Algebra - I
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ALGEBRA I

1. Lecture 75: Smith Canonical Form for a Euclidean domain

Theorem 1.1. *Let R be a Euclidean domain. Every $m \times n$ matrix A with entries in R is equivalent to a matrix of the form $diag(d_1, \ldots, d_r, 0, 0)$ for some $r \leq min\{m, n\}$, where the $(d_1) \supseteq \ldots \supseteq (d_r)$. The factors $d_1, \ldots, d_r$ are unique upto multiplication by units.*

Proof. Firstly we show that you can start with any matrix over a Euclidean domain and then reduce it to a Smith canonical form. And then we show that this Smith canonical form is unique (upto multiplication by units). The algorithm proceeds by induction. If A is 0 then we are done. Otherwise by interchanging rows and columns, we can assume that $a_{11} \neq 0$ and is minimal among the columns. So we have $a_{1j} = qa_{11} + r$ and the operation $C_j = C_j - qC_1$ so that $a_{1j} = r < a_{11}$. Then we exchange the columns C_j to obtain $a_{11} = r$ updating the entry with a lower value. Such a process cannot continue indefinitely; after finitely many steps, we would have cleared out all the entries in the first row except for the $(1, 1)$ entry- the prime d_1. we get a matrix which is d_1 in the $(1, 1)$ spot which is the only nonzero entry in the first row and the first column. Let A' be the submatrix of A excluding the first row and column. d_1 divides all the entries of A'.By induction on A' we obtain the Smith canonical form.

Given equivalent matrices

$$A = \text{diag}(d_1, \ldots, d_r, 0, 0, 0)$$
$$B = \text{diag}(e_1, \ldots, e_s, 0, 0, 0)$$

We show $(d_i) = (e_i)$ and $r = s$. In fact we need only show (since induction) that $(d_1) = (e_1)$. We use the result that all nonzero k by k minors of A are either 0 , or of the form $d_{i_1}, \ldots, d_{i_k}$, where $i_1 < \cdots < i_k$. If A is equivalent to B ; then the ideal generated by k by k minors of A is equal to those k by k minors of B. Recall from the lecture on a tensor algebras and exterior algebras that $\wedge^k A$ has as entries the k by k minors of A. If A is equivalent to B, then we have $B = GAH$ for some matrix $G \in GL_m(R)$ and $H \in GL_n(R)$ and $\wedge^k B = \wedge^k G \circ wedge^k A \circ \wedge^k H$. So, the ideal generated by k by k minors of A has to be equal to the ideal generated by k by k minors of B. So $(d_1 \ldots d_r) = (e_1 \ldots e_r)$ and for $k > r$ we have $(e_1 \dots e_k) = (d_1 \dots d_k) = 0$ so $r = s$. We have $(d_1) = (e_1)$ so that $d_1 = ue_1$ for a unit u. $\square$