itself a subset of J therefore, if you extend I to $U^{-1}R$ it is going to be a subideal of J. But already the subset G_1 generates J. So, therefore, I also generates J. So, this is what we need in this proof . So, every ideal is an extended ideal.

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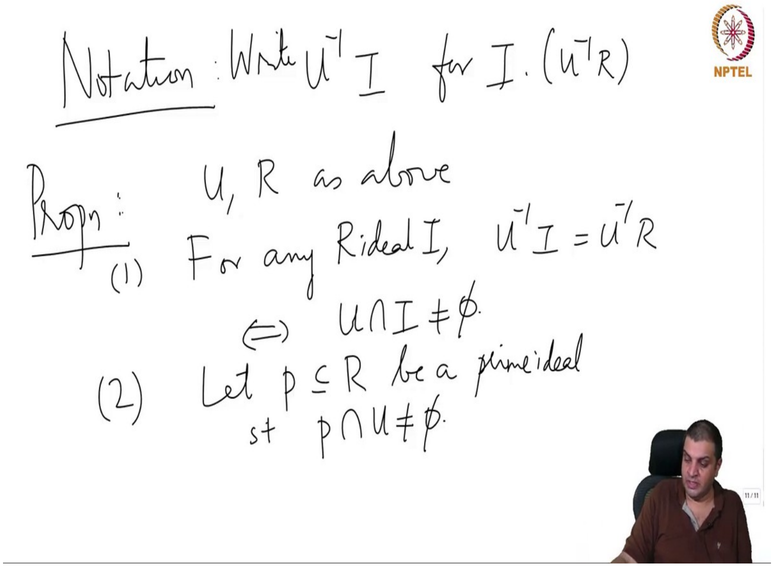

So, again one more piece of notation. We will write $U^{-1}I$ for I extended to $U^{-1}R$ and later we will see also for modules. So, here some important proposition, I mean basic properties of this which keeps which we get keep using all the time.

So, U and R as above.

1) For any R-ideal I, $U^{-1}I = U^{-1}R$ if and only if $U \cap I \neq \emptyset$. If this is true then there is will be some element inside I and 1 over that would also be inside this ring.

So, some $\frac{a}{a}$ that will give 1. So, that is what one will have to check.

2. Let $P \subset R$ be a prime ideal such that $U \cap I = \emptyset$ So, $U^{-1}P$ is not going to be the whole ring ok.

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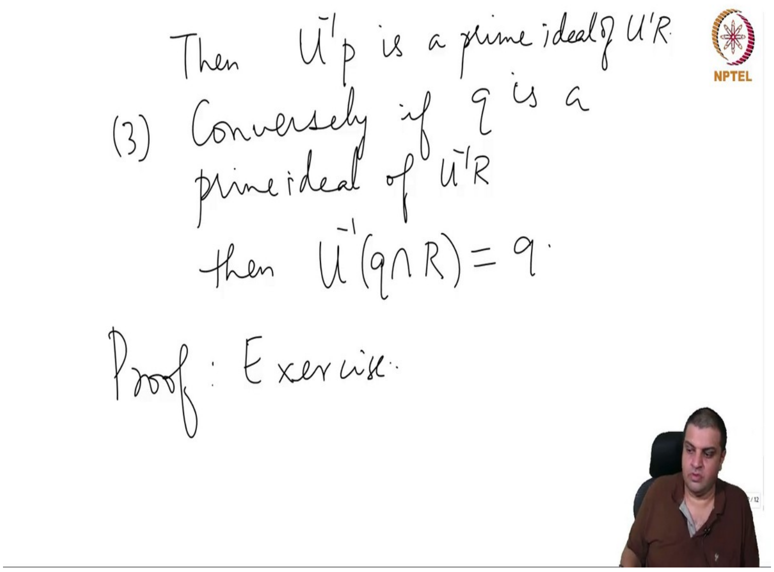

Then $U^{-1}P$ is prime ideal of $U^{-1}R$. So, this is what we have.

3. Conversely if q is a prime ideal of $U^{-1}R$, then we can look at the contraction of q to R, which is a prime ideal of R and then $U^{-1}q \cap R = q$.

And, so these are the statements that can be proved relatively without difficulty. So, proof is left as an exercise. So, the point that one gets from these is the following.

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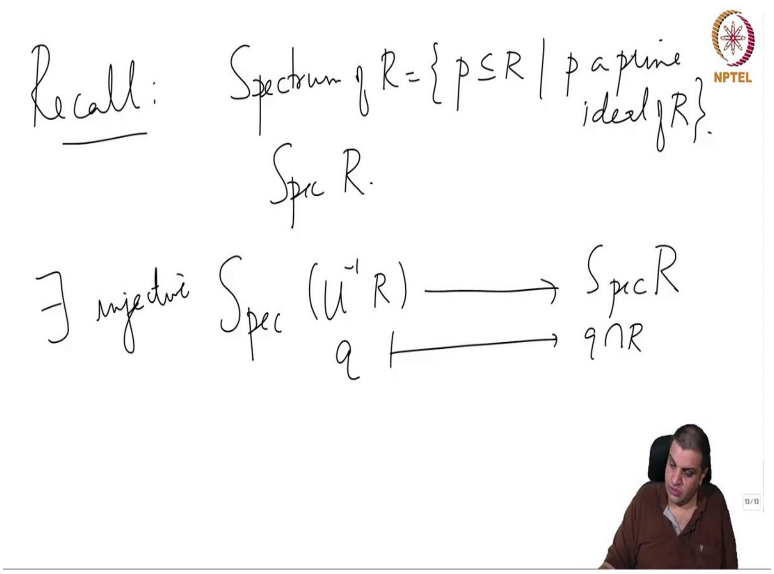

So, recall that what we called the prime spectrum of R $spec(R) = \{p \subset R : p \text{ is a prime ideal}\}$, This we you did this in the exercises which we denote by spec (R).

Now what does the previous proposition say it says that there exist an injective map $spec(U^{-1}R) \rightarrow spec(R)$. And the map here is $q \rightarrow q \cap R$ and that is a prime ideal here and know from the last part if q and q' are different then the target will also be different that is what the last part is saying. So, this is injective. So, this is one application.

So this sort of observations I have to do with giving a topology on the set spec(R) which we will do soon, but that is where that is where the sort of discussion goes to.

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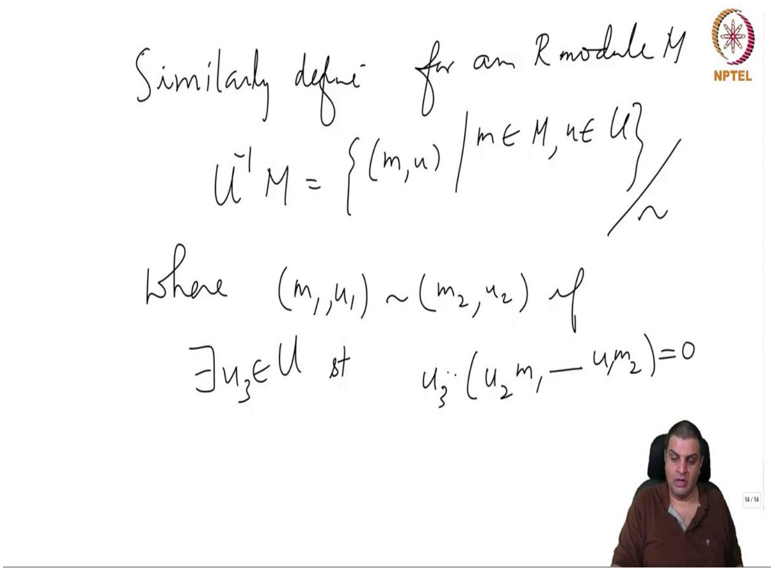

So, just now we can talk about localising modules. We can similarly define for an R module M, $U^{-1}M = \{(m, u) : m \in M, u \in U\}$ modulo an equivalence relation, where $(m_1, u_1) \sim (m_2, u_2)$ if there exists some $u_3 \in U$ such that $u_3(u_2 m_1 - u_1 m_2) = 0$.

So, again it says the two fractions are same and we check this by multiplying cross multiplying the denominators that is the idea behind this. To work in to make it work in all generality not just with rational numbers one needs to also put this inside here.

So, this is a $U^{-1}M$ is $U^{-1}M$ -module.

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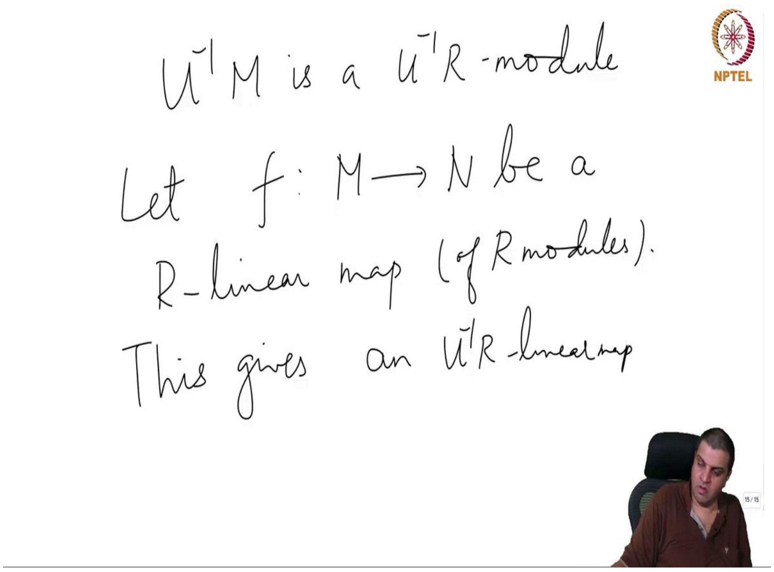

And so, let $f: M \to N$ be a an R-linear map of R-modules. This gives an $U^{-1}R$ - linear map .

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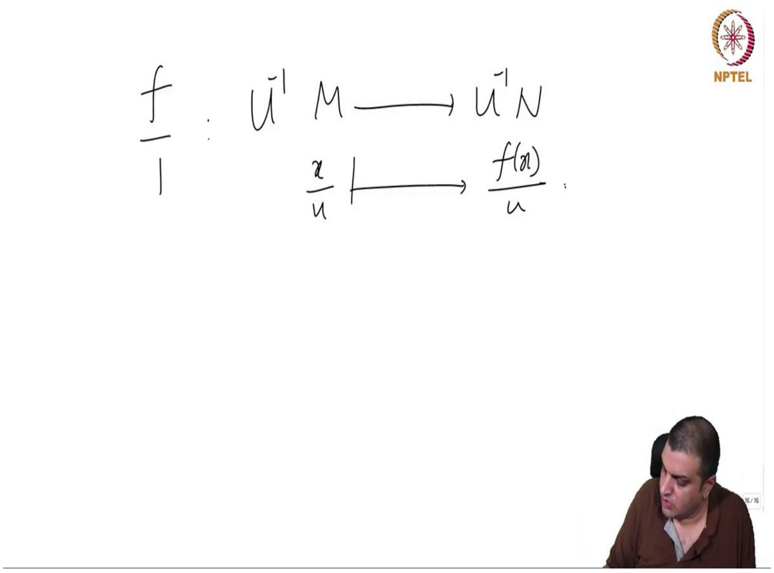

Which we will define like a fraction $:U^{-1}M \to U^{-1}N$. what is a map? So, here are elements of the form $\frac{x}{u}$ and this gets mapped to $\frac{f(x)}{u}$. So, this is a $U^{-1}R$ - linear. So, any such f gives such a map.

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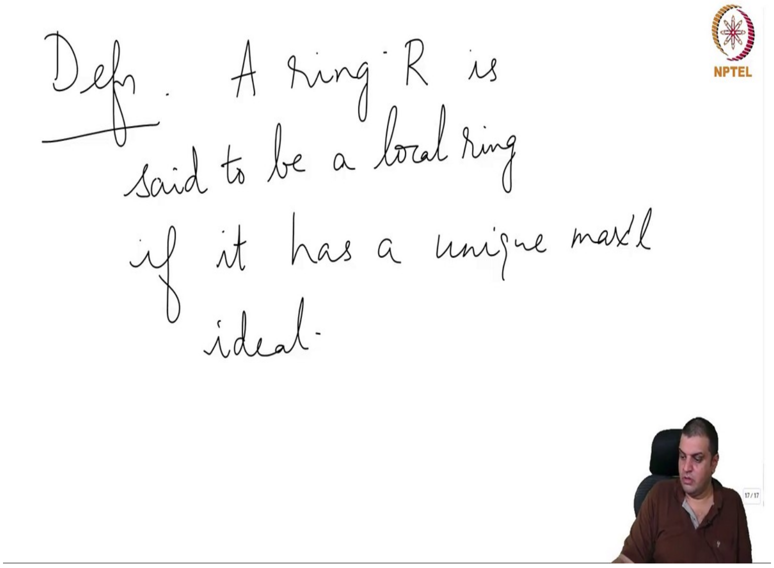

So, one more definition in this context, a ring is said to be a local ring if it has a unique maximal ideal.

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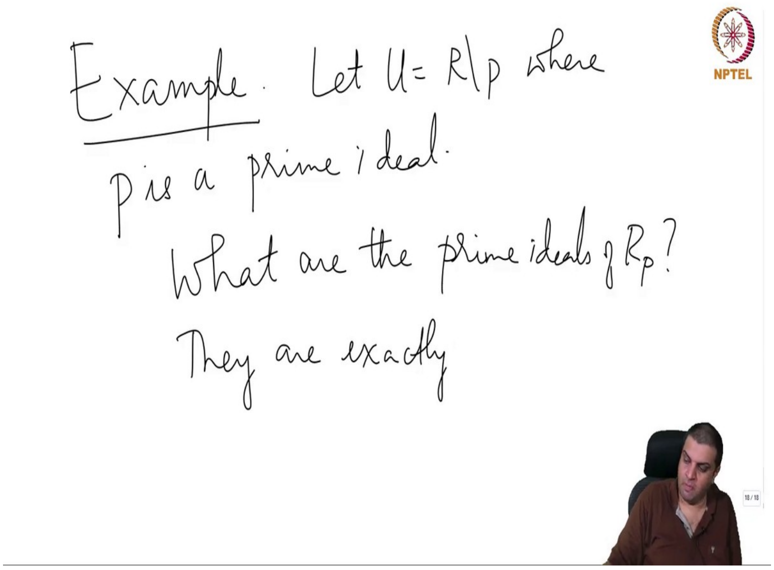

For example, so let us look at the example that we saw. Let $u = R \setminus p$, where p is a prime ideal. Of course, fields are local rings, but again that is not if these were the only local rings it map this notion may not be of much use. So, there are other things.

Now what are the prime ideals of R_p ? So, the earlier proposition which in part we proved in path within proof gives the following, which is that they are exactly those of the form

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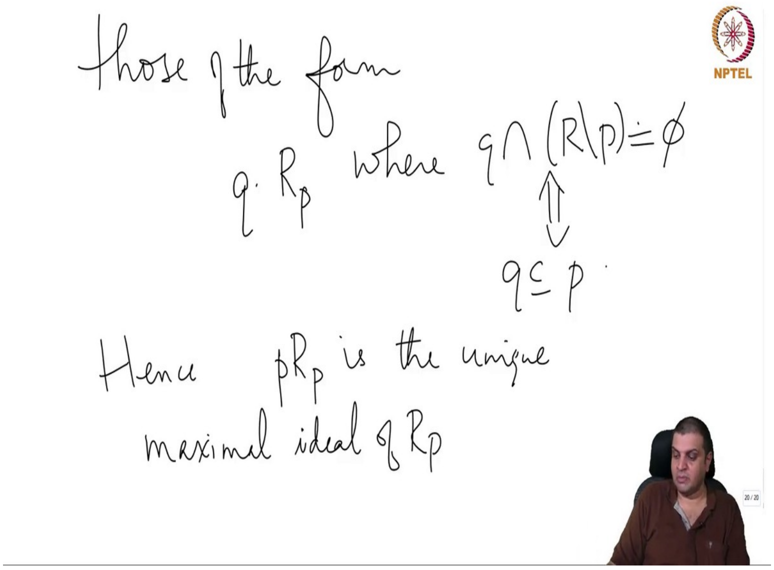

qR_p , where $q \cap U = \varnothing$, and this condition is just the same thing as $q \subset p$.

So, this is what that proposition said that, the prime ideals that do not become the whole ring after localising are exactly the prime ideals that do not that do not intersect the multiplicative closed set and when we extend those things you will get prime ideals and these are the only prime ideals of the localised ring of $U^{-1}R$. So, these are exactly these form. So, hence the extension of pR_p is the unique maximal ideal of R_p .

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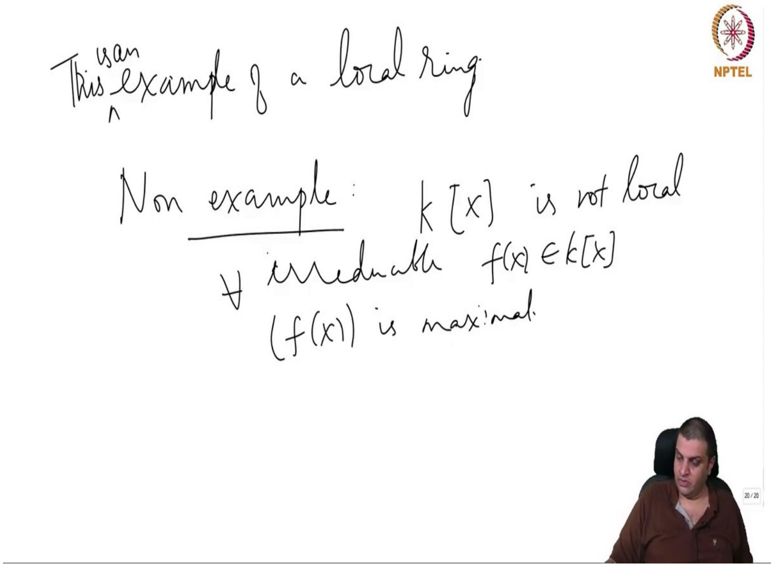

So, this is an example of a local ring. There are other local rings also which does not come out as localisations like this immediately.

Another example and this is very closely related to a polynomial rings in a way. So, let me just say non example, k[x] is not local. why? For every irreducible $f(x) \in k[x]$, $(f(x))$ is maximum and there are infinitely many irreducible polynomials.

So, therefore, there are enough. So, this is not a I mean one of the rings that you will become familiar with in the course of this these lectures is actually not local.

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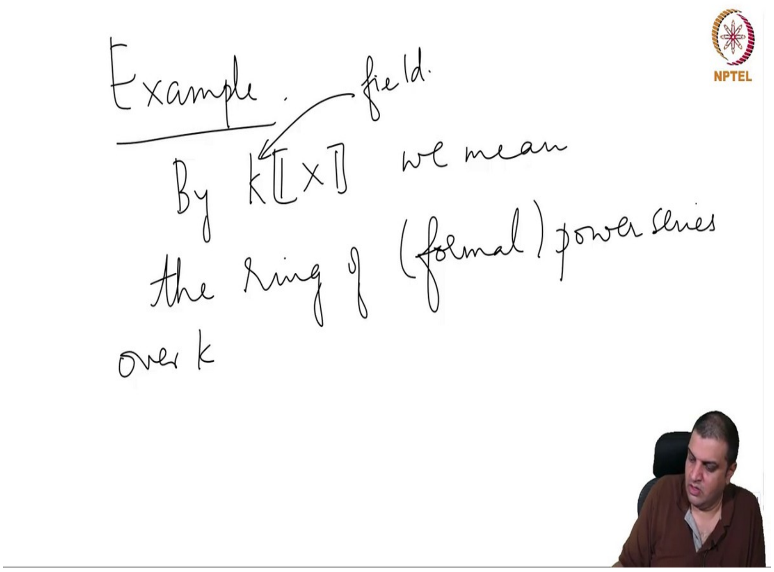

But an example, by and by one can do it for more variables, but just to get familiar let us do one, we mean the ring of what is called formal power series over k. So, let us again put k to be a field .

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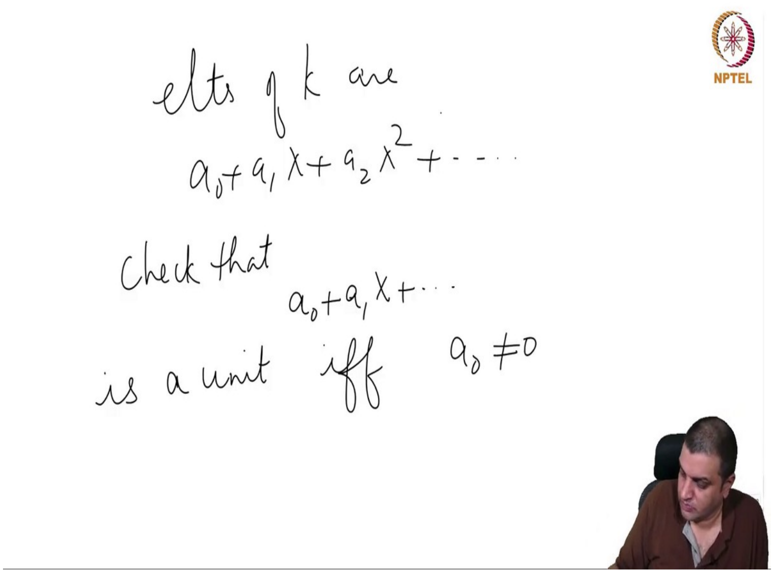

So, what are these? This is just say elements of k are of the form $a_0+a_1x+a_2x^2+\dots$ and no statement like, so unlike for a polynomial for this to be a polynomial it has to be 0 after a while, in finitely many stages it has to become 0. Here there are no such conditions. And then one can check that $a_0 + 1 a_1 x + \ldots$ is a unit iff and only if $a_0 \neq 0$. So, that is one statement one can check.

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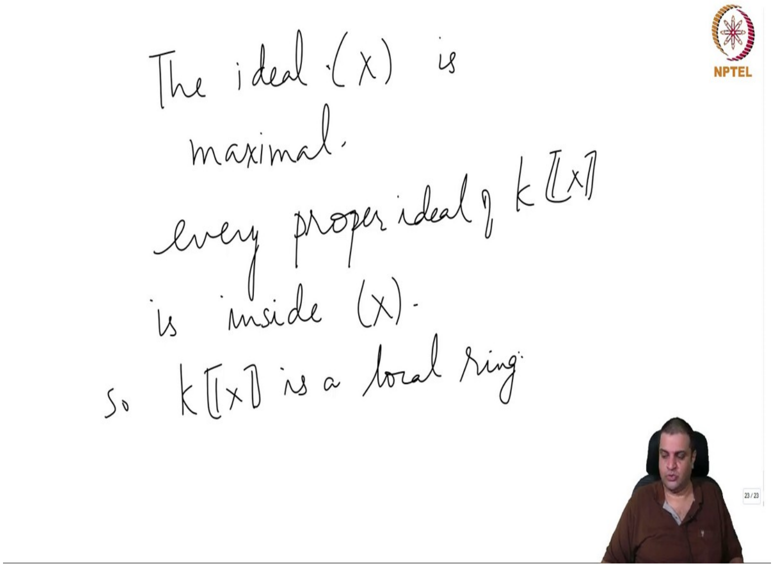

The ideal (x) is maximal. And every proper ideal of k[[x]] is inside (x). So, this is a local ring. So, various algorithms that one can work in polynomial ring can also be extended to such power series rings, but that is not something that we will pursue we see at all in this course.

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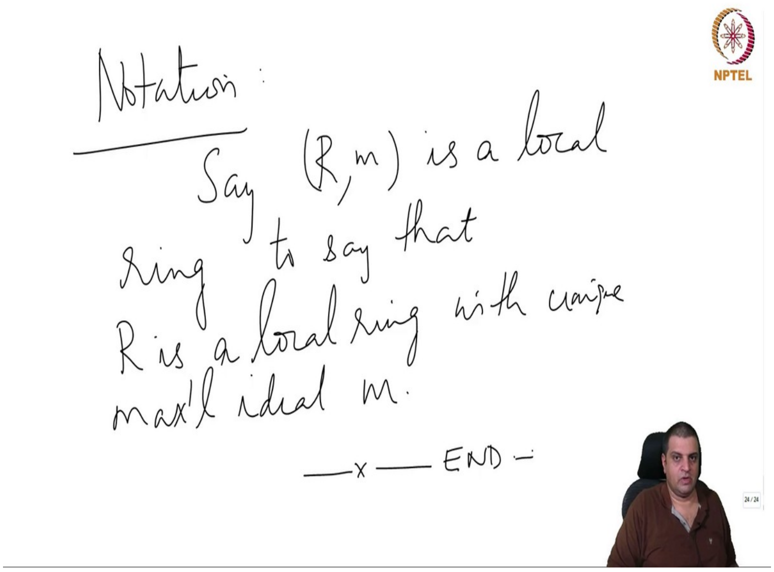

So just one piece of notation before we finish. By when we say, we say we say (R,m) it is a local ring to say that R is a local ring with unique maximal ideal m .

So, that is the end of this lecture and in the next lecture we will look at what is called determinantal trick and Nakayama's lemma and then we will try to understand some idea of the topological aspects of spec(R).