Computational Commutative Algebra

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Lecture – 15
Modules – Part 1

Welcome to lecture 15. So, in this lecture and the next, we will discuss about Modules and very basic properties of these things; some of these, some other properties we will do in the exercises.

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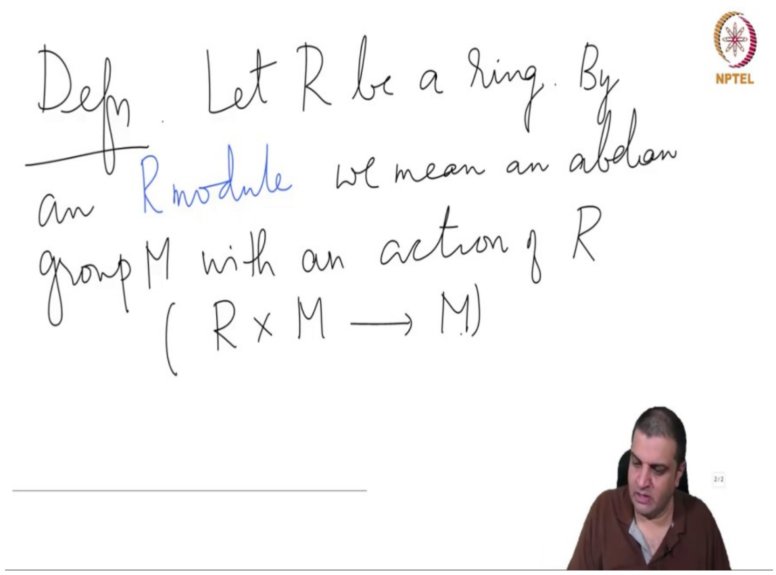

Let R be a ring. So, we have just come out of the computational part; the initial discussion of computational parts; so now from here onwards R is some arbitrary ring, commutative. By an R module, we mean an abelian group M with us, with an action of R, satisfying the following.

So, what exactly we mean by an action of $R \times M \to M$; some map from $R \times M \to M$ satisfying certain properties.

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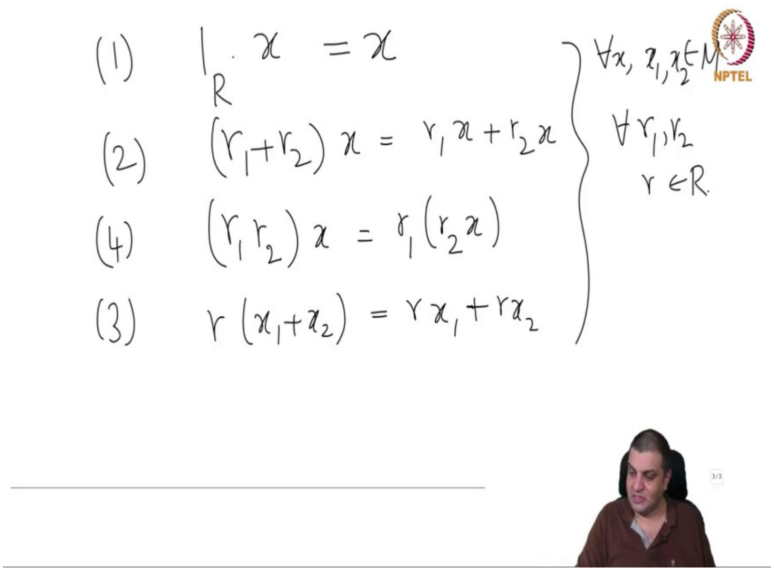

One, the element 1 of R should act on an element x of M as identity; so this is a notation. So, $r \cdot m$ is just a notation; one can think of it as scalar multiplication. And second, it should respect the addition in R. Three, $r \cdot (x_1 + x_2) = r \cdot x_1 + r \cdot x_2$. And four; it should respect the multiplication in R.

And this is $\forall x_1, x_2 \in M$ and $\forall r_1, r_2, r \in R$. So, if you think about it precisely; the definition how we would formally syntactically the same as the definition of a vector space over a field.

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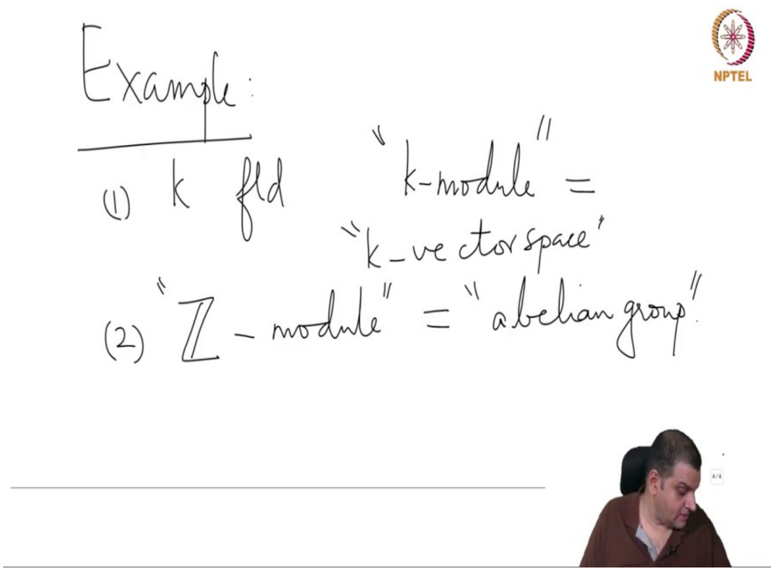

So, that is the most basic example. If, k is a field then the word k module is synonymous with k vector space; with a set is a k vector space precisely when it is a k module. 2. If for Z; what are modules over Z? The word Z module is synonymous with abelian group. So, M is an abelian group and giving it an action of Z respecting these properties, does not give it any extra structure.

So, Z module is precisely an abelian group and one of the advantage of study expressing various things in the language of modules as opposed to just studying ideals and quotient rings, is that first of all there are things that we must address and not just ideal. I mean there are objects that are genuinely modules and not ideals and quotient rings. So this is a uniform language to state and prove many theorems.

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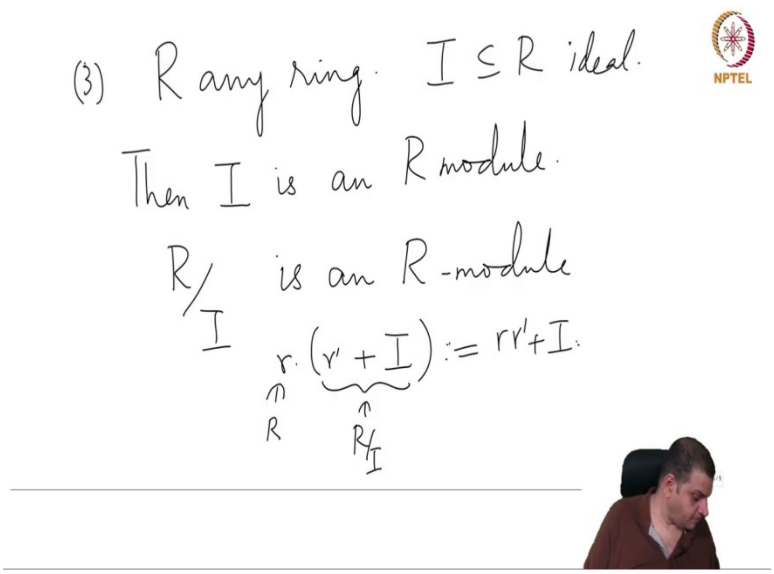

So, let me explain in a minute R any communicative ring; I an ideal. Then I is an R module and $\frac{R}{I}$ the quotient ring is an R module; in the natural way; $r \cdot (r' + I) = r r' + I$ So, this is an element of R, this is an element of $\frac{R}{I}$.

So, one advantage of discussing many things in modules is that of course, one has to worry about them in general. Even otherwise, if you usually just wanted to study, makes prove theorems about ideals and quotient rings; the language of modules gives us a uniform way of answering or stating or proving or going about proving them in a uniform way.

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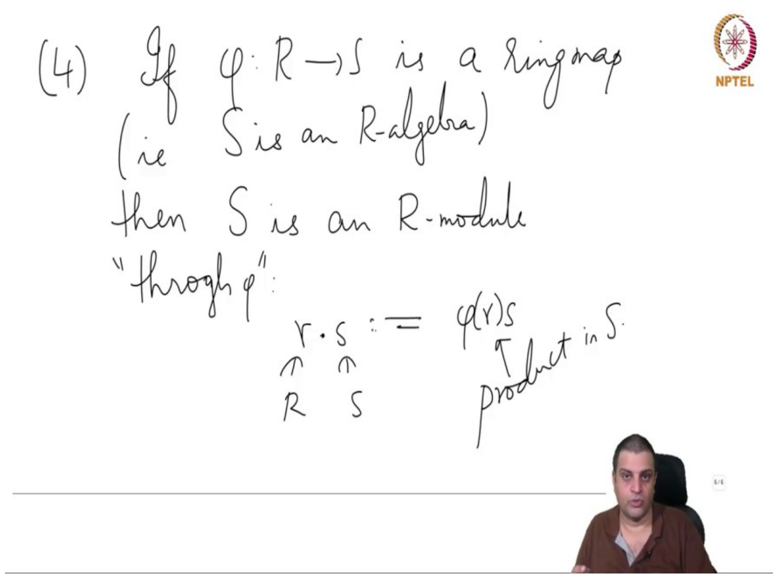

Four, If $\phi: R \to S$ is a ring map; in other words that is S is an R algebra. Then S is an R module in the following natural way. So, we will say through ϕ the map in the natural way. So, we have to say what $r \cdot s$ is; so this $r \in R$; this is inside S. We define this to be $\phi(r) \cdot s$, this is actually an element inside S.

So, here we are defining what this dot means and here this is the product. So here this is the product inside S which is already given to us. So if you have a ring map, in other words if you have an algebra then that automatically gives a module structure.

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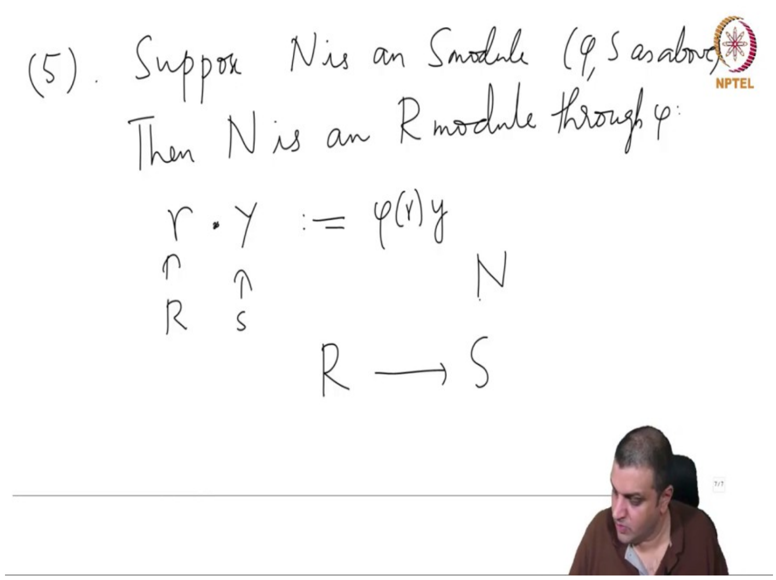

We can generalize that a little bit, suppose N is an S module. So, ϕ, S as above, then N is an R module through ϕ. In other words, if you have $r \cdot y$; this is inside R, this is inside S; again we want to define what this multiplication means.

Define it to be $\phi(r)y$. Now, this is just the S module structure; this is an element of S this is an element of N. So, we can think of it this way; we have some ring R here and a ring S here and some module over that. We can think of we can look at this abelian group as an R module through this arrow. So, this is one another observation about modules.

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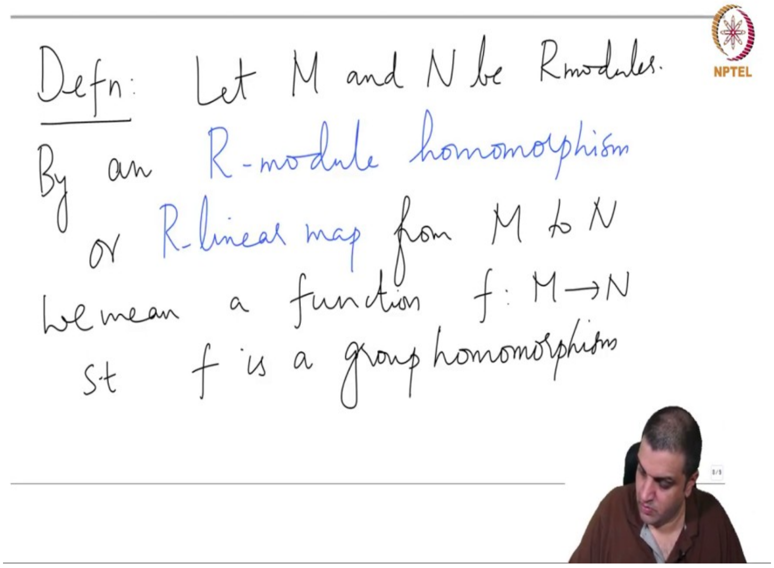

Definition: So, as soon as we introduces new structure, new class of sets; we would like to understand the maps between the functions between that set, those sets of that type which respect that structure.

Let M and N be R modules. By an R module homomorphism or a homomorphism of R modules, or just plainly R linear map from M to N; we mean a function $f: M \rightarrow N$, but M and N are not just sets they have such that there were extra structure.

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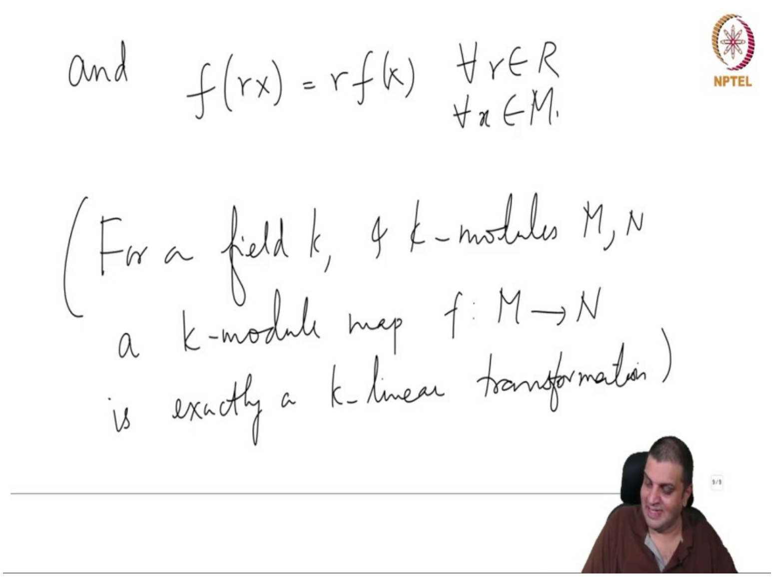

So, first of all they have an underlying structure of abelian groups such that f is a group homomorphism and $f(rx) = rf(x)$.

So, this times is in M, this times is in N, but it is all will defined $\forall r \in R$ and $\forall x \in M$. So, if you had like $r_1 x_1 + r_2 x_2$ then it is an abelians group of homomorphism. So, we can just take out the plus outside and then we can apply this. So just a remark here; for a field k and k modules M and N; a k module map $f: M \rightarrow N$ is exactly a k linear transformation.

So, really we are not doing anything I mean at least definition wise, we have not done anything new, except just restating definitions in linear algebra.

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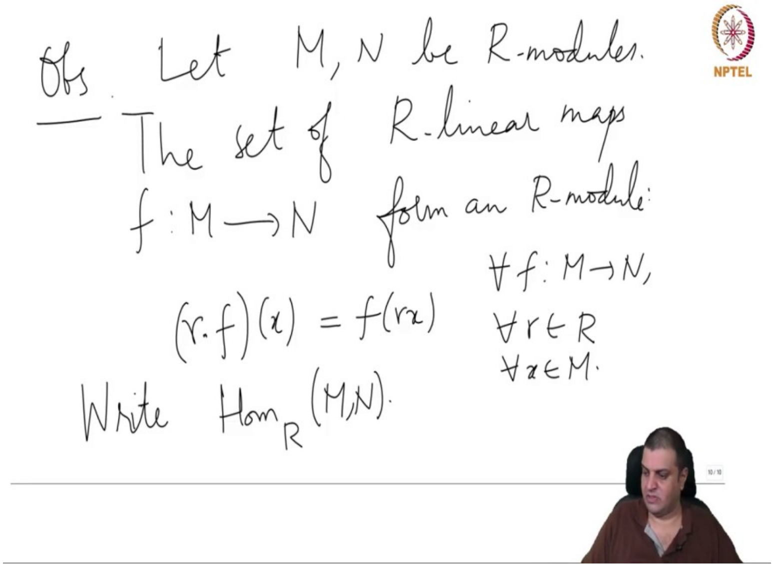

So, that is another observation that you would like to make is that the set of homomorphisms, let M and N be R modules. The set of R linear homomorphisms form an R module.

So, we have to say what the map is; it is multiplication that we have to define. We have to check that it satisfy the part of the definition, $r \cdot f$ itself is a homomorphism from M to N. So, this is going to take some x and you have to say what this x is; this is really $f(rx)$, $\forall f: M \to N$, $\forall r \in R$ and $\forall x \in M$. So we will write $Hom_R(M, N)$.

So, this is like the statement that the set of linear transformations between two vector spaces from one vector space to another vector space is a linear vector space itself. So one has to check that this satisfies the definitions. Now, we want to look at special classes of modules; so the first thing is what is called a free module.

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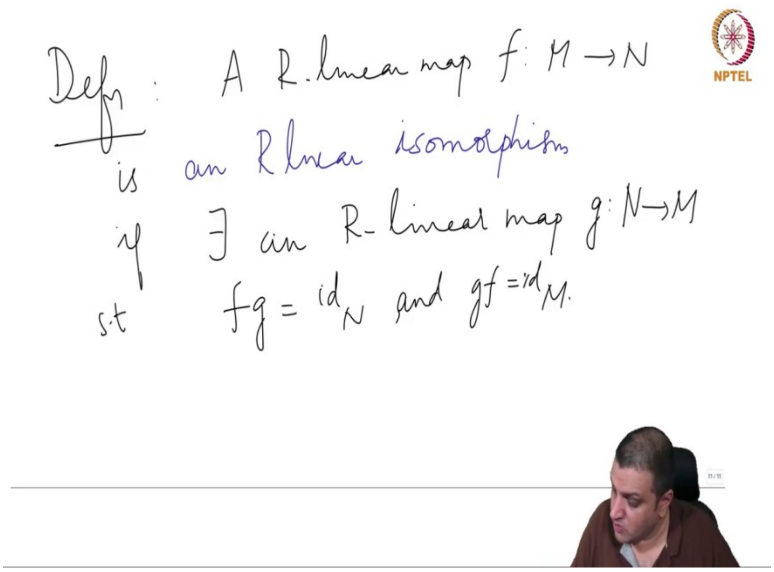

Before that one more definition. A R linear map $f: M \to N$, so I am just continuing the notation M and N are R modules is an R linear isomorphism if there exist an R linear map $g: N \to M$ such that $fg = Id_N$ and $gf = Id_M$.

So, fg is a map from N to itself; so this composite is identity and gf is a map from M to itself; this is the definition, so this is an isomorphism.

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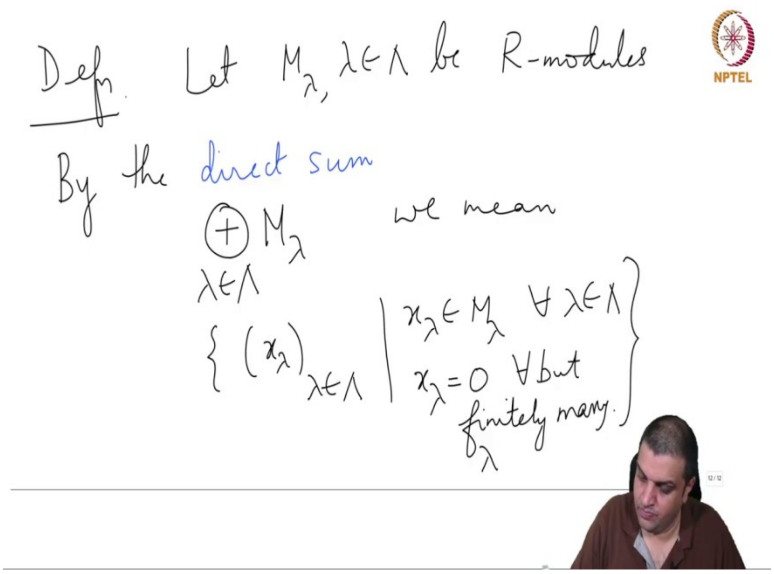

Let $M \in \lambda, \lambda \in \Lambda$ be R modules, by the direct sum which is denoted as this; so this is what would sometimes we call the external direct sum. So, we mean $\left\{ (x_\lambda)_{\lambda \in \Lambda} \middle| x_\lambda \in M_\lambda \forall \lambda \in \Lambda, x_\lambda = 0 \forall \, but\ fintely\ many\ \lambda \right\}$. By the direct sum, we mean the set which is really just take a long sequence I mean as many as there are elements in Λ, but do not take arbitrary elements everywhere, take with this condition. So, at the λ-th coordinate; you take an element from M_λ, subject to this condition.

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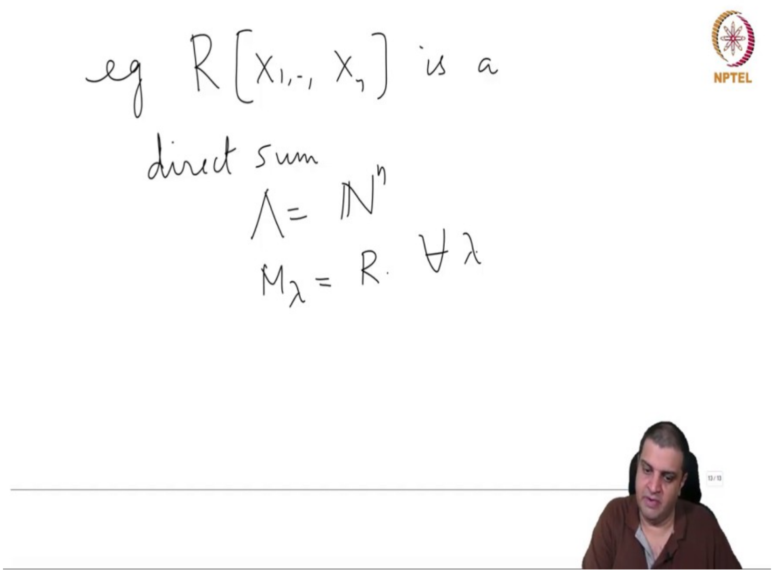

So, for example, if you take a polynomial ring in some finitely many variables is a direct sum, where the indexing set $\Lambda = N^n$ and M_λ is just R itself for all λ; so this is one example.

So, you could have an infinite infinitely many factors in the direct sum, but every polynomial will have only finitely many terms; so this is what (Refer Time: 19:36).

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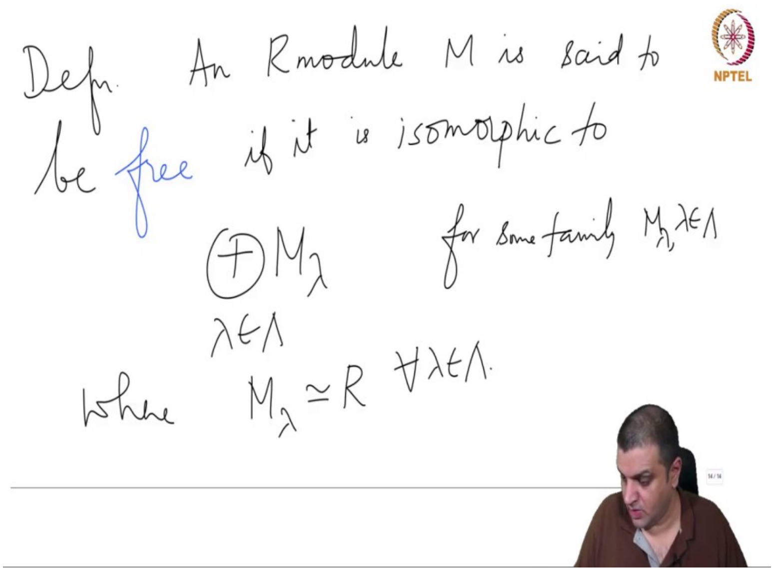

Definition, an R module M is said to be free, if it is isomorphic to a direct sum, $¿ \lambda \in \Lambda M_{\lambda}$, where M_{λ} is isomorphic to R $\forall \lambda \in \Lambda$. So, it is just a direct sum of some number of copies; not necessarily finite of the ring itself.

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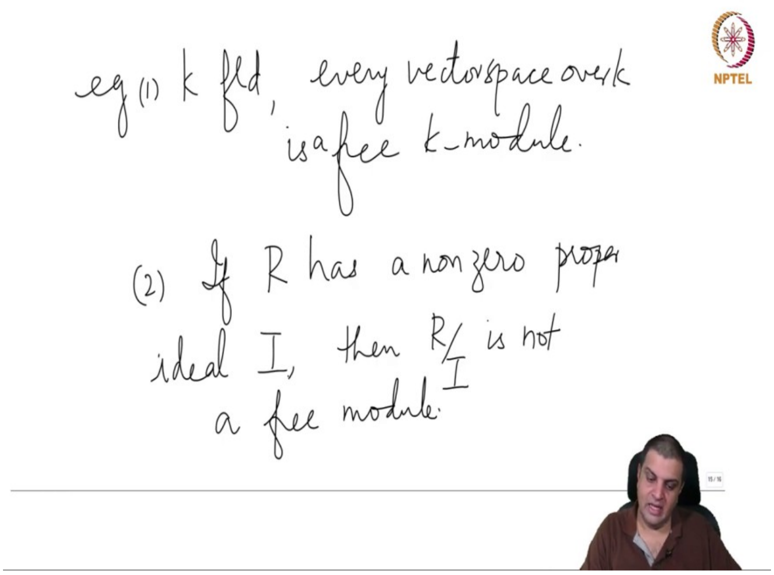

So, for example k field; every vector space over k is a free k module. So, even this definition does not introduce; it does not bring us anything new, when we discuss vector spaces over free. Two, if R has a non zero proper ideal I, for example, if R is not a field; it will have a maximal ideal different from 0.

So, any the second statement applies to everything that is not a field, then $\frac{R}{I}$ is not a free module. So, the reason is that, if you look at the definition of a free module because it is a copies of R, there are elements in it. For example, you could take one in one factor and 0s everywhere, that cannot be multiply to 0 by any non zero element of the ring.

Well, that is not true here; so it is a statement about free module. Free modules is always some element which cannot be multiplied to 0, by a non zero element of the ring. However, if you look at $\frac{R}{I}$ non zero ideal I, any element of from the definition, any nonzero element of I will multiply every element here to 0. So, this cannot be free.

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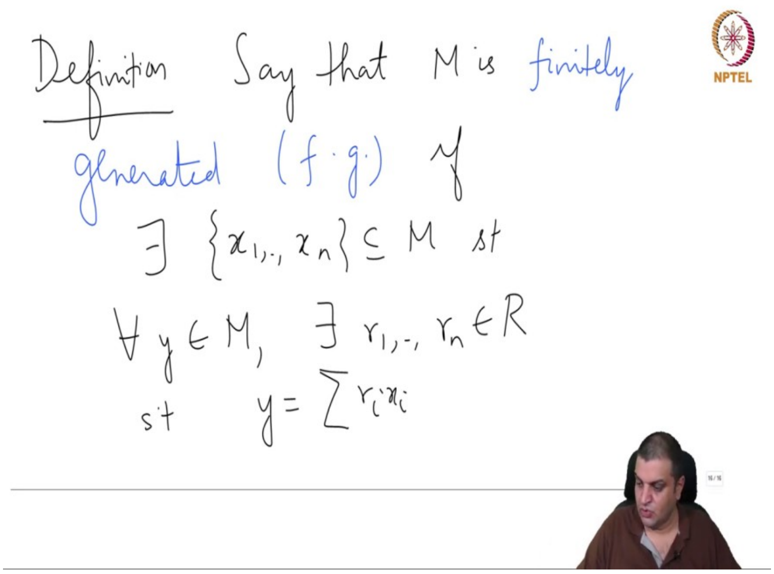

Definition: Say that M is finitely generated, we will abbreviate this as f.g. So, this is the third context in which we are using the same abbreviation to mean the same.

If there exists some finite subset $\{x_1,\ldots,x_n\}\subseteq M$ such that $\forall\, y\in M$, $\exists\, r_1,\ldots,r_n\in R$ such that $y=\sum r_i x_i$. So such a thing is called a generating set and we are saying that there is a finite generating set; so then we say M is finitely generated.

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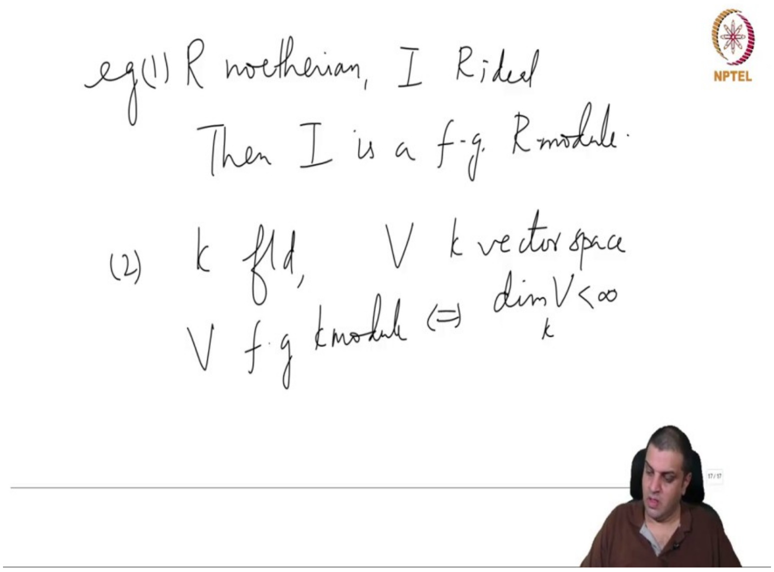

So, for example, R noetherian, I and R ideal, then I is a finitely generated R module. Two, k field, V is a k vector space, then V is finitely generated k module if and only if V has finite dimension.

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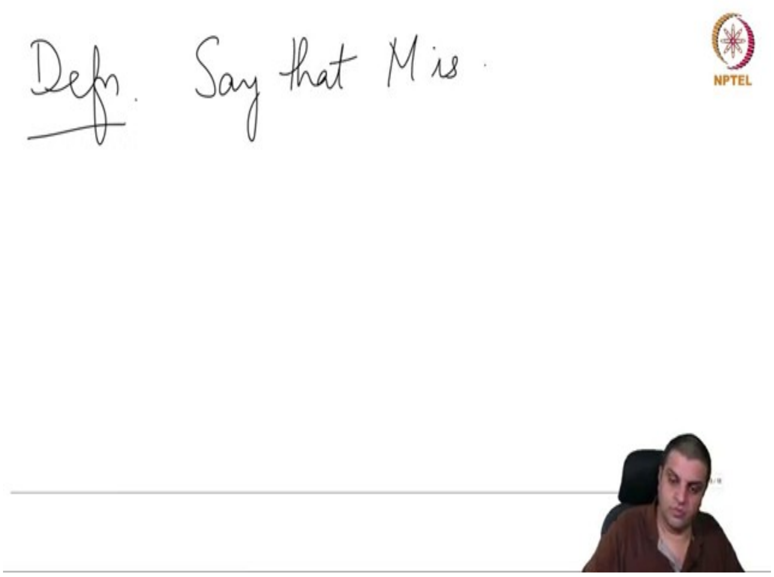

Another definition: Say that M is finitely presented; sorry before this definition, just a remark.

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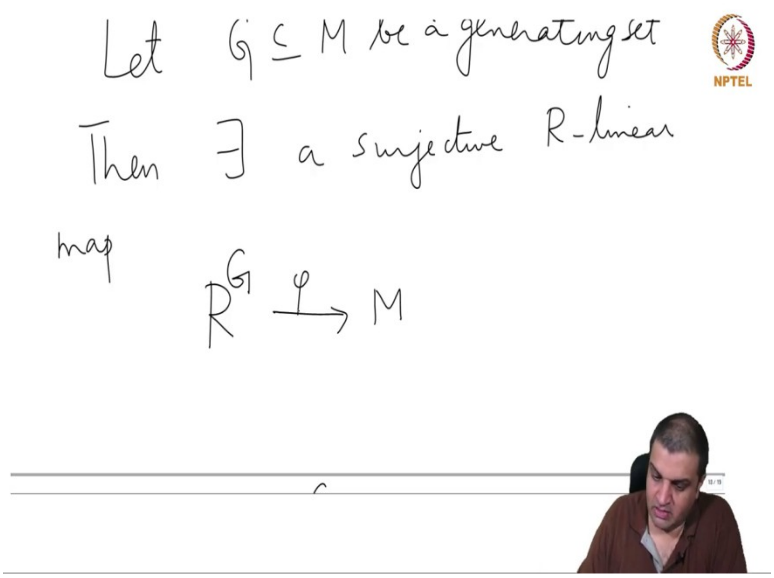

So, let G inside M be a generating set; then, there exist a surjective R linear map $R^G \to M$, where by R^G we mean the free R module with basis G.

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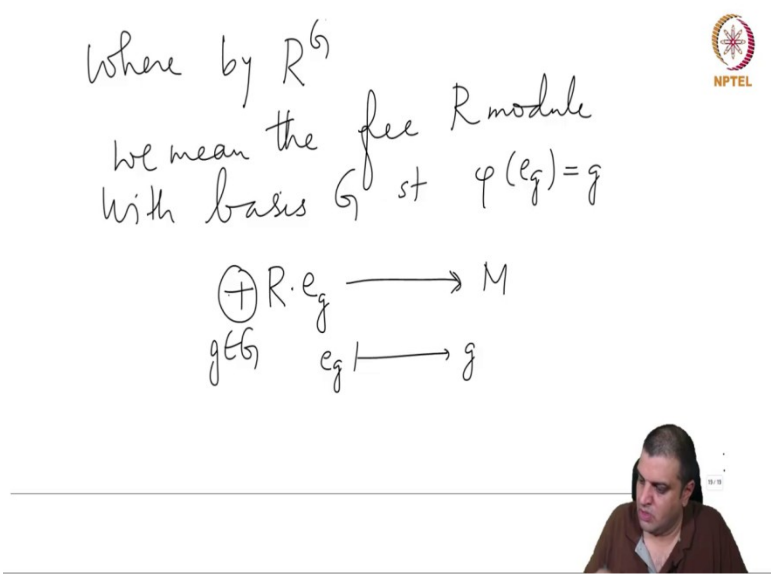

So, I did not define basis now, it will be in the exercises. So, immediately after this lecture; we should also work out that part about the basis. So, in other words it is a direct sum of R with some basis element that is indexed by G to M; such that $\phi(e_g) = g$.

So, the map is e_g here goes to g; g is an element inside M. e_g is just a dummy element indexed by the generating set. So, once you identify a generating set, then there is a map from a free module of that rank to of that many copies of R to M.

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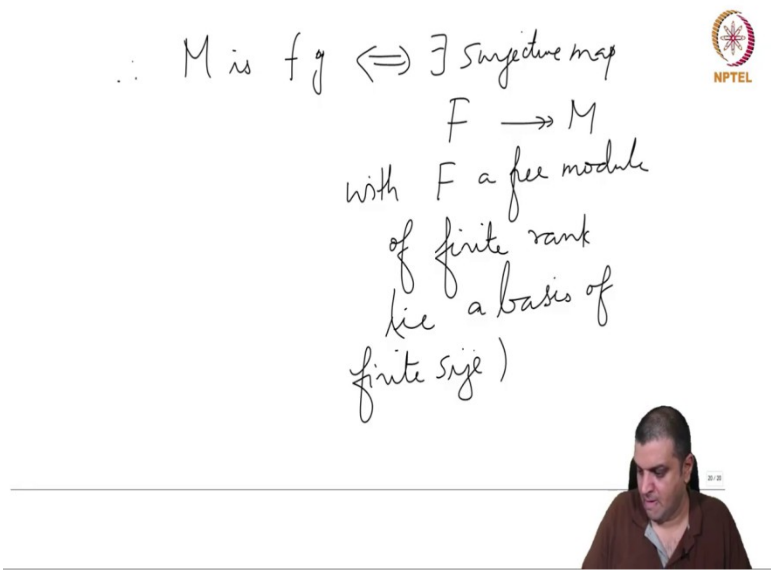

So M is finitely generated if and only if there exist a surjective map $F \to M$ with F, a free module of finite rank; that is a basis of finite size. Rank is the cardinality of the basis and if you have a base of cardinality; if you have a basis of finite size, then it is called a free module of finite rank or a finitely generated free module. And M is finitely generated precisely when there is such a map.

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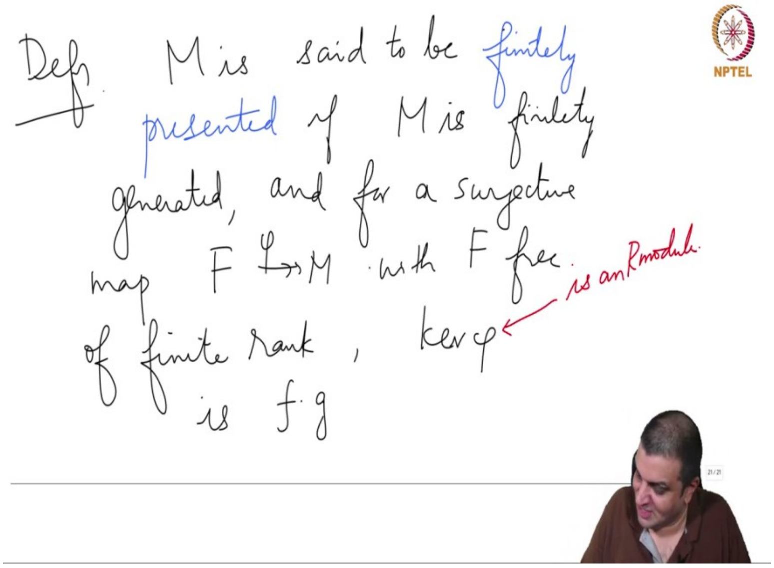

Definition: M is said to be finitely presented, if M is finitely generated and for a surjective map F to M, where F free of finite rank. So, such a thing exist because it is finitely generated; for a surjective map $\phi: F \to M$, kernel of phi is finitely generated. I should clarify, kernel of a R linear map is an R module. I did not explicitly state this, I did not actually even define what kernel is. So, let us actually do that now, so it is just with regard to the previous line.

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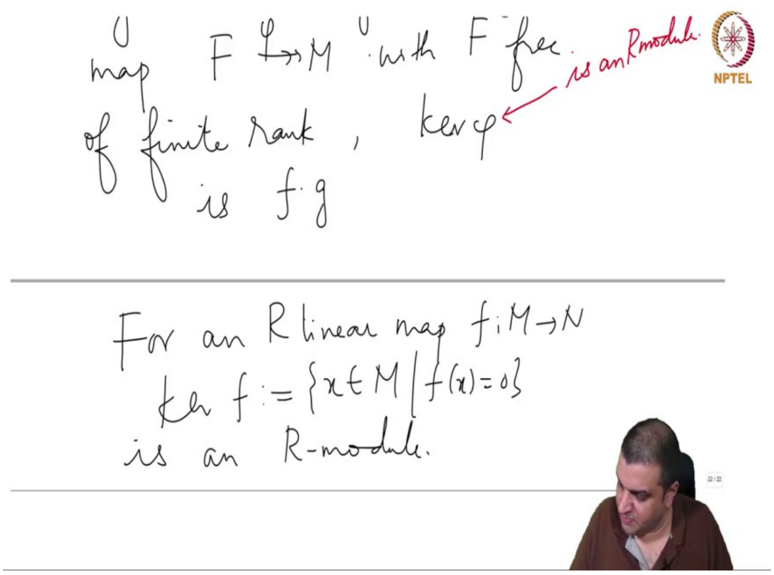

For an R linear map $f: M \to N$; kernel of f which is the stuff $\{x \in M | f(x) = 0\}$ is an R module.

So, the observation that we would like to make and we will quickly see an example of that is the following.

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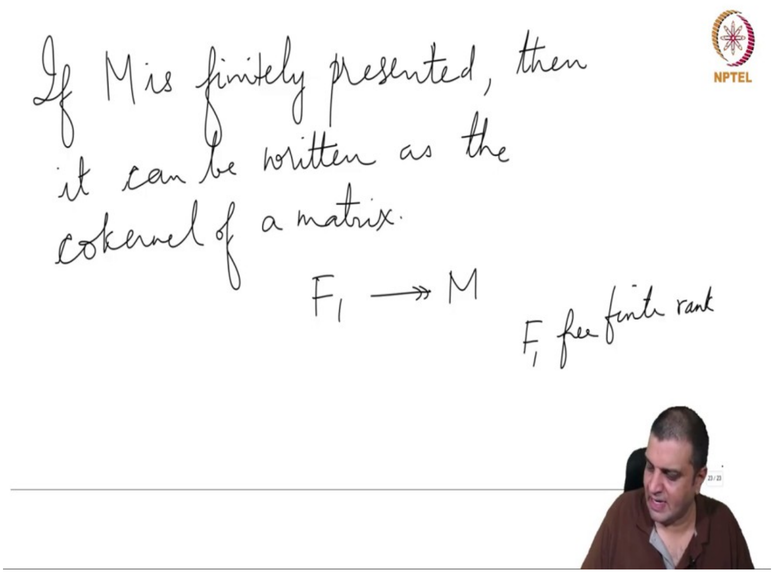

If M is finitely presented, then it can be written as the co kernel of a matrix. How? so let us assume M is finitely presented. Then there is some F_1 free module surjecting onto M. F_1 is free of finite rank. Finite rank just means like vector spaces, finite dimension; dimension is not a word used in this context, it means something else; so, this is ϕ.

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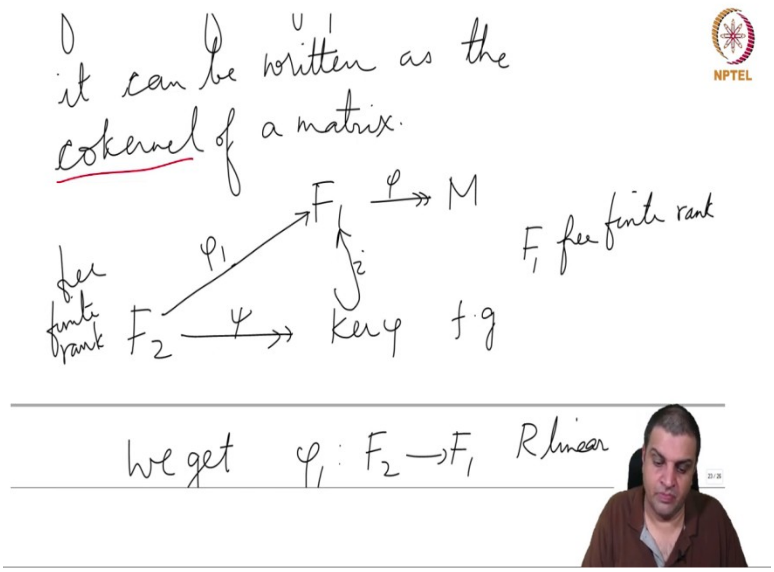

Now, kernel of ϕ is again finitely generated. Therefore, we can take some F_2 which maps like this. Now, let us look at the these maps. So, this is a kernel sits inside here; so this is an inclusion, F_2 is finite rank free; I mean both F_1 and F_2 are finite rank free. So, this one surjects onto this kernel; kernel sits inside F_1 and the composite here; so we can take the composite. So, let us call this map ψ and let us call this map i and let us call this map ϕ_1.

So, ψ is surjective, i is injective and we can take the composite; so we get a map like that. So, we get a map $\phi_1 : F_2 \to F_1$, this is R linear. So, you should check that this is R linear and so these are free modules of finite rank.

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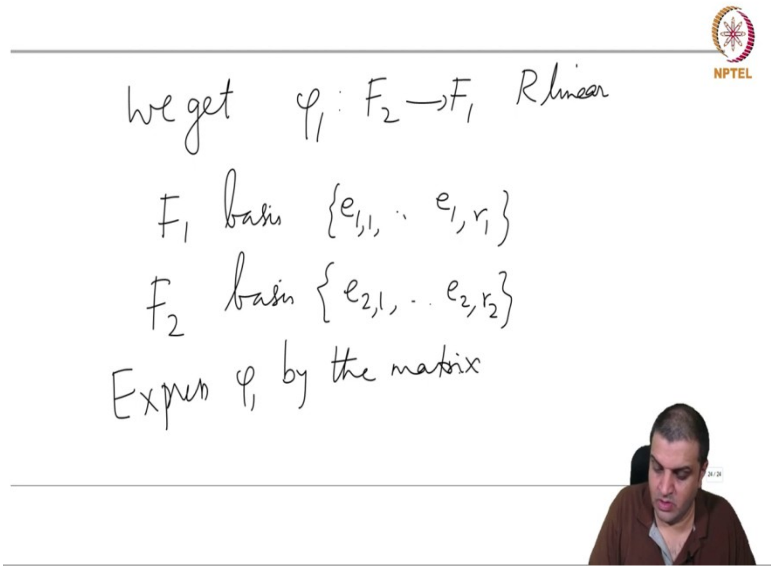

So, F_1 has basis; let us say $\{e_{1,1}, \ldots, e_{1,r_1}\}$ this is the rank of F_1. F_2 has basis $\{e_{2,1}, \ldots, e_{2,r_2}\}$, r_2 is the rank of F_2.

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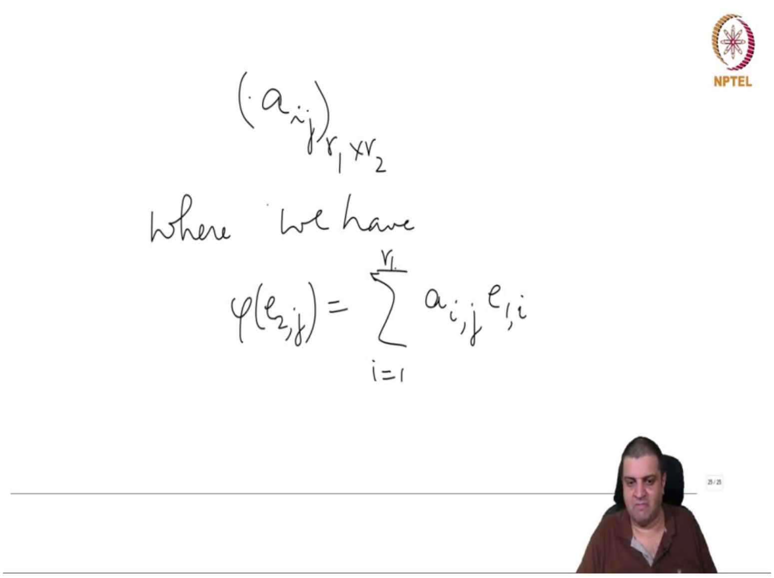

And now we can write this expression the same way we do for vector spaces; express ϕ_1 by the matrix $(a_{ij})_{r_1 \times r_2}$. So this will be a $r_1 \times r_2$ matrix where we have $\phi(e_{2,j}) = \sum_{i=1}^{r_1} a_{i,j} e_{1,i}$; so this describes the j-th column. So, this is one of the reasons; so for most of the rings that we will work with most of the modules will be finitely presented. One of the advantage is that it can be expressed in some finite data. So, what is co kernel? So, we will continue this discussion in the next lecture, but let me just finish this; what is co kernel.

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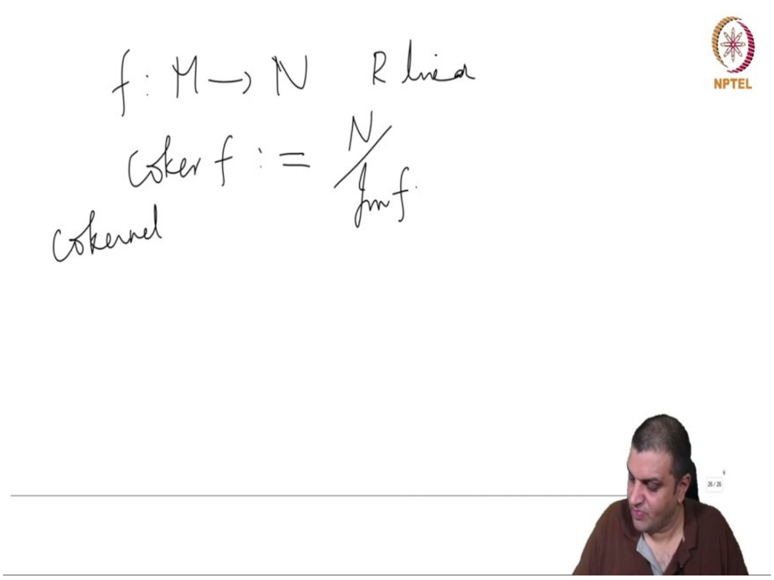

Co kernel means, if $f: M \to N$, R linear; then the co kernel of f is by definition $\frac{N}{\Im f}$. And the point here is that M is F_1modulo the kernel; because of this surjectivity and injectivity image of ϕ_1 is exactly the image of i; which is the kernel.

So, here what we should observe is that image of ϕ_1 is image of i, which is equal to kernel of this ϕ. Therefore M is isomorphic to the co kernel of this map. So there is some advantage in working with these modules. So, we will continue our discussion about modules in the next lecture.