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ALGEBRA I

1. Lecture 74: Equivalence of Matrices

Definition 1.1. *Let R be a commutative ring. Then for $X, Y \in M_{m\times n}(R)$ we say that X is equivalent to Y if there exists two matrices $A \in GL_m(R)$ and $B \in GL_n(R)$ such that $Y = AXB$.*

This is an equivalence relation.

Theorem 1.2. *If $X \sim Y$ then $R^m/\mathcal{C}(X) \cong R^m/\mathcal{C}(Y)$ as R-modules, where $\mathcal{C}(X)$ is the column space of X.*

Two matrices A, B are equivalent if they are both also equivalent to the canonical representative of their equivalence class. The algorithm for finding this representative when R is a field is as described below. If A is 0 the algorithm terminates.

If not, we ensure a_{11} is nonzero by interchanging rows and columns. We eliminate all other entries in the first row by $R_i \to R_i - \frac{a_{1i}}{a_{11}} R_1$. We eliminate all the entries in the first column by $C_i \to C_i - \frac{a_{1i}}{a_{11}} C_1$, and $a_{11} \to 1$. After doing this we rearrange the remaining rows and columns to make the a_{22} entry nonzero and repeat this process. At the end of the algorithm we obtain a diagonal matrix $\operatorname{diag}(1, \ldots, 1, 0, \ldots, 0)$. The number of 1s is the rank of the canonical matrix. The rank of a matrix X is the rank of the canonical representative in its class. Two matrices are equivalent if they have the same rank.