Algebra - I
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ALGEBRA I

1. Lecture 79: Structure of finitely generated abelian groups

Let R be a PID and M be a finitely generated R-module generated by $\{x_1, \ldots, x_m\}$. Define a surjective homomorphism $\phi : R^m \to M$ by

$$\phi(e_i) = x_i, \text{ for } i = 1, \ldots, m.$$

Then $M \cong R^m/\ker\phi$. Suppose $\ker\phi$ is generated by $\overrightarrow{v_1}, \ldots, \overrightarrow{v_n}$. Let $A \in M_{m\times n}(K)$ be defined by

$$A = (\overrightarrow{v_1}| \ldots |\overrightarrow{v_m}).$$

Then $M \cong R^m/\mathcal{C}(A)$.

Suppose A has Smith normal form

$$D = \begin{pmatrix} d_1 & 0 & 0 & \ldots & 0 & \ldots \\ 0 & d_2 & 0 & \ldots & 0 & \ldots \\ 0 & \ldots & \ddots & 0 & \ldots & \ldots \\ 0 & \ldots & 0 & d_r & 0 & \ldots \\ 0 & 0 & \ldots & 0 & 0 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix},$$

with $(d_r) \subseteq (d_{r-1}) \subseteq \ldots \subseteq (d_1)$.

Then note that $M \cong R^m/\mathcal{C}(A) \cong R^m/\mathcal{C}(D)$ and thus

$$M \cong R/(d_1) \oplus \ldots \oplus R/(d_r) \oplus R^{n-r}.$$

Theorem 1.1. *Every finitely generated module M over a PID R is isomorphic to*

$$R/(d_1) \oplus \ldots \oplus R/(d_r) \oplus R^f,$$

for $(d_r) \subseteq \ldots \subseteq (d_1) \subset R$ and $f \geq 0$. Moreover r, f and the ideals $(d_1), \ldots, (d_r)$ are unique.

Note that $\mathbb{Z}-$modules $\leftrightarrow$ abelian groups. The action is given by $\pm ka = \pm(\underbrace{a + \ldots + a}_{\text{k times}})$.

Theorem 1.2 (Structure theorem for finitely generated abelian groups)**.** *Every finitely generated abelian group A is isomorphic to a unique abelan group of the form*

$$A \cong \mathbb{Z}/d_1\mathbb{Z} \times \ldots \times \mathbb{Z}/d_r\mathbb{Z} \times \mathbb{Z}^f.$$

Note that $|A| < \infty \iff f = 0$, and if $f = 0$ then $|A| = d_1 \dots d_r$.

Example 1.3. *How many isomorphism classes of finite abelian groups of order 12? For each such class we have the decomposition:*

$$A \cong \mathbb{Z}/d_1\mathbb{Z} \times \dots \times \mathbb{Z}/d_r\mathbb{Z},$$

with $d_1 | \dots | d_r$ *and* $d_1 \geq 1$ *and* $d_1 \dots d_r = 12$. *There are two such possibilities:*

- $A \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}$
- $A \cong \mathbb{Z}/12\mathbb{Z}$.

Note that $A \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$ *is not a valid decomposition.*

Given a prime $p \in R$ define

$$M_p = \{m \in M | p^k m = 0 \text{ for some } k > 0\}.$$

M is p-primary if $M = M_p$. $M \cong \oplus_{(p) \text{ is prime ideal}} M_p$.

Lemma 1.4. *The module* $R/(d)$ *is* p*-primary iff* $(d) = (p^r)$ *for some* $r \geq 0$.

Proof. $R/(d)$ is p-primary $\implies p^k(1 + (d)) = (d)$ for some $r \geq 0$. So $p^k \in (d)$ for some $k > 0$ which means $d | p^k$ for some $k > 0$. Thus $(d) = (p^r)$. $\square$

If M is a finite p-primary R-module then

$$M \cong R/(p^{k_1}) \oplus \dots \oplus R/(p^{k_r}),$$

for some $k_1 \leq \dots \leq k_r$.

For a finitely generated abelian group

$$A \cong A_{\text{tor}} \times \mathbb{Z}^f,$$

for some f. $A_{\text{tor}} \cong A_{p_1} \times \dots \times A_{p_r}$ for some prime numbers $p_1, \dots, p_r$ and p_i-primary modules A_{p_i}.

Note that a finite abelian group is p-primary iff its order is a power of p. Such groups are called abelian p-groups.

Example 1.5. *How many nonisomorphic finite abelian groups are there of order 360?*

Since $360 = 2^3 3^2 5$, *we have*

$$A \cong A_2 \times A_3 \times A_5,$$

and $|A_2| = 8, |A_3| = 9, |A_5| = 5$. *For the* 2*-primary and* 3*-primary components we have*

$$A_2 \cong \mathbb{Z}/2^{k_1}\mathbb{Z} \times \dots \times \mathbb{Z}/2^{k_r}\mathbb{Z}$$

$$A_3 \cong \mathbb{Z}/3^{l_1}\mathbb{Z} \times \dots \times \mathbb{Z}/3^{l_s}\mathbb{Z}$$

with $\sum_i k_i = 3$ *and* $\sum_i l_i = 2$. *There are three possibilities for* A_2, *namely*

$$\begin{aligned}&\mathbb{Z}/8\mathbb{Z}\\&\mathbb{Z}/4\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}\\&\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}\end{aligned}$$

and two for A_3, *namely*

$$\begin{aligned}&\mathbb{Z}/3\mathbb{Z}\times\mathbb{Z}/3\mathbb{Z}\\&\mathbb{Z}/9\mathbb{Z}.\end{aligned}$$

A_5 *is* $\mathbb{Z}/5\mathbb{Z}$.

Thus there are $3 \times 2 \times 1 = 6$ *isomorphism classes of abelian groups of order 360.*

Definition 1.6. *A partition of* n *is a nonincreasing finite sequence of positive integers* $(k_1, \ldots, k_s)$ *such that* $\sum_i k_i = n$. *The set of partitions of* n *is denoted* $Par(n)$.

Lemma 1.7. *The number of isomorphism classes of an abelian* p*-group of order* p^n *is equal to the number of partitions of* n.

Theorem 1.8. *Let* $n = p_1^{n_1} \ldots p_r^{n_r}$ *be the prime factorisation of* n. *The number of isomorphism classes of abelian groups of order* n *is*

$$\prod_j Par(n_j).$$