Computational Commutative Algebra
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Lecture - 10
Nullstellensatz – Part 1

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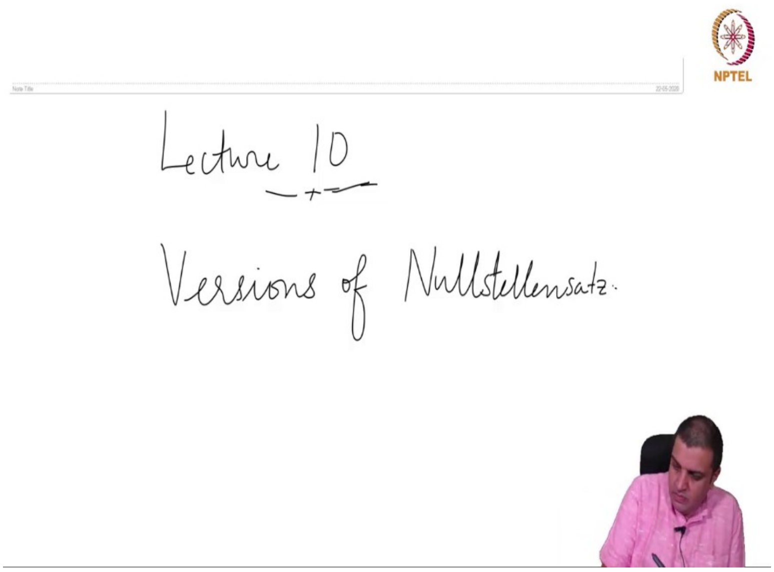

This is the 10th lecture in this course on Computational Commutative Algebra. So, now, so in this lecture we look at Versions of Nullstellensatz.

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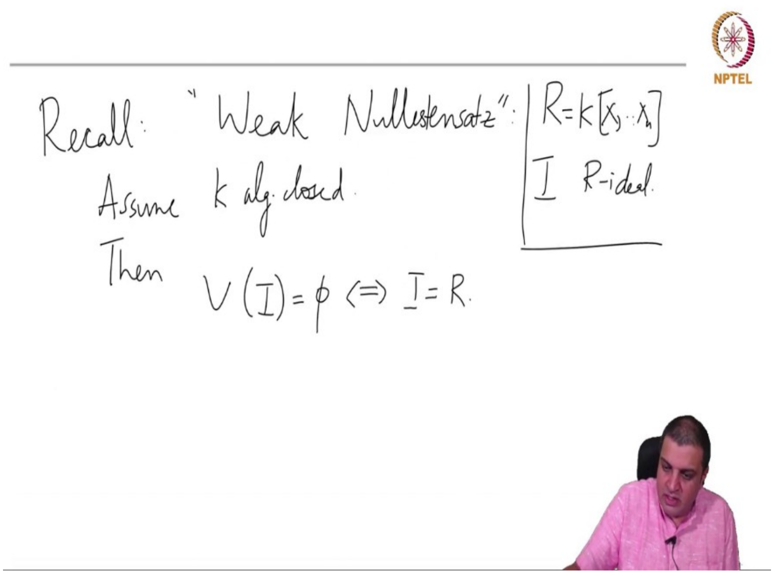

So, recall, what we call the weak Nullstellensatz. This, so throughout this lecture again sorry for saying this again; k is a field and R is a polynomial ring in some finitely many variables over k and I is an ideal of R.

So, with that notation, assume k is algebraically closed. Then $V(I) = \varnothing$ if and only if I = R . So, this is one version of the Nullstellensatz.

So, we will do another version.

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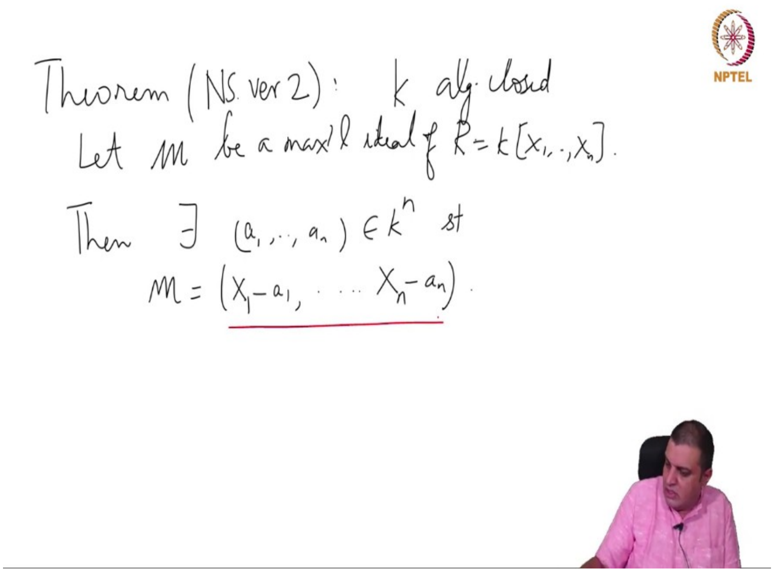

So, let us write down another version. So, I will just abbreviate as NS for Nullstellensatz version 2. So, this might look completely unrelated to the earlier version, but we will see that there is some connection. Let k be an algebraically closed field.

Let m be a maximal ideal of $R = k[x_1, \ldots, x_n]$; then there exists a point $(a_1, \ldots, a_n) \in k^n$, such that $m = (x_1 - a_1, \ldots, x_n - a_n)$. So, let us think about this ideal for a minute. W e will prove that the weak Nullstellensatz that we define here that, we recall here something about the variety of an ideal be empty is equivalent to the statement; although the way it is written, they look very different.

So, let us try to understand this ideal m here.

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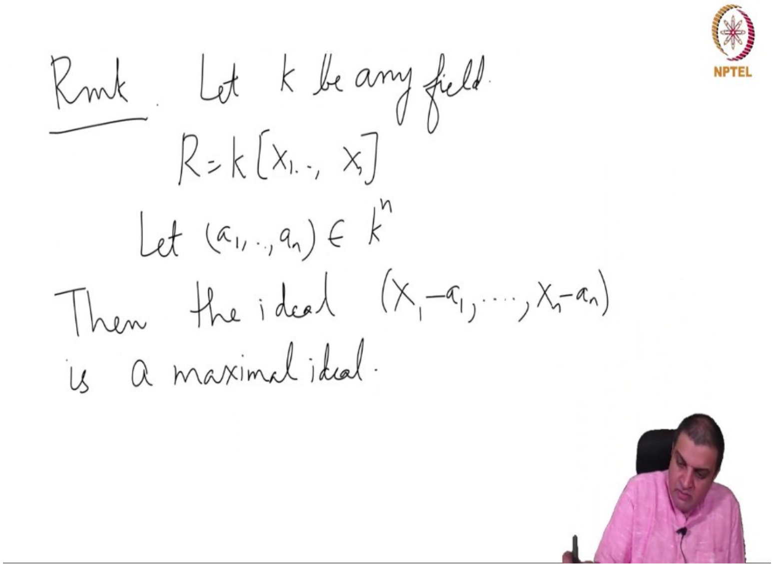

Remark: let k be any field, not necessarily algebraically closed, $R=k[x_1,\ldots,x_n]$ and let $(a_1,\ldots,a_n)\in k^n$. Then $(x_1-a_1,\ldots,x_n-a_n)$ is a maximal ideal. I will explain it in a minute.

Let us go back to the statement of the theorem, in the theorem it says that every maximal ideal looks like this; here we are say over an algebraically close field, here we are saying that over any field, this is a maximal ideal. And why is that so?

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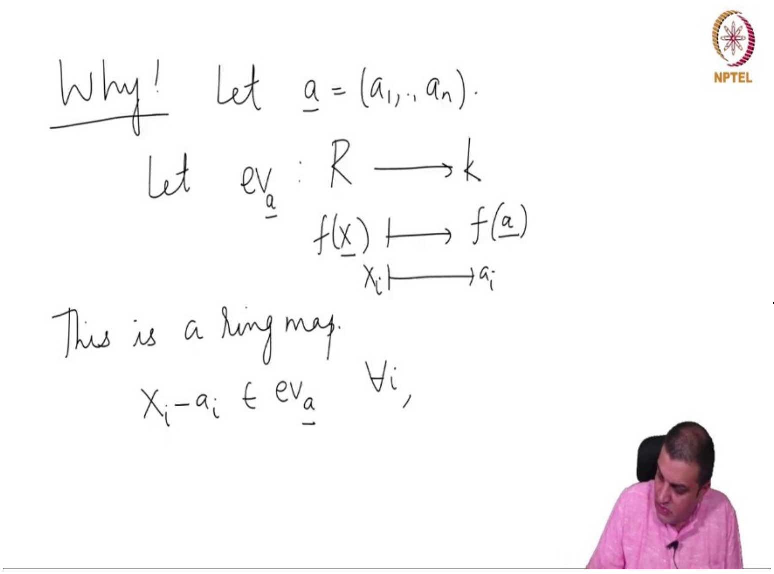

$\frac{a}{\square}$

So, let us understand why? So, if we want to show that something is a maximal ideal, we can just show that it is a kernel of a map to a field, a surjective map onto a field; in other words the ring modulo of that ideal is a field.

So, let us say $\underline{a}$ denotes $(a_1, \ldots, a_n)$ and let ev_a: R $\rightarrow$ k where f($\underline{x}$) $\rightarrow$ f($\underline{a}$) and $x_i \rightarrow a_i$. So, this is a ring homomorphism. Check! $x_i - a_i \in Ker(ev_a)$ for all i, So, that is one observation,

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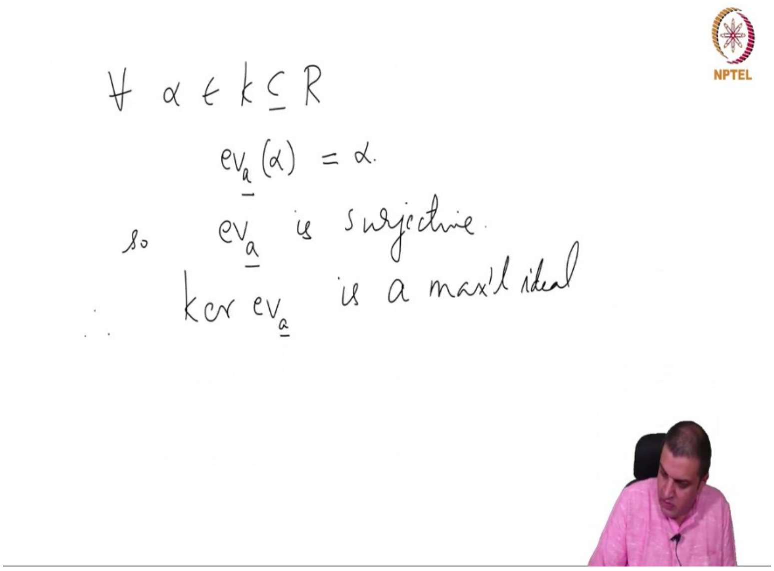

And $\forall\, \alpha \in k \subseteq R, ev_a(\alpha) = \alpha$. *So*, ev_a *is surjective*. $Ker(ev_a)$ *is a maximal ideal*.. And what we will show is that, it is the ideal generated by these linear polynomials.

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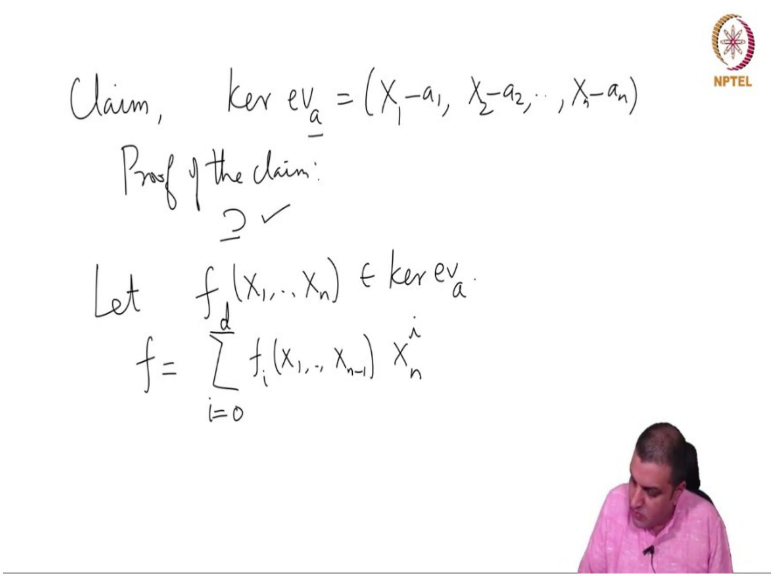

So, claim $: Ker\, ev_a = (x_1 - a_1, \ldots, x_n - a_n)$. So, now, let us ask, why this is true? . We already observed that this inclusion $\supseteq$is true; that each one of them is in the kernel, so the ideal generated by that by them is in the kernel.

So, now, what we want to prove is the other way around. So, $f(x_{1,\ldots,}x_n) \in Ker(ev_{\underline{a}}.)$

f = $\sum_{i=0}^{d} f_i(x_i, \ldots, x_{n-1}) x_n^i$.

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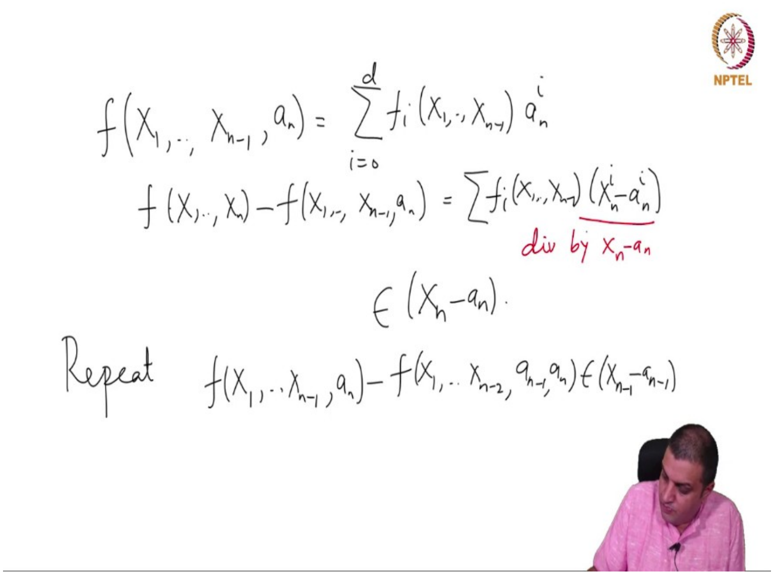

Now, consider $f(x_{1,\dots,}x_{n-1},a_n)=\sum_{i=0}^{d} f_i(x_i,\dots,x_{n-1})a_n^i$

. let us go back, this is just expression of f; now we are just evaluating it here. So, this is, if you take the difference.

So, this is the polynomial $f(x_{1,\dots,}x_n)-f(x_{1,\dots,}x_{n-1},a_n)=\sum_{i=0}^{d} f_i(x_i,\dots,x_{n-1})(x_n^I-a_n^i)$. ; but this term is divisible by $a_n{}^i$, this is divisible by x_n-a_n. Therefore, this element is inside the ideal generated by x_n-a_n and we can repeat this. Now evaluate this polynomial.

So, let me just do it once one more step $f(x_{1,\dots,}x_{n-1},a_n)-f(x_{1,\dots,}x_{n-1},a_{n-1},a_n)\in(x_n-a_n)$. .

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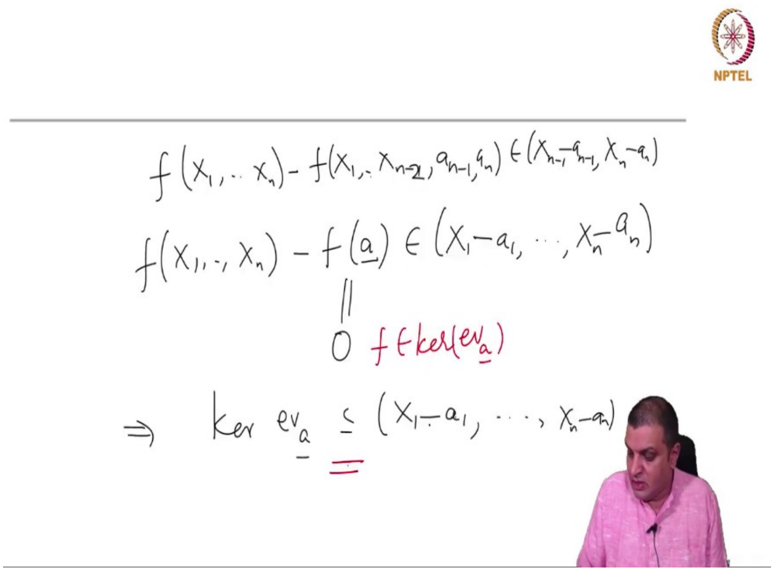

So, therefore, so repeat; so therefore, we will get that $f(x_{1,\dots,}x_n)-f(x_{1,\dots,}x_{n-1},a_{n-1},a_n)\in(x_{n-1}-a_{n-1},x_n-a_n)$

. And now repeat this to conclude that, $f(x_{1,\dots,}x_n)-f(\underline{a})\in(x_1-a_1,\dots,x_n-a_n)$.

But notice that $f(\underline{a})=0$; this is the thing since $f(x_{1,\dots,}x_n)\in Ker(ev_{\underline{a}}.)$, . So, that proves that : $Ker\,ev_a\subseteq(x_1-a_1,\dots,x_n-a_n)$ and the hence it is equality here.

So, this is the this is the remark that, over any field ideal such as this are maximal ideals. The content of the version 2 of the Nullstellensatz is that, over an algebraically closed field every maximal ideal looks like this, ok. And let us just make one more remark.

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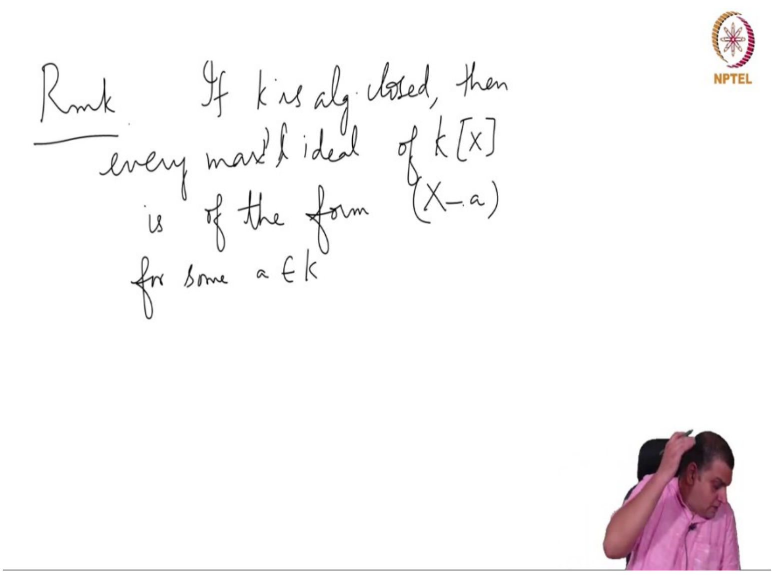

Another remark, if k is algebraically closed; then every maximal ideal of, maximal ideal of k[x] is of the form (x - a) for some $a \in k$.

So, what the Nullstellensatz, the version 2 of the Nullstellensatz is does is to generalize this statement to larger number of variables. And this is prove how, how does one prove this? Well, if a maximal ideal, then it has to be generated by any reducible polynomial; and if it is any polynomial in over k splits into linear factors, so a reducible polynomials are precisely the linear polynomials and that is.

So, any proof of any version of the Nullstellensatz requires considerable amount of work, and for this course we will postpone the proof to a later stage, where we will prove this version 2 that we just stated; that will come after we discuss integral extensions and Noether Normalization Lemma. One can directly prove the weak Nullstellensatz using elimination and resultants; there are other proofs of stronger statements from which these two will follow.

So, what we will do today now is to prove that the 2 versions that we stated; what we call the weak Nullstellensatz and which is about the emptiness of V of I. And the version 2 which is about maximal ideals they are equivalent to each other; that we will see now.

The proof of either, proof of version 2 will be given later, so that would establish the; only then we will have proved Nullstellensatz, till then I mean, however we will continue using them. It is just that we postpone the proof till we develop sufficient techniques to prove it. But one should not wait till that is done to see how it is applied.

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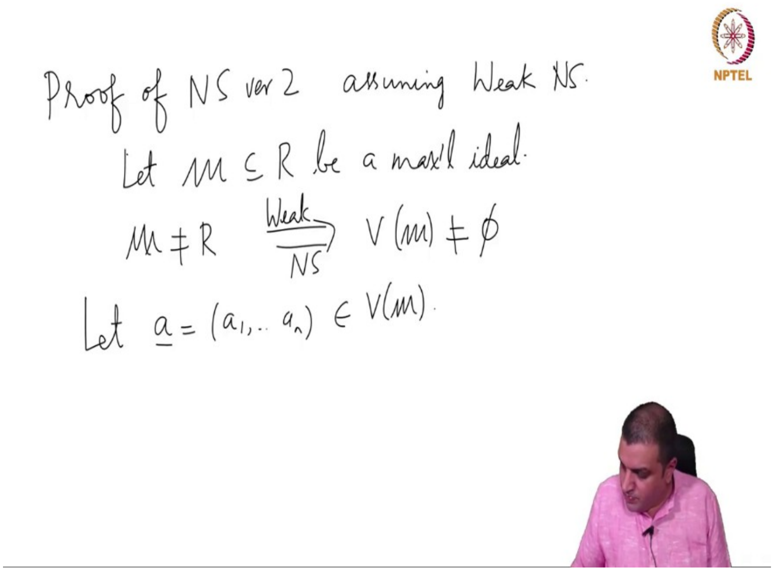

So, now proof of Nullstellensatz version 2 which is the statement about maximal ideals, assuming the weak Nullstellensatz. So, we will assume that, we will assume this statement about the emptiness of V(I) and prove this. So, what do we want to show? So, let $m\subset R$ be a maximal ideal, then m = R by definition; therefore by weak Nullstellensatz $V(m)\neq\emptyset$. So, take a point, let a, which is $(a_{1,},\ldots,a_n)\in V(m)$.

So, let us, consider what is the set of polynomials vanishing at that point? So, let us sorry let us go back here; . So, now, we want to look at this as a variety.

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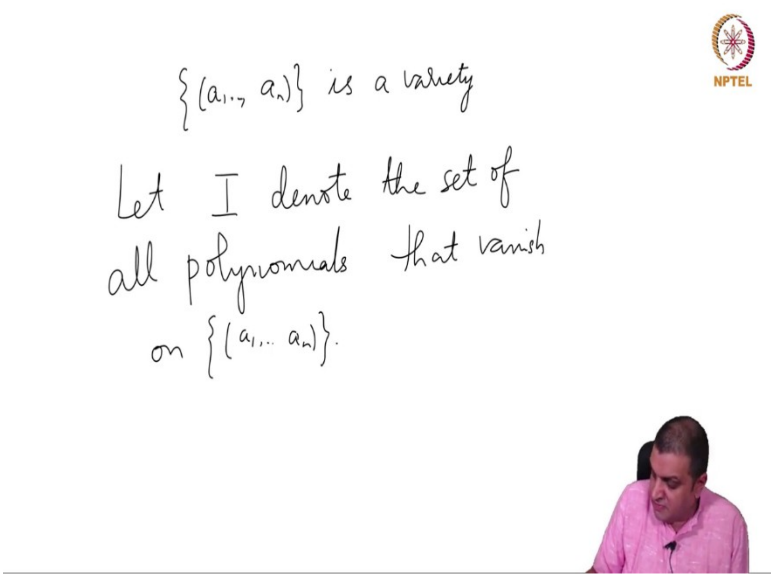

So, this singleton point inside $\{(a_{1,\ldots},a_n)\}\in k^n$ is a variety.

Now we talked about the ideal of a variety; the set of all polynomials that vanish in that I on that variety. So, this is a variety. Let I denote the set of all polynomials that vanish on $\{(a_{1,\ldots},a_n)\}$; in other words that vanish at $\underline{a}$. So, then we observed the following.

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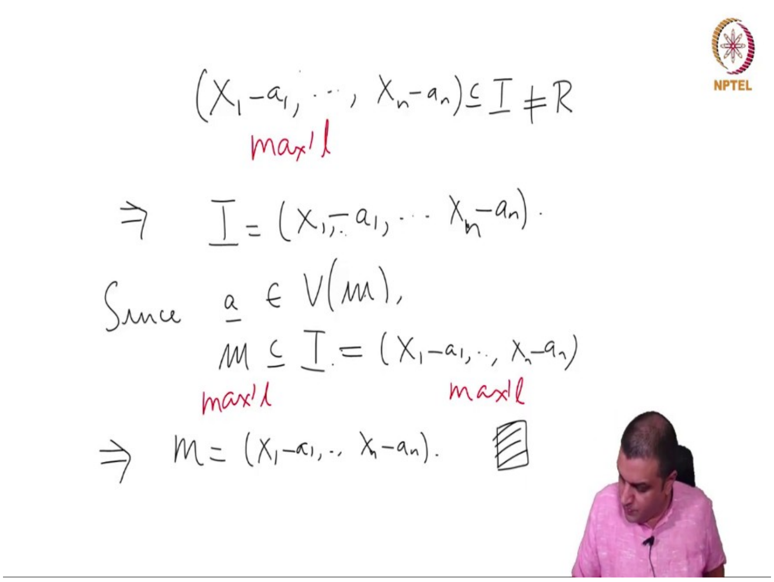

Then $(x_1 - a_1, \ldots, x_n - a_n) \subseteq I$; and the constant polynomial 1 does not vanished at that point; so this is not equal to R. This is the maximal ideal that, we proved any such ideal is maximal over any field this we proved in the remark. So, therefore, $(x_1 - a_1, \ldots, x_n - a_n) = I$

. So, let us go back what this ideal is.

So, this is the ideal, this is the set of all polynomials that vanish on at that point. So, this is the observation that we need. So, in particular, since this point $\underline{a} \in V(m)$, $m \subseteq I$, that let us just think about from it; m is some ideal which some collection of polynomials which vanish at $\underline{a}$, I is the set of all polynomials that vanish at $\underline{a}$; therefore it must be inside I.

But again, again this is remember this is equal to $(x_1 - a_1, \ldots, x_n - a_n)$. Again this is maximal meaning not equal to the whole ring, this is also maximal; but maximal is a property under containment and this cannot be a strict inequality then, this now implies that $m = (x_1 - a_1, \ldots, x_n - a_n)$. So, this proves the version 2 of the Nullstellensatz from version 1, which was the weak a Nullstellensatz.

Let us we can quickly take a look at a Macaulay 2 example, which shows about this; I mean just to see that it is a maximal ideal.

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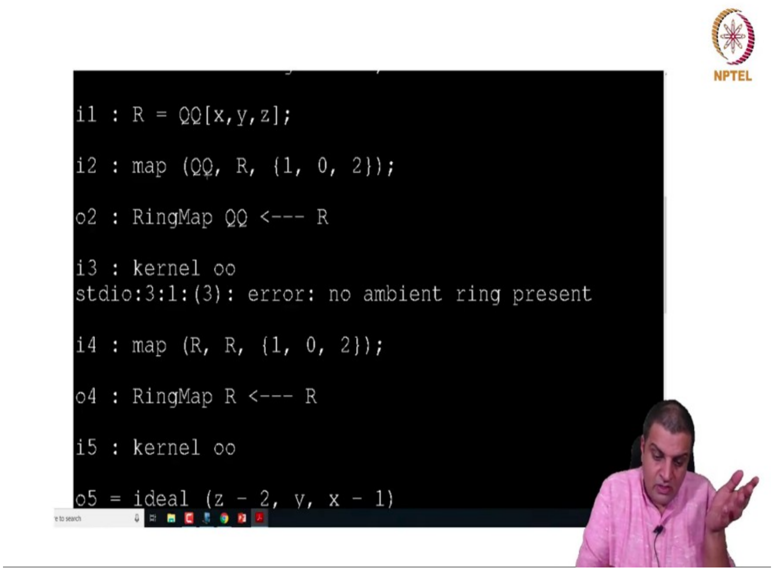

So, we define a polynomial ring in three variables x, y, z, we map from $R\,tend\,to\,Q$; so this says $x \to 1, y \to 0, z \to 2$.

So, it is evaluating a polynomial at 1, 0, 2; but then [vocaiszed-noise] Macaulay 2 throws up some error, that is because in order to compute these thing through algorithms, all these rings must have certain structure and in this case QQ is not does not satisfy that. So, it has to be some algebra where there are coefficients and then there are generators and QQ is not defined like that in Macaulay 2.

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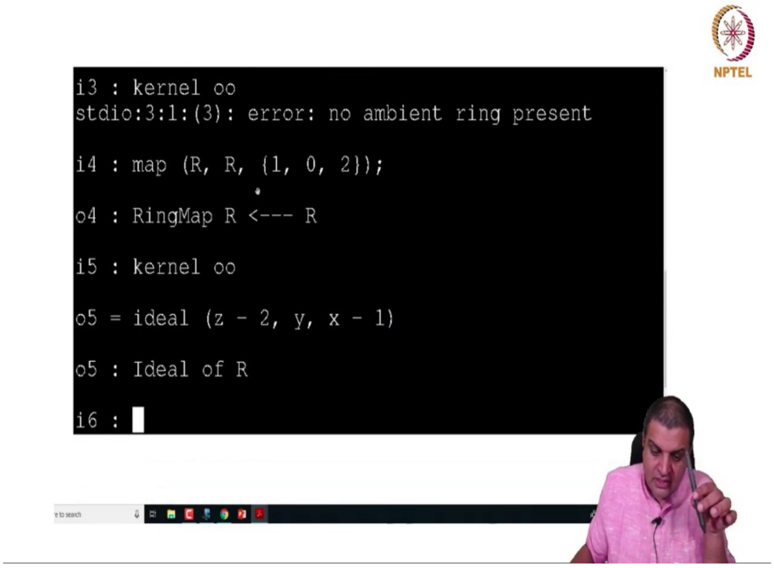

So, one must be wary of these issues. So, the way one to get out of that is, just define a map from $R \to R$ itself. So, remember Q sits inside as the sub ring of constant polynomials. So, if you map like this, the image of this map is actually Q; that is because these are constants. So, this is indeed we doing the same map as earlier.

And if you ask for it is map, now the algorithm is able to run and it will produce what is expected (z - 2, y, x - 1),. So, we will prove the other direction also.

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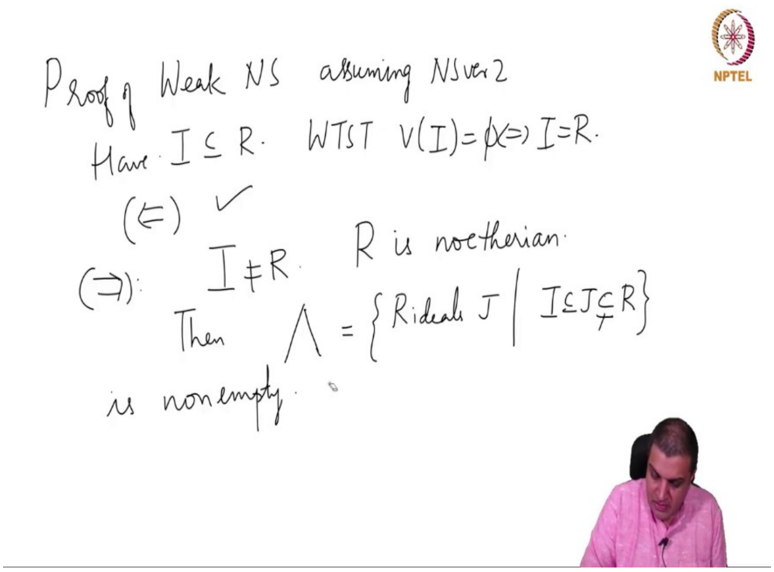

So, proof of the weak Nullstellensatz, assuming the Nullstellensatz version 2. So, this is the description of maximal ideals, this is the description of V (I). So, what, what do we have? We have an ideal $I \subset R$.

So, what do we want to show? $V(I) = \varnothing \; iff \; I = R$. So, one direction does not mean anything, I mean this is automatic. If I = R, then there is no point at which every element of I vanishes, the constant the nonzero constant polynomials would not vanish anywhere. So, which is here that we need.

we assume that if I is R, then there is nothing prove; now we assume $I \neq R$ and then prove that this is not empty. So, remember that R is a Noetherian ring that we have proved. Then the collection of ideals J, such that $\Lambda = \{R - ideal\, J : I \subseteq J\} \neq \varnothing$.

For example, I itself is inside here; this implies that Λ has a maximal element.

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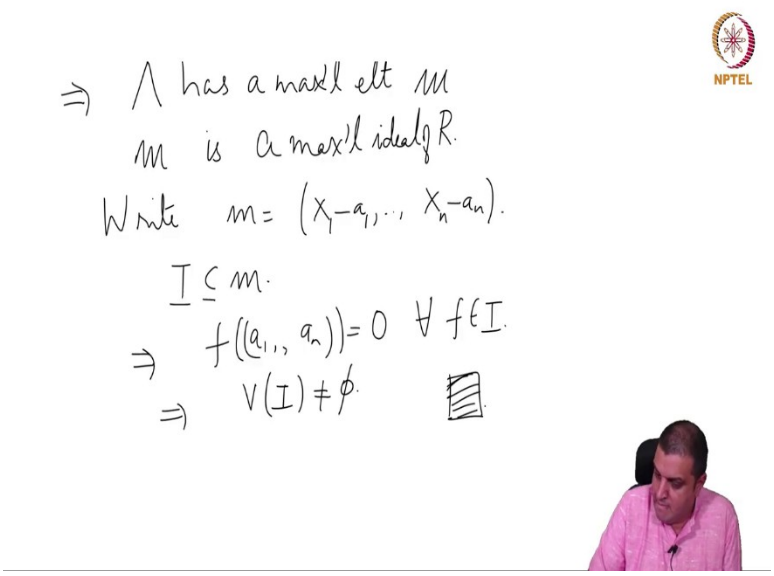

This is one of the characterizing properties of Noetheiran rings; every non empty collection of ideals has a maximal element, this implies that Λ has a maximal element. Let us call it m, . As such it is just a maximal element of Λ; however the way, because what lambda is m is a maximal ideal of R.

So, remember that the word maximal element; this refers to maximal element in this posit, here it is a maximal ideal. So, write $m=(x_1-a_{1,}..x_n-a_n)$. So, then $I\subseteq m$; then again we can show that this is the kernel of the evaluation map . So, therefore, this implies that $f(a_{1,},\ldots,a_n)=0$ for all $f\in I$; in other words $V(I)\neq\varnothing$.

So, that is the end of this.