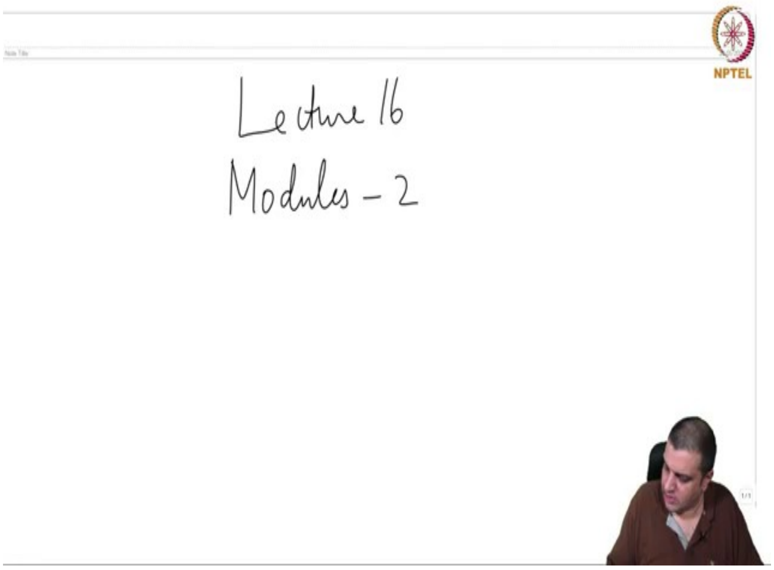
Computational Commutative Algebra

Prof. Manoj Kummini
Department of Mathematics
Chennai Mathematical Institute
Indian Institute of Technology, Madras

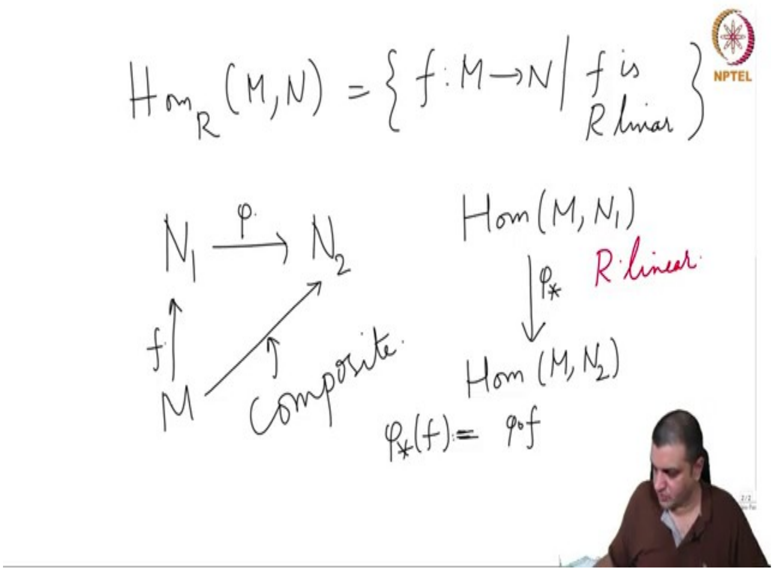
Lecture – 16
Modules – Part 2

(Refer Slide Time: 00:18)



So, we continue our discussion about Modules and we will also see some Macaulay examples .

(Refer Slide Time: 00:29)

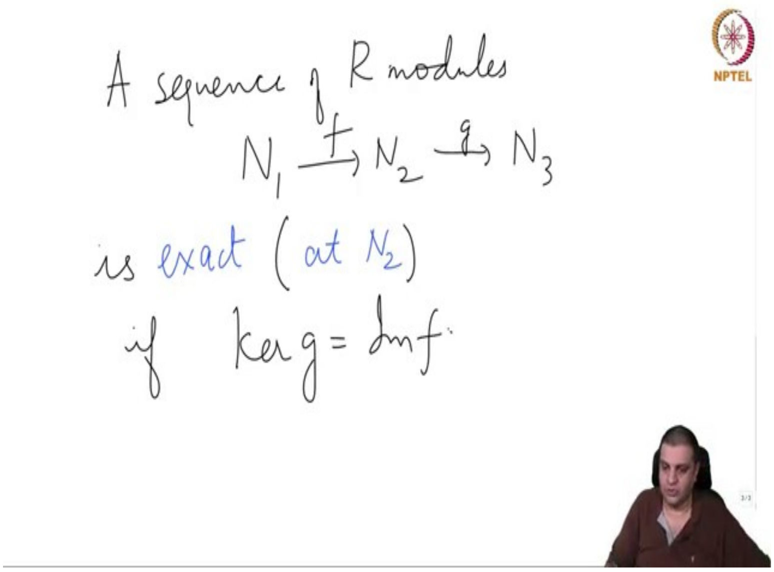


So, the first thing that we would like to observe using the Hom module that we saw last time $Hom_R(M, N) = \{f: M \to N : f \text{ is } R-linear\}$ is set of R linear map from M to N.

So, suppose we had $\varphi : N_1 \to N_2$ and we have a map from $f: M \to N_1$. If we compose this we would get a map $M \to N_2$. So, this is the composite , what it does is think of this as something fixed that N_1 to N_2 is some fixed map, then for any f there exist a map here which is a composite. So, what have we obtained? We have obtained a map from $\varphi_{star} : Hom(M, N_1) \to Hom(M, N_2)$.

$\varphi_{star}(f) = \varphi f$ is you first apply f to get to N_1 and then you apply phi. So, this is the definition of what φ_{star} means and one can check that this is R-linear. So, this itself is. So, remember these are R-modules and this is an R-linear map. So, given this data one gets for every M one gets a map like this. So, this is one observation.

(Refer Slide Time: 02:49)

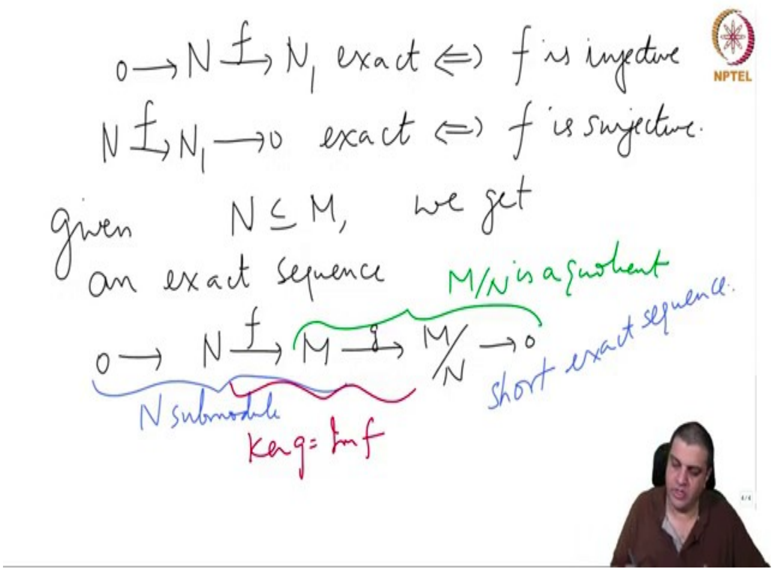


So, now we would like to define what is called an exact sequence of R-modules $N_1 \to N_2 \to N3$ is exact.

So, whenever we say exact it only it only applies at a place where there is a map coming in and a map going out. So, right now sorry I am just trying to define this term exact (at N_2) one cannot talk about exactness here or there the way we have.

If sorry we need some names for these things called this thing f and call this thing g $ker(g)=im(f)$. So, this is what we have.

(Refer Slide Time: 03:53)

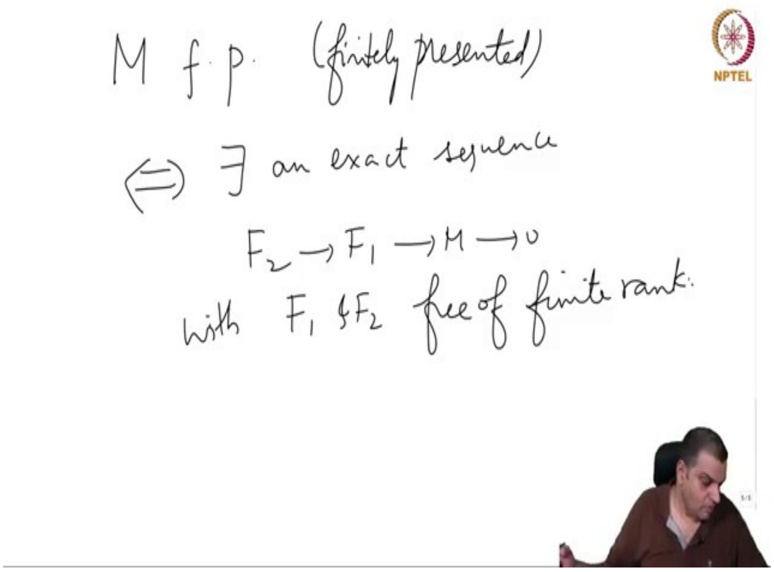


So, $0 \rightarrow N \xrightarrow{f} N_1$ exact means its exact at N because that is the only place where there is a map coming in and going out if and only if f is injective.

Similarly, some $N_2 \xrightarrow{f} N\,1 \rightarrow 0$ exact again exact in the mid, that is the only place where the question make sense if and only if again let us call this map f; f is surjective and given $N \subset M$ we get an exact sequence $0 \rightarrow N \xrightarrow{f} M \xrightarrow{g} \frac{M}{N} \rightarrow 0$ that is just to say it is an submodule first map is injective and then $\frac{M}{N} \rightarrow 0$. So, this just says $M \rightarrow \frac{M}{N}$ is surjective. This much says $\frac{M}{N}$ is a quotient module and the middle part says. So, let us label this map f and g ,middle part says $Ker(g) = Im(f)$ which is precisely what it is the $ker(M \rightarrow \frac{M}{N})$ = N. So, given a submodule inside a module we get an exact sequence like this.

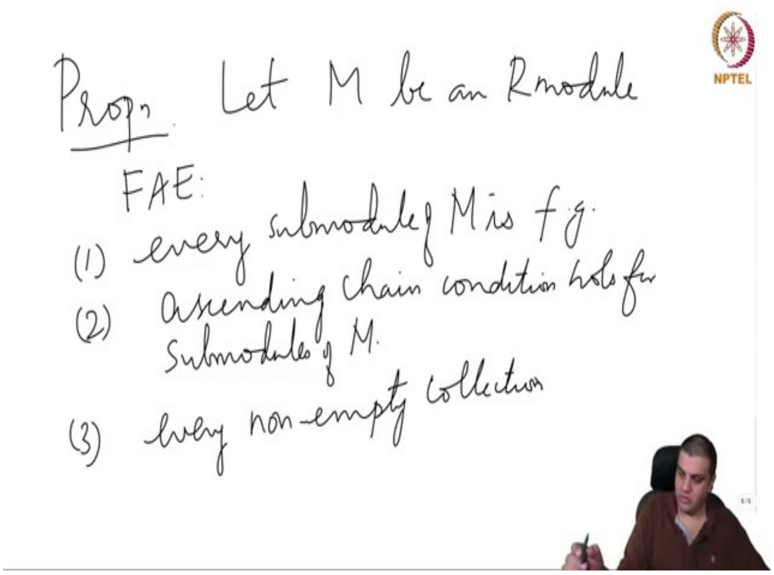
This such a thing is called short exact sequence to say that to denote that there only three at most three nonzero terms. The left 0 is to indicate that this is injective the right 0 is to indicate that this is surjective. So, such a thing is called a short exact.

(Refer Slide Time: 06:29)



So, if you remember what we did in the finite presentation. So, M is finitely presented means that there exist an exact sequence $F_2 \to F_1 \to M \to 0$ with F_1 and F_2 both are free and of finite rank. So, this is the we just restating what we know yeah in this language.

(Refer Slide Time: 07:37)



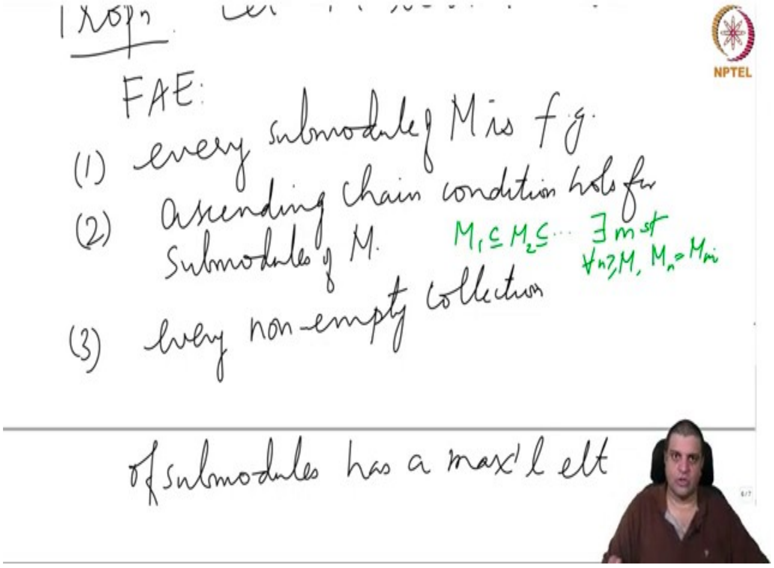
So, one reason one would like to study such things is the following proposition, let M be an R-module then the following are equivalent.

1) every submodule of M is finitely generated.

2) ascending chain condition holds for M for sub modules of M.

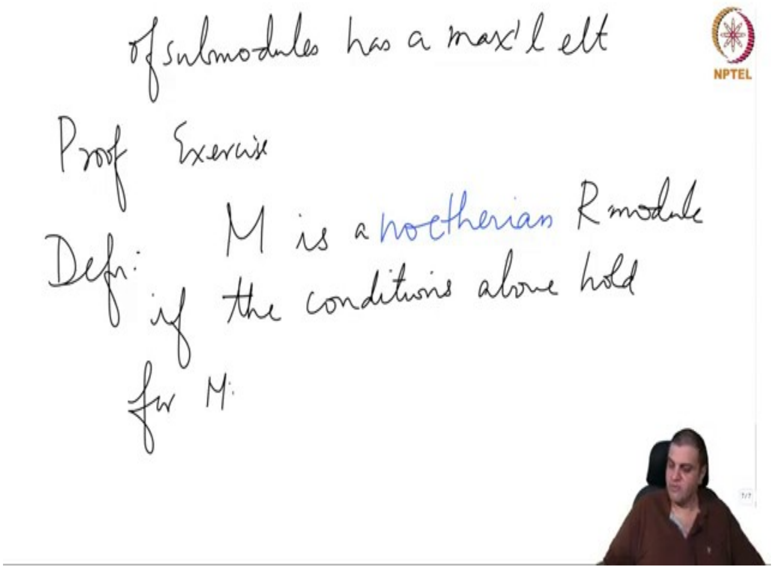
3) every non-empty collection of submodules has a maximal element.

(Refer Slide Time: 08:44)



So, ascending chain condition means, what we similar to what we discussed for ideals if you have $M_1 \subset M_2 \subset \dots$ let us that is called an ascending chain, then there exist m such that for all $n \geq m$, $M_n = M_m$ after a while it stays constant.

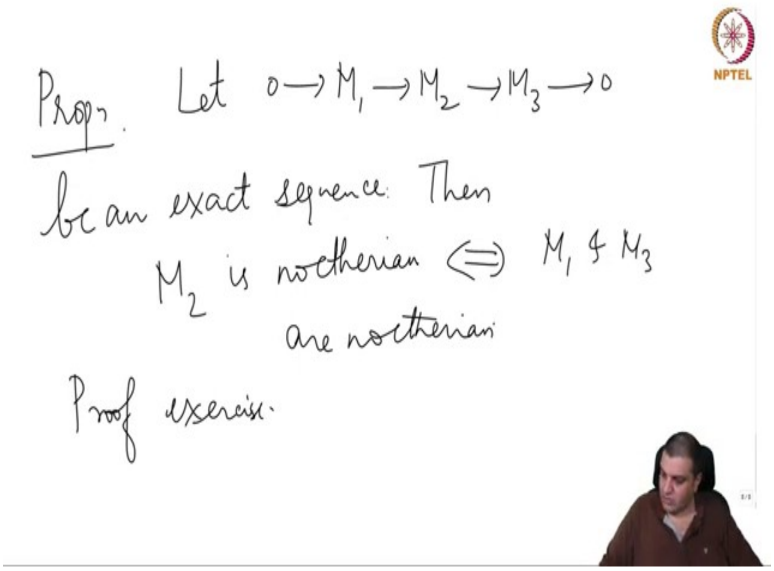
(Refer Slide Time: 09:25)



So, I the proof is the same identical to the proof there we did for ideals. So, this is an exercise and then now we define M is said to be noetherian. So, we are not saying anything about R we just saying M is noetherian. So, we say M is noetherian R-module if the conditions above hold for M. I mean if any single one holds the other two also will hold.

So, any one of them holds for M. So, then M is said to be noetherian. So, why did we set up all I mean one of the reasons its sometimes easy to describe in terms of exact sequence is the following proposition.

(Refer Slide Time: 10:21)

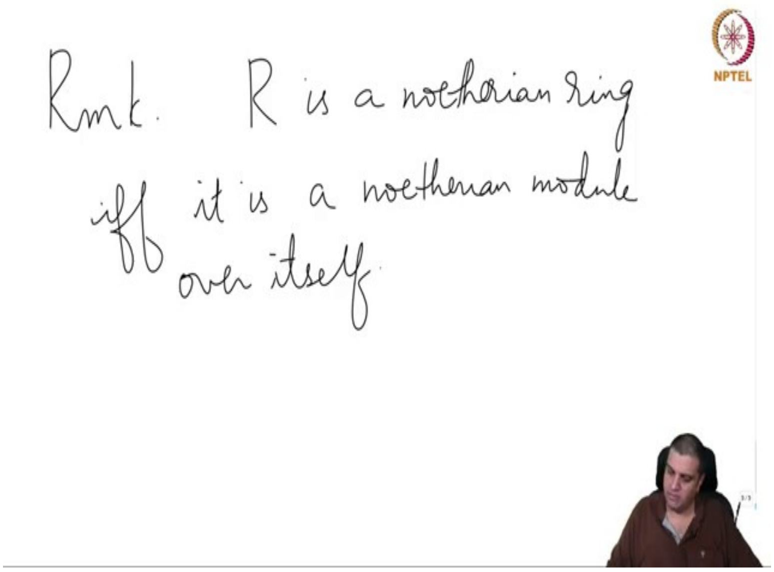


Let $0 \to M_1 \to M_2 \to M_3 \to 0$ be a short exact sequence short because they are only three at most three nonzero terms you can say.

Then, M_2 is noetherian if and only if M_1 and M_3 are noetherian proof is an exercise not very difficult. If M_2 is noetherian every submodule is finitely generated a submodule of M_1 is also submodule of M_2.

So, every submodule of M_1 is say finitely generated; similarly if you have a submodule of M_2 its inverse image here will be a submodule of M_1 and then you can that is finitely generated and that finite generating set is enough to generate its image inside M_3 and then your work a little bit to prove the other direction. So, all that is an exercise.

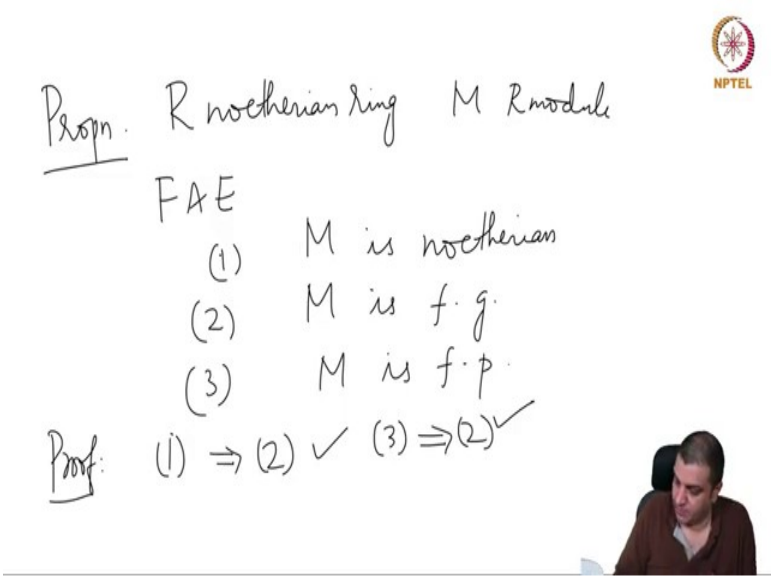
(Refer Slide Time: 11:40)



So, now just one remark R is a noetherian ring, this is something that we defined a while ago to me that every ideal is finitely generated if and only if it is a noetherian module over itself. I mean if we consider it as a module over itself then it is a noetherian module.

That is that is because the R-submodules of R are precisely the ideals. So, in this study of commutative algebra as well as its applications to algebraic geometry, number theory, noetherian rings and noetherian modules are quite important .

(Refer Slide Time: 12:41)



So, here is a proposition which gives us a large class of noetherian modules as which are also I mean in the important in applications.

So, let us say R is a noetherian ring, M an R-module then the following are equivalent:

1) M is noetherian,

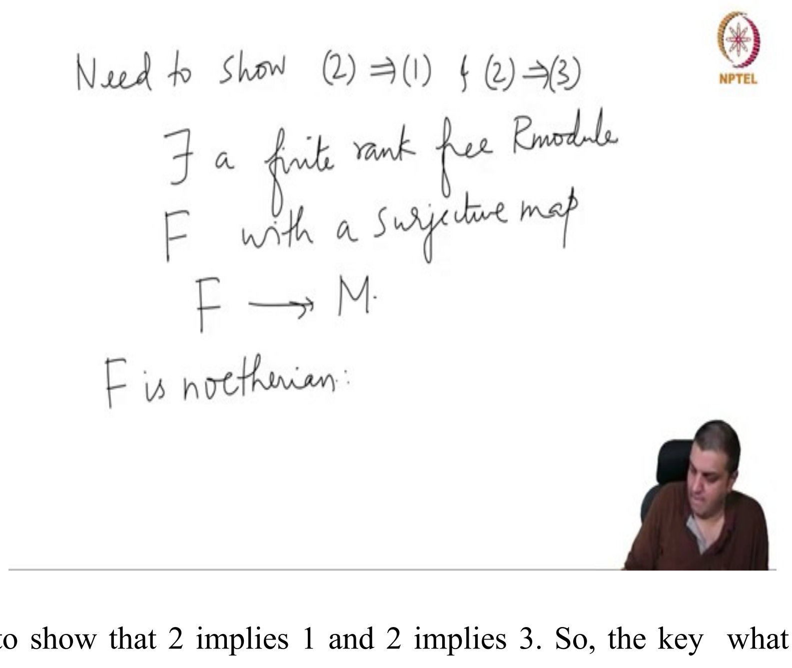
2) M is finitely generated.

3) M is finitely presented.

So, there are some implications that are immediate from the definitions if M is noetherian then every submodule of M is finitely generated. So, in particular 1 implies 2 M itself is a submodule of M. So, it is finitely generated by definition.

Similarly, finitely presented assumes its already finitely generated. So, that implies 3 implies 2 hence we need to show that 2 implies 1 and 2 implies 3.

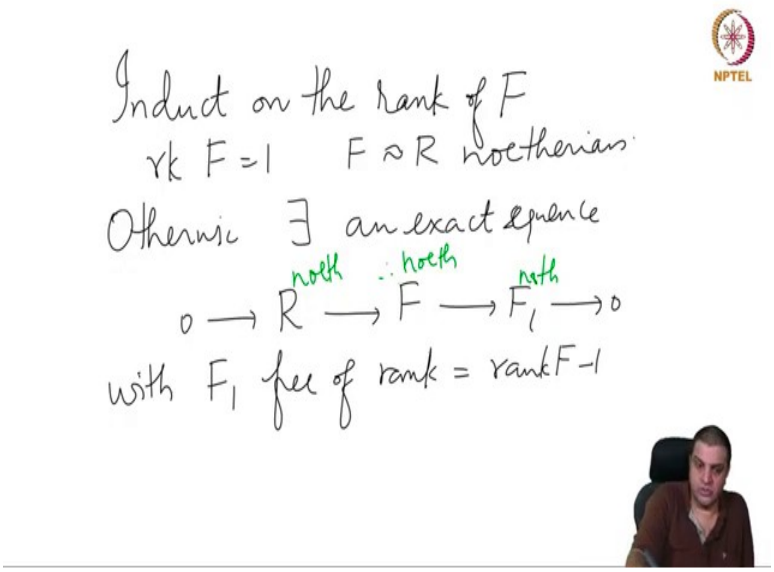
(Refer Slide Time: 14:08)



So, we need to show that 2 implies 1 and 2 implies 3. So, the key what do we know about a finitely generated R-module that there exists a finite rank free R-module F with a surjective map $F \to M$. So, this is what hypothesis 2 tells us.

So, the first observation that we would like to make is F is noetherian why is that so?

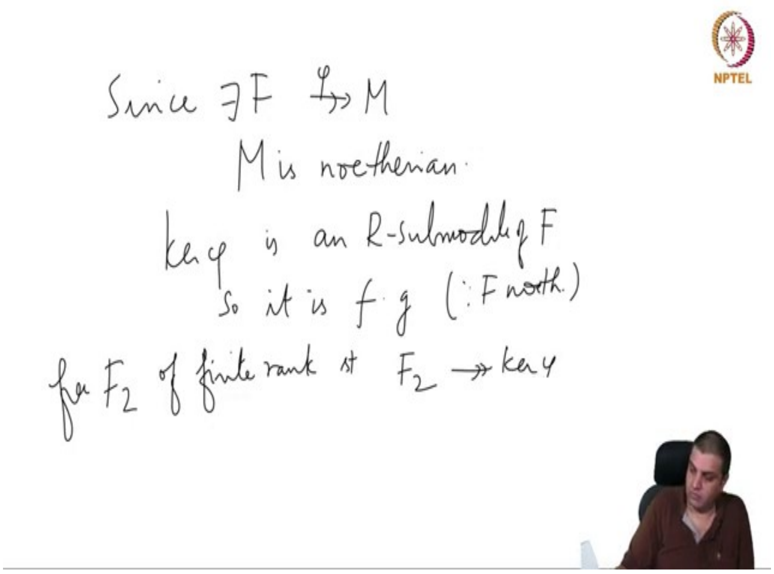
(Refer Slide Time: 15:25)



So, this is induct on the rank of F. By assumption if $rk(F)=1$, then F is isomorphic to R which is noetherian otherwise there exist an exact sequence $0 \to R \to F \to F_1 \to 0$. So, this you should prove and you will prove this is what is called as split exact sequence with F_1 free of rank equals $rk(F)-1$.

So, its exactly like vector spaces in this situation. So, R is noetherian; F_1 is noetherian therefore, we can conclude that F is also noetherian.

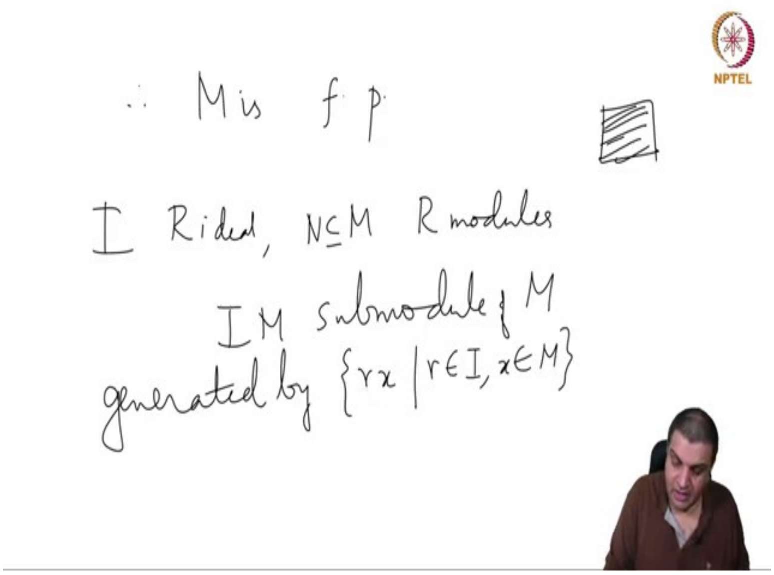
(Refer Slide Time: 16:47)



Since there exist a surjective map $F \to M$, M is noetherian.

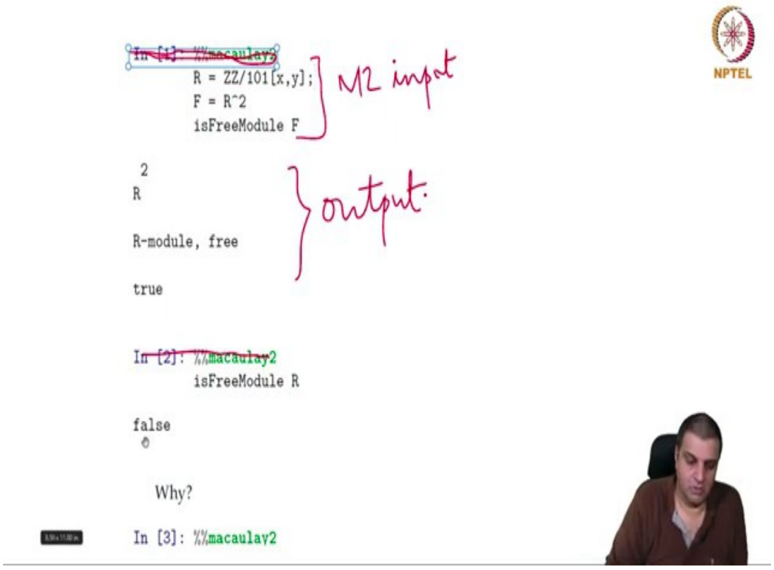
Now, let us call this map . kernel of φ is an R-submodule of F. So, it is finitely generated that is because F is noetherian, again we are using the fact that F is noetherian So, therefore, there is an F_2 free of finite rank such that F_2 surjects on to $ker(\varphi)$.

(Refer Slide Time: 17:55)



So, therefore, M is finitely presented. So, this is the proof that over a noetherian ring the three notions of a module being noetherian, module being finitely generated and module being finitely presented are the same. This is an important situation where many questions that we would like to understand or we study the situation would be some variation of this thing.

(Refer Slide Time: 18:26)

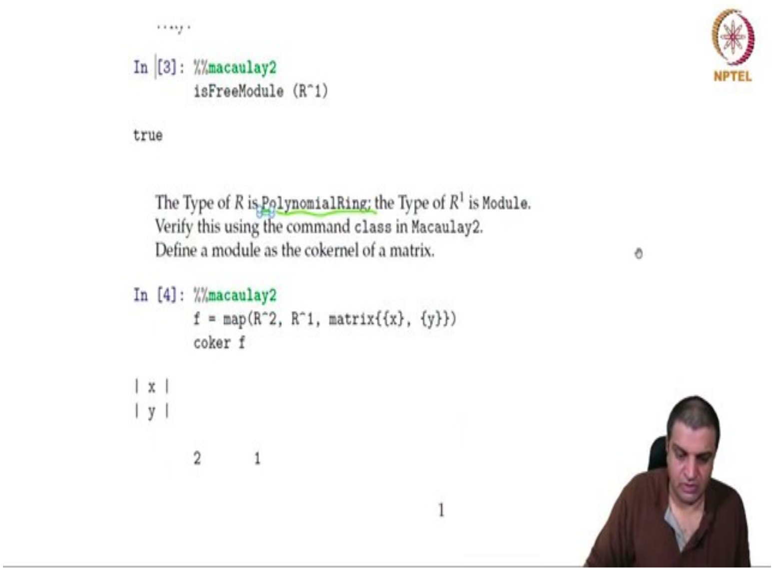


So, let us look at Macaulay example, here again let me just emphasize this first line should be just ignore altogether, this is part of the code in the right part of the program that writes the code takes the output and writes the documentation. So, just ignore that altogether and this is the input part the one that is the offset from the left is input, Macaulay2 input and this is the ah. So, this much in now type set are found is the output.

So, ignore this thing altogether. So, similarly that is just some comment line at the top ignore that. So, let us see what are we doing we ask $R=\frac{Z}{101}[x,y]$ then the way we specify a free module is a caret and a number. So, this is a free module of rank 2, then we ask is free module.

So, this is a function which takes an input module and returns Boolean output and here it says it is free. So, then we ask isFreeModule R it says false and why?

(Refer Slide Time: 20:07)



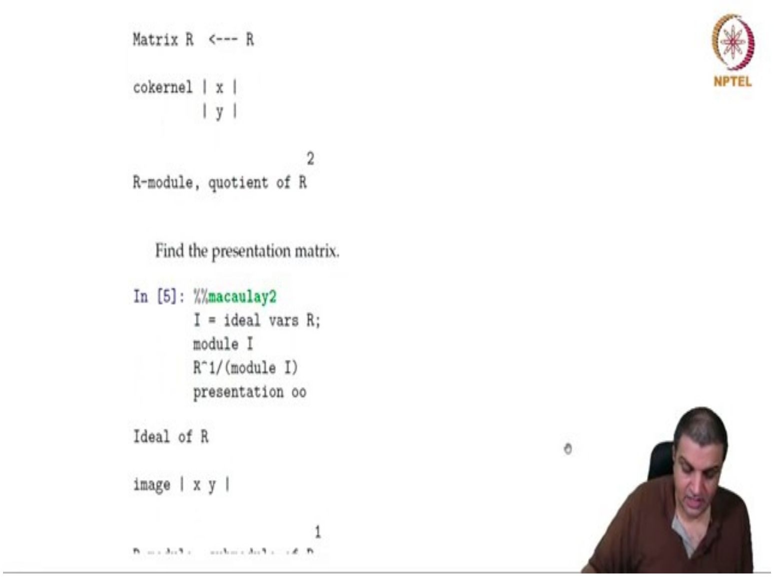
Well, So, R itself is a Polynomial Ring not a module I mean the type of R in Macaulay's R and so, you would give R to the 1. So, we ask its FreeModule R, I mean R super I mean R carat caret 1 like a superscript then it says it is true.

So, one can verify this using the command to check what the type of some quantity is class and if you check this you will you will get these two one thing. Sometimes if some functions give unexpected answers its a good idea to check these things or if you are using a function then its better to check what type of the input should have and we are going to use output what type it has.

So, now, we can define. So, we saw earlier that if you have a finitely presented module over it can be written as a cokernel of a matrix. So, here this is what we do we define a and f = map(R^2, R^1, matrix{{x}, {y}}) to be a map to the free module R^2 from the free module of rank 1. So, remember the maps arrows always in the right to left in Macaulay and the. So, this is a matrix its a matrix. So, each. So, a matrix is a list of lists each individual list in the list is the rows.

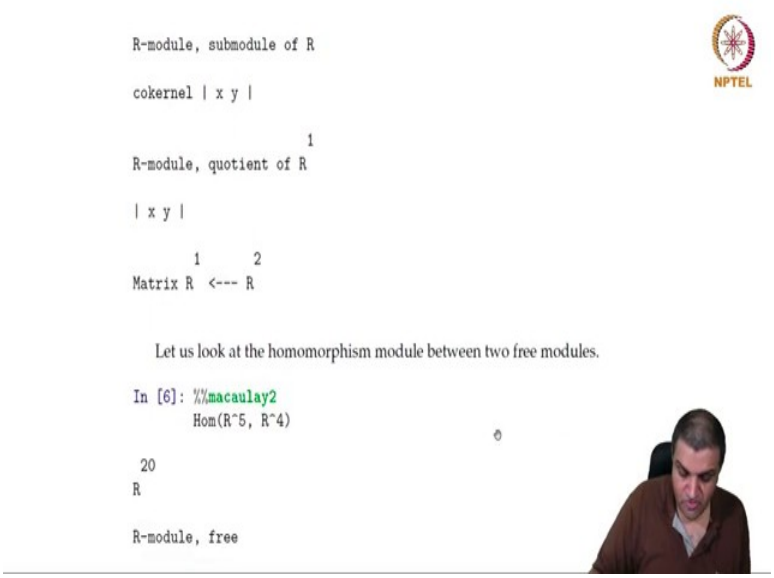
So, this is a 2 by 1 matrix. So, here its 2 by 1 matrix. So, this is what it does and yeah.

(Refer Slide Time: 21:55)



So, here it just constructed the cokernel of this matrix.

(Refer Slide Time: 22:03)

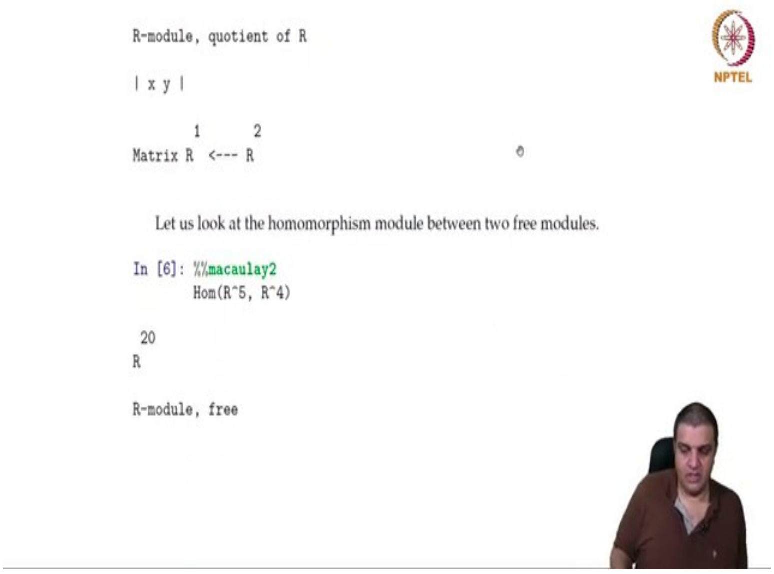


We can given a matrix we can ask for its presentation matrix these are over noetherian. So, anything finitely generated its finitely presented. So, it will it tell us of the presentation matrix. So, here we define I to be an ideal of R and we write this command called module of I which is to which is to construct the module corresponding to I.

Its the same its the same abelian group, but now its type has changed to module and its something new. So, these are required. So, to keep then we write R_1/ (module I). So, this is again this is this is still $\frac{R}{I}$, but now thought of as a module and not a quotient ring and we asked for his presentation matrix and here the ideal I was the idea generated by the variables of R. So, it is a maximal ideal in this case and here we get R- submodule of R^1.

Sorry I jumped one step. So, the output of module I is this thing. So, if you we gave I. So, its an ideal of R then we asked for module of I its an image of this R-submodule of R^1, then we asked R^1/ module(I), then it said R-module quotient of R^1 and it just gave the matrix for which it is a coefficient its a its said its a cokernel and then we asked for its presentation matrix it showed as the matrix.

(Refer Slide Time: 23:45)

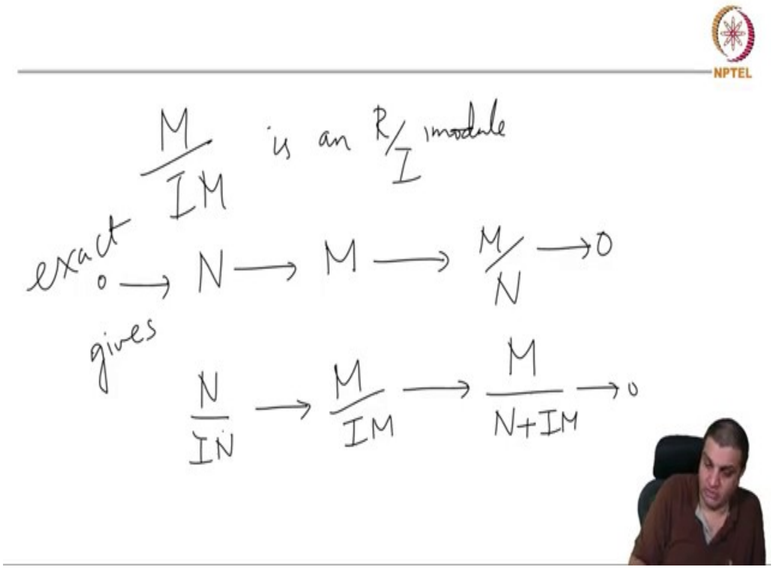


So, we asked here this is the input line Hom(R^5, R^5) what is it? It says it says R^20 its a free module of rank 20. So, this will you will do as an exercise if you have a free module of rank a and a free module rank b Hom between them is of rank a b free of rank a b this would be one of the exercises.

Let us discuss a few more things if you have an ideal. So, I an R-ideal, $N \subset M$ R-modules. So, this means I also implicitly mean that it is an R-submodule, now not just a subset. So, if you have this then we can define $\frac{M}{I\,M}$. So, observe that IM is a submodule of M generated by $\{r\,x : r \in I, x \in M\}$

So, its it is like taking the square of an ideal, but here its two different one is an ideal one is a module again just taking the products is not going to give us a submodule, we have to take the submodule generated by them meaning finite R-linear combinations of such things. This $r \in I$, but when you take combination is the coefficient can come from R.

(Refer Slide Time: 25:46)

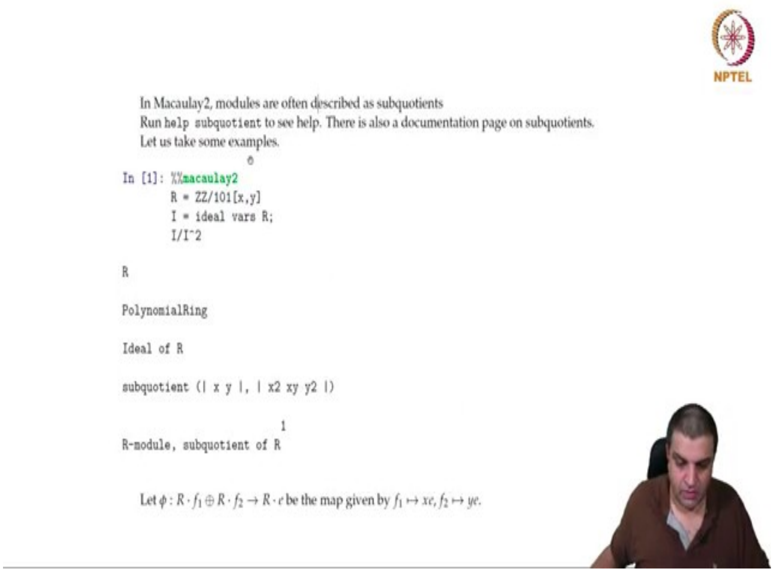


So, this IM is a submodule. So, we can talk about $\frac{M}{I\,M}$ is an $\frac{R}{I}$-module in a natural way.

So, we are in exact sequence $0 \to N \to M \to \frac{M}{N} \to 0$. So, now, what if we further go modulo the submodule generated by I inside them. So, we will get $\frac{N}{I\,N} \to \frac{M}{I\,M} \to \frac{M}{N+I\,M} \to 0$

So, in addition we are setting IM to 0. So, this exact . Notice that this was injective, but here there is no guarantee that this is injective, but one would get such a sequence just look at one more Macaulay case.

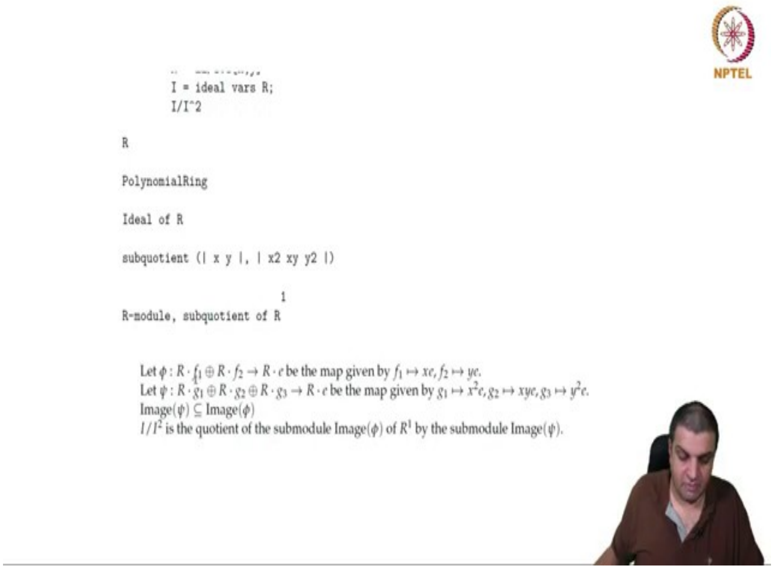
(Refer Slide Time: 27:13)



So, one just one thing that I needed to explain while using Macaulay here is what of the notion of what is called a subquotient. So, there is a help page on sub quotients. So, please read that let us just quickly explain with an example here just take an ignore this line. So, R is just a polynomial in two variables, I is the ideal generated by the variables then we ask I/ I ^2.

I/ I ^2 is not an ideal. So, it would just it is written as subquotient of this and its given two matrices then its a subquotient and it says R-module subquotient of R^1. So, what exactly does that mean?

(Refer Slide Time: 28:00)



So, let us take a map φ from a free module of rank 2 with basis $\{f_1, f_2\}$ and basis $\{e\}$. So, rank 2 here and rank 1 here in which this matrix describes the map in other words f_1 goes to xe and f_2 goes to ye that is what the first matrix here means similarly here its a map from rank 3 to rank 1.

So, $\{g_1, g_2, g_3\}$ basis and say $\{e\}$. So, this g_1 goes to $x^2 e$, g_2 goes to xye and g_3 goes to y^{2e} from here and notice that the image of the second map ψ is inside the image of φ and both of these are submodules of R^1 they the map has target R^1.

So, this image is a sub module and it says I/I^2 is the quotient of the larger submodule by the smaller submodule that is what its called a subquotient and this is a way Macaulay describes many modules and one can convert it to the usual description of quotients cokernel of a map there are commands to that we will do and as we proceed we will see these things. So, this is the end of this lecture.