Algebra - I
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ALGEBRA I

1. Lecture 71: Counterexamples to the Noetherian condition

In this lecture we are going to look at counter examples to the Noetherian Condition.

Example 1.1. *Take $R = K[x_1, x_2, \dots]$ where K is a field. Then the chain $(x_1) \subset (x_1, x_2) \subset \dots$ does not terminate. When R is not a Noetherian ring you can always find a finitely generated R - module which has a sub module that is not finitely generated. Take as an R - module R itself which is finitely generated. Let $I = \cup_r (x_1, \dots, x_r)$ which is a sub-module of R.*

Example 1.2. *So, take R to be any ring and take M to be the infinite direct sum of R. Let $M_n \subset M$ consisting of sequences $(r_1, r_2, \dots)$ such that $r_i = 0$ for $i > n$. Then $M_1 \subset M_2 \subset \dots$ is a strict ascending chain which does not terminate.*

Example 1.3. *Take R to be the ring of integers and take M to be the rational numbers as a R module. Take M_n to be all rational numbers which are of the form $2^{-n}k$ for some integer k. Then $M_1 \subset M_2 \subset \dots$ is an infinite ascending chain. So $\mathbb{Q}$ is not a Noetherian $\mathbb{Z}$- module.*

Lemma 1.4. *If U is a finitely-generated module of $\mathbb{Q}$ then it is generated by a single element.*

Proof. Suppose it is generated by say $q_1, \dots, q_n$. Let g be the gcd of the numerators of $q_1, \dots, q_n$ and let l be the lcm of the denominators, then $q = \frac{g}{l}$ generates the ideal $(q_1, \dots, q_n)$. $\square$