Algebra - I
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ALGEBRA I

1. Lecture 80: Similarity of Matrices

Let us start with the finite dimensional vector space V with basis $u_1, \ldots, u_n$ and T linear map from V to V. Let A be the matrix of T with respect to this basis. Suppose we have another basis, $v_1, \ldots, v_n$. Let us say $v_j = \sum_{i=1}^{n} X_{ij} u_i$, and define $X = (X_{ij})$.

Theorem 1.1. *Let B denote the matrix of the transformation T with respect to the basis $v_1, \ldots, v_n$. Then*

$$B = X^{-1}AX.$$

Definition 1.2. *Two matrices A, B are similar if there exists a matrix X such that $B = X^{-1}AX$.*

Two matrices are similar if and only if, they represent the same linear transformation with respect to different basis. Similarity is an equivalence relation.

Any property of a matrix A that is defined in terms of its linear transformation is invariant under similarity. For example, the rank of a matrix A.

Definition 1.3. *A matrix A is idempotent if $A^2 = A$.*

Lemma 1.4. *If A is idempotent and B is similar to A then B is idempotent.*

Definition 1.5. *A matrix A is nilpotent if there exists some positive integer k such that $A^k = 0$.*

Given a polynomial $p(t) = a_0 + a_1 t + \ldots + a_n t^n \in K[t]$, define $p(T) = a_0 I + a_1 T + \ldots + a_n T^n$. Define the ideal I of all polynomials $p \in K[t]$ such that $p(T)$ is the linear transformation. Since $K[t]$ is a principal ideal domain there must be a polynomial $q \in K[t]$ such that $I = (q)$. This polynomial q is called the minimal polynomial of A.

Lemma 1.6. *If A is similar to B, then the minimum polynomial of A is equal to the minimal polynomial of B.*

We also have that the trace of similar matrices are equal, as are the determinants.

Definition 1.7. *The characteristic polynomial of a (square) matrix A is*

$$det(tI - A).$$

Lemma 1.8. *Similar matrices have the same characteristic polynomial.*